ABSTRACT

COMMUTATORS AND OPERATOR IDEALS

by

Gary Lynn Weiss

Chairman: Allen L. Shields

This work concerns itself with ideals of operators on separable Hilbert space and commutators of these operators. It is intended to be a complete and self-contained source for what is presently known on this subject. The following are two examples of the kinds of results we obtain.

We prove that every trace class operator with trace 0 which is also contained in a Schatten p-class for some $p < 1$ is the sum of 14 or fewer commutators of Hilbert-Schmidt operators.

We prove that if I is the two-sided ideal of operators which is the union of all the Schatten p-classes for which $p < 2$, then the operators in I^2 whose traces are 0 are precisely the finite linear combinations of commutators of operators in I.

We further prove that if every trace class operator with trace 0 is a finite linear combination of commutators of Hilbert-Schmidt operators, then the number of such commutators needed is 14 or fewer.

The central result is that any diagonal operator with entries $-d, d_1, d_2, \ldots$ satisfying $d_n \downarrow 0$, $\Sigma\, d_n = d < \infty$,

and $\Sigma (\log n) d_n < \infty$ is a finite linear combination of commutators of Hilbert-Schmidt operators. This result is important in view of another of our results which is that every trace class operator with trace 0 is a finite linear combination of commutators of Hilbert-Schmidt operators if and only if every diagonal operator with diagonal entries $-d, d_1, d_2, \ldots$ satisfying $d_n \downarrow 0$ and $\Sigma d_n = d < \infty$ is a finite linear combination of commutators of Hilbert-Schmidt operators.

We generalize Fuglede's Commutativity Theorem in a context relating to operator ideals. We show that in some cases the generalization holds true, and in others it fails.

We prove that the 1973 Brown, Douglas, and Fillmore characterization of $\mathcal{N} + K(H)$ does not extend, as is, to other operator ideals.

We introduce a new "stretch" axiom for Calkin ideal sets, and investigate some of its properties. We mention that independently of this thesis, we recently used ideas related to this axiom to develop a new characterization of all ideals of L(H) in terms of increasing sequences of positive integers. With this characterization and assuming the continuum hypothesis, we have shown in joint work with Andreas Blass that K(H) properly contains two two-sided ideals of L(H) whose sum is precisely K(H).

In regard to our work on commutators and a few other topics, the point of view is largely matricial and many of the techniques we develop are infinite matrix techniques.

COMMUTATORS AND OPERATOR IDEALS

by

Gary Lynn Weiss

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
1975

Doctoral Committee:

Professor Allen L. Shields
Professor Frederick Beutler
Assistant Professor Andreas Blass
Professor Carl Pearcy
Professor Melapalayam S. Ramanujan

To my Mommy

ACKNOWLEDGMENTS

I would like to express my thanks to Allen Shields, Norberto Salinas, Carl Pearcy, Hugh Montgomery, Andreas Blass, Layne Watson and Dennis Jespersen for their help and support in this endeavor. I would also like to thank Maxwell O. Reade and Rolf Turner for their help and support earlier in my career.

TABLE OF CONTENTS

INTRODUCTION

Bounded linear operators acting on a separable, infinite-dimensional complex Hilbert space have been objects of intense study for over 60 years. Some of the deepest and most useful results of the last 40 years were found by exploiting multiplicative commutativity relations between operators (e.g. the spectral theorem for normal operators, the theory of von Neumann algebras, and the theory of invariant subspaces). L(H), the set of all bounded linear operators acting on such a Hilbert space H, is a B*-algebra which is itself noncommutative, but which contains many important commutative and "almost" commutative classes. To understand the structure of L(H) we must study how operators commute as well as how they do not commute. One of the best ways to study noncommutativity is to study commutators, that is, operators of the form AB-BA. The operator AB-BA may be viewed as a measure of the commutativity of A and B. We say A and B almost commute if the operator AB-BA is "small". "Small" can have one of several qualitative or quantitative meanings. For example, an operator may be said to be small if it belongs to a specified class of operators, usually an ideal in L(H) such as the compact operators, or simply if its operator norm is small. Compared to the large body of knowledge on commuting operators, surprisingly little is known about noncommuting and almost commuting operators. Only in the last 10 years have many deep and important results been discovered in this area.

Originally, the study of commutators received its impetus from quantum mechanics. The Heisenberg uncertainty principle when formulated in mathematical terms involves the statement that the identity operator I is the commutator of two unbounded operators. Furthermore, physical considerations make it natural to conjecture that in order to have I = AB-BA, either A or B must be unbounded. That is, the identity is <u>not</u> a commutator of two bounded linear operators. This was first proved by Wintner [39] in 1947. Two years later, Wielandt [38] showed that the same result holds true in any normed algebra with an identity. In the 1950's, Halmos initiated the first purely mathematical study of commutators in [15], [16] and [17]. He noted that Wielandt's proof showed that an operator of the form $\lambda I + K$ (where $\lambda \neq 0$ is a complex number and K is a compact operator) is not a commutator of bounded operators, and he asked which operators are. He also introduced a number of basic techniques which remain extremely useful. In [16], he stated the following conjecture due to Kaplansky: If B = AX-XA and AB=BA, then B is quasinilpotent. Later, this was proved independently by Kleineke [21] and Shirokov [35]. In the early 1960's Pearcy [25] and Brown, Halmos and Pearcy [5] refined the Halmos techniques to give partial results on the structure of the class of commutators of L(H). Finally, in 1965, Brown and Pearcy [6] completely characterized the class of commutators of L(H) with the remarkable result that the only noncommutators are operators of the form $\lambda I + K$, where $\lambda \neq 0$ and K is a compact operator. At the same time, but using quite different techniques, Radjavi [28] characterized the class of self-commutators, that is, operators of the form $A^*A - AA^*$. In 1967 Putnam [27] published a short book containing most of what was known about commutators at that time. In 1971, Anderson [1] and Anderson and Stampfli [2] found an elegant and much shorter proof of the Brown and Pearcy characterization. Also in 1971, Pearcy and Topping [26] began to study the commutator structure of special classes of operators. In particular, they considered commutators of operators in various two-sided ideals of L(H) such as the class of compact operators and the Schatten p-classes. They proved that every compact operator is the sum of 10 (or fewer) commutators of compact operators. However, they left unsolved the question: Is every compact operator a commutator of compact operators. This remains unsolved today and provides much of the impetus for the current studies on compact operators. The techniques of the 1960's which were developed to characterize the commutator class of L(H) appeared to have little use when considering the commutator structure of classes of compact operators. Indeed, it appeared that new techniques needed to be developed to handle commutators of compact operators. In 1972 Salinas [31], in a sequel to [26], used the Pearcy and Topping techniques to extend their results. Since then, several peripheral papers have appeared extending the results in [26] to various Banach spaces (e.g. the L^p-spaces), but no further progress has been made on the essential structure questions of commutator classes. The 1970's has seen a flurry of activity in related areas. Deep results on 'derivations' (a branch of commutator theory) were discovered. In 1973, a surprisingly large number of important papers were published in which compact operators and commutators play central roles. Some of the most striking papers were Brown, Douglas and Fillmore [8], Berger and Shaw [4], and Helton and Howe [20]. It is especially interesting to note that the trace class and the trace play major roles in [4] and [20] and will also be crucial in this treatise.

In this thesis we study compact operators, operator ideals and commutators from a constructive point of view depending heavily on the use of sequences and infinite matrices. In the following remarks we summarize the main results of each chapter.

We first establish some notation. Let H be a separable, infinite-dimensional complex Hilbert space. Let L(H) denote the algebra of all bounded linear operators on H, and let K(H) denote the two-sided ideal in L(H) of all compact operators. Furthermore, let C_p denote the Schatten p-class, so that C_2 and C_1 are the Hilbert-Schmidt and trace class ideals, respectively, of compact operators in L(H). For each ideal I in L(H), let C(I) denote the class of commutators AB-BA where $A,B \in I$. Let $[I,I]$ denote the finite linear span of C(I). Let I^2 denote the ideal generated by $I \cdot I$ and, if $I \subset C_1$, let I^0 denote the class of operators in I whose trace is 0. Finally, let (N) denote the class of all normal operators in L(H).

We mentioned earlier that in 1965, Brown and Pearcy [6] characterized C(L(H)), and in 1971, Pearcy and Topping [26] began to study commutators of compact operators. In [26] the authors proved that $[K(H),K(H)] = K(H)$ and $C_1 \subset C_p = [C_{2p},C_{2p}]$ for every $p > 1$, and asked whether or not $C(K(H)) = K(H)$ (or even whether or not $P \in C(K(H))$, where $P = (a_{ij})_{i,j=1}^{\infty}$ and $a_{ij}=0$ for every i,j except $a_{11}=1$). They also asked whether or not $C(C_{2p}) = C_p$ for $p > 1$, $C(C_2)=C_1^o$, or $[C_2,C_2] = C_1^o$. In 1972 Salinas [31] attempted to generalize the Pearcy and Topping techniques and results.

In Chapter 1 of this work, we study commutators of compact operators. In Section 3, we give a short proof of the results of Pearcy and Topping, and Salinas, we generalize their techniques, and we mention that Salinas erred when he claimed to have improved the Pearcy and Topping techniques (He has acknowledged this fact). In fact, we mention that their techniques do not generalize naturally. In Section 4 (the main section) we develop infinite matrix techniques together with several infinite series and elementary number theoretic results to yield results on the structure of $[I,I]$ when $I \subset C_2$ (in which case $[I,I] \subset (I^2)^o$). We show that a certain class of well-known non-trivial ideals satisfy $[I,I] = (I^2)^o$. The main result yields, among others, the main structural relation

$$C_q^o \subset (\bigcup_{p<1} C_p)^o = [\bigcup_{p<2} C_p, \bigcup_{p<2} C_p] \subset [C_2, C_2] \subset C_1^o, \text{ for every } q<1.$$

This relation contains the set theoretic equation $[I,I] = (I^2)^o$ for $I = \bigcup_{p<2} C_p$, which is the largest ideal we have which we know satisfies this identity. Furthermore, we mention that in joint work with John Conway we have shown that these techniques cannot be generalized. In Section 5, we develop necessary conditions and sufficient conditions on an ideal I in order that $C_1 \subset [I,I]$. We further show that $\bigcup_{p<1} C_p \subset [I,I]$ if and only if $P \in [I,I]$, where P is the rank one projection operator mentioned earlier. In Section 6, we show that $P \neq A^*A-AA^*$ for any $A \in K(H)$ and we produce $A,B \in K(H)$ for which the matrix entries of AB-BA are the same as those of P except for two types of diagonals off the main diagonal on which the entries are of the order of the sequence (1/n). In Section 7, we show that if we let $[I,I]_n$ denote the set of all sums of n or fewer commutators in C(I), then $[I,I] \oplus 0 \subset [I,I]_3$. Furthermore, $(I^2)^o = [I,I]$ if and only if $[I,I]_{14} = [I,I] = (I^2)^o$, $I^2 = [I,I]$ if and only if

$[I,I]_8 = [I,I] = I^2$, and in particular, $[C_2,C_2] = C_1^o$ if and only if $[C_2,C_2]_{14} = [C_2,C_2] = C_1^o$. In other words, if the commutator equations in question were true, then it would suffice to consider 14 or fewer commutators. In Section 8, we pose several concrete problems for commutators of finite matrices and give partial results in this direction. It appears that a positive or negative solution to any of our commutator problems for finite matrices could yield the corresponding answer to the question 'Is $[C_2,C_2] = C_1^o$?'. In Section 9, we interpret the results of Chapter 1, expressing the likelihood that $[C_2,C_2] \neq C_1^o$ and likewise for the other equations under consideration.

Chapter 2 deals with generalizations of the Fuglede Commutativity Theorem (and the Fuglede-Putman generalization) [27] for normal operators. We ask which ideals I have "the generalized Fuglede property" that whenever $A \in (N)$, $B \in L(H)$ and $AB-BA \in I$, we then obtain $A^*B-BA^* \in I$ (The case $I = \{0\}$ is Fuglede's Theorem). We show that K(H) has this property, F(H) (the class of all finite rank operators) does not have this property, and C_2 has this property at least for those A that are also diagonalizable (We have recently shown, independent of this thesis, that C_p also fails to possess this property when $0 < p < 1$). Furthermore, if C_2 should fail to possess this property for some normal operator A, then it would follow that A cannot be expressed as the sum of a diagonalizable operator and a Hilbert-Schmidt operator. This would answer, in the negative, a well-known and important question of Halmos and Berg [3].

In Chapter 3, Section 1 we answer two questions of Brown, Douglas and Fillmore [8, especially pp. 123-124]. We prove that not every Hilbert-Schmidt operator is the sum of a normal operator and a trace class operator. Furthermore, there exist operators $A \in C_2$ so that $A \oplus Q \notin (N) + C_1$ for every $Q \in L(H)$. The significance of these results is that they show that the Brown, Douglas, Fillmore characterization of $(N) + K(H)$ [8, especially Theorem 11.2] in conjunction with the Helton and Howe [20] vanishing of the trace invariant fails to characterize $(N) + C_1$. In addition, we show that these characterizations fail to hold for $(N) + I$ for every ideal I for which $I \neq I^{1/2}$. We also construct two Hilbert-Schmidt weighted shift operators which are not trace class operators; one is the sum of a normal and a trace class operator and the other is not. This answers a question recently posed to me by Ciprian Foias, among others. In Section 2, we investigate the relationship between the diagonal sequences of matrix representations of operators and the membership of these operators in ideals. The main result is that if $1 \leqq p < \infty$, then $T \in C_p$ if and only if, for every basis $\{e_n\}$, the diagonal sequence $((Te_n, e_n))_{n=1}^{\infty} \in \ell^p$.

Chapter 4 deals with infinite series rearrangements of a new type. Let $x_n \uparrow \infty$, $y_n \underset{\neq}{\downarrow} 0$, $x = (x_n)$, $y = (y_n)$, and let $S(x,y) = \{ \Sigma\, x_{h(n)} y_n$: h(n) is any rearrangement of the set of positive integers}. We examine the structure of $S(x,y)$ in R^1. The main result, contributed to by Paul Erdos and Hugh Montgomery, states that, for every such y, $S(x,y) = [a,\infty)$ for some $a > 0$, if and only if x_{n+1}/x_n is uniformly bounded. (Recently W.A.J. Luxembourg has pointed out to us related work by Peter W. Day [10])

Chapter 5 deals with several topics. In Section 1, we give matricial representations for the normal operators $M_z \in L(L^2([-\pi,\pi]x[-\pi,\pi]))$ and $M_{z+w} \in L(L^2(T^2))$ (where T^2 denotes the torus), each having some continuous spectrum of positive planar area. Hence, according to Berg [3], both of these operators are candidates for normal operators which are not the sum of a diagonalizable and a Hilbert-Schmidt operator. In fact, they are candidates for which it is hoped that one of them shall fail to possess the generalized Fuglede property in regard to C_2. In Section 2, we introduce the "stretch" axiom to create new classes of ideals together with a new way of looking at ideals. (Recently this point of view has led to a solution of a problem posed by Brown, Pearcy and Salinas [7] in 1971. Independently of this thesis, we use the stretch axiom to find a new characterization of all ideals in L(H) and we use this together with transfinite induction and the continuum hypothesis to prove that K(H) is the join of two proper ideals (i.e. K(H) = I + J where I,J are ideals in L(H) and $I,J \subsetneq K(H)$). The transfinite induction argument is due to Andreas Blass.) Finally, in Section 3, we prove that every admissible function is equivalent to a continuous (even n-differentiable) admissible function (in the sense of Brown, Pearcy and Salinas [7]). (This result was obtained simultaneously with, and independently of Salinas [32]).

NOTATION AND PRELIMINARY FACTS

We shall introduce notation as the need arises. However, for the reader who wishes to read sections out of sequence, we provide here a brief summary of the notation we most often use.

Let H_1 and H_2 denote two separable, complex Hilbert spaces. If for $A \in L(H_1)$ and $B \in L(H_2)$ there exists a unitary operator $U: H_1 \longrightarrow H_2$ such that $A = U^{-1}BU$, then we say A and B are unitarily equivalent and we denote this relation by $A \cong B$. We will not distinguish the case when A and B act on the same Hilbert space.

Let $x = (x_n)_{n=1}^{\infty} = (x_n)$ denote an arbitrary sequence of complex numbers. Let $D(x) = D((x_n))$ denote the diagonal operator with diagonal entries $(x_1, x_2, \ldots)$. It is easily shown that if $y = (y_n)$ is a rearrangement of $x = (x_n)$, then $D(y) \cong D(x)$. This holds in a more general sense. If A has a complete orthogonal set of eigenvectors in H_1 and B has a complete orthogonal set of eigenvectors in H_2, where H_1 and H_2 have the same dimension, then $A \cong B$ if and only if the set of eigenvalues of A is the same as that of B, counting multiplicities. For example, $D(x) \cong D((x_1, x_3, \ldots)) \oplus D((x_2, x_4, \ldots))$ where the left-hand operator acts on H, with respect to the standard basis, and the right-hand operator acts on $H \oplus H$, with respect to its standard basis. For the sake of simplicity, we shall also let $D(x_1, x_2, \ldots)$ denote $D(x) = D((x_n)) =$

$D((x_1, x_2, \ldots))$.

Let F(H) and K(H) denote the two-sided ideals in L(H) of the finite rank operators (operators with finite-dimensional range) and compact operators, respectively. Let I,J denote nontrivial two-sided ideals in L(H). It is well-known that $F(H) \subset I \subset K(H)$. Also the quotient algebra L(H)/K(H) is known as the Calkin algebra. It is also a B*-algebra with identity.

Let C(I) denote the set of all commutators AB-BA where A and B $\in$ I. Let [I,I] denote the linear span of C(I). Let $[I,I]_n$ denote the set of linear combinations of n or fewer commutators of I (equivalently, the set of all sums of n elements of C(I)).

Let C_p denote the Schatten p-ideals and $\|\cdot\|_{C_p}$ denote the C_p-norm. Let ℓ^p denote the sequence space of all p-summable sequences and let $\|\cdot\|_{\ell^p}$ denote the ℓ^p-norm. Therefore the trace class is C_1 and the Hilbert-Schmidt class is C_2.

If $I \subset C_1$, let I^o denote the set of operators in I with trace 0.

In the case when the elements of a set are real or complex numbers, we shall sometimes, depending on the context, use standard set notation to mean something different than usual. Namely, if (d_n) is a sequence of real or complex numbers, then $\{d_n\}$ ordinarily means simply the set of numbers occurring in (d_n) and ignores multiplicities. However it is very important that we do not ignore multiplicities. Therefore, depending on the context, we shall sometimes define $\{d_n\}$ to be the unordered set of numbers, counting multiplicities, corresponding to the sequence (d_n).

If $\{A_\alpha\}$ is a set of operators in L(H), then $\Sigma_\alpha \oplus A_\alpha$ denotes the direct sum of $\{A_\alpha\}$ and acts on $\Sigma_\alpha \oplus H$.

Let $\mathbb{C}$ denote the complex numbers, R the real numbers, and Z the integers.

Let Z^+ denote the set of positive integers, let $\mathcal{S}$ denote the set of finite-tuples of positive integers, and let $\mathcal{S}^+$ denote the set of all finite-tuples of positive integers that are greater than or equal to 2.

Let $M(T_{11},T_{12},T_{21},T_{22})$ denote the 2x2 matrix with operator entries T_{ij} in the (i,j) position. It is well-known that this is a general form for all operators in $L(H \oplus H)$.

If I is an ideal in L(H), let I^2 denote the ideal generated by $I \cdot I$.

For each ideal I in L(H), let Calk(I) denote the well-known Calkin ideal set of I (see [9]) which is precisely $\{(|x_n|): D((x_n)) \in I\}$.

If H_1 and H_2 are two Hilbert spaces of the same dimension, then there are many isometric isomorphisms U (i.e. unitary transformations) mapping H_1 onto H_2. Each such unitary transformation U induces a canonical *-isometric isomorphism F_U mapping $L(H_1)$ onto $L(H_2)$ via the map $F_U: A \longrightarrow UAU^{-1}$. It is clear that each such induced F_U preserves compactness, positivity, and eigenvalues (counting multiplicities). From this it is clear that each such induced F_U takes each ideal in $L(H_1)$ into the unique ideal in $L(H_2)$ that has the same Calkin ideal set. That is, we induce a multiplicative lattice isomorphism between the lattice of ideals in $L(H_1)$ and the lattice of ideals in $L(H_2)$. What is more, this induced lattice isomorphism is the same one for all such F_U. Hence we can identify ideals in $L(H_1)$ with ideals in $L(H_2)$. In light of this identification, an ideal I will be thought of as lying in either $L(H_1)$ or $L(H_2)$, whichever is convenient at the moment. For example, if I is an ideal in L(H), then (the associated ideal) I is also contained in $L(H \oplus H)$. Furthermore, it is easy to prove that this associated ideal I in $L(H \oplus H)$ is also precisely the set $\{M(T_{11},T_{12},T_{21},T_{22}): T_{ij} \in I \subset L(H)\}$.

Let $T = U|T|$ denote the polar decomposition of an operator T, where $|T| = (T^*T)^{1/2}$ is the positive part and U is a partial isometry which satisfies $U^*U|T| = |T|$.

We shall use the symbol $\downarrow$ to mean non-increasing. For example, $x_n \underset{\neq}{\downarrow} 0$ means that x_n is decreasing (but not necessarily strictly) to 0, and is never equal to 0.

If T_n is a sequence of operators in L(H) and $T \in L(H)$, we shall say $T_n \longrightarrow T$(SOT) if T_n approaches T in the strong operator topology. That is, for every $f \in H$, $\|T_nf - Tf\| \longrightarrow 0$ as $n \longrightarrow \infty$. Also, we shall say $T_n \longrightarrow T$(WOT) if T_n approaches T in the weak operator topology. That is, for every $f,g \in H$, $(T_nf,g) \longrightarrow (Tf,g)$ as $n \longrightarrow \infty$.

CHAPTER 1

COMMUTATORS IN IDEALS OF COMPACT OPERATORS

1. Introduction

Let H be a separable, infinite-dimensional complex Hilbert space and let L(H) denote the algebra of all bounded linear operators on H (all operators herein are bounded and linear). Denote by K(H) and F(H) the two-sided ideals in L(H) of compact operators and finite rank operators, respectively. Let I and J represent any non-trivial two-sided ideals in L(H) (all ideals herein are two-sided) and so, as is well-known, $F(H) \subset I \subset K(H)$ and I is closed in the operator norm topology if and only if $I = K(H)$.

A commutator is an operator of the form AB-BA where A and B are operators in L(H). Such an operator is said to be the commutator of A and B. A commutator of I is an operator of the form AB-BA where A and B are operators in I. Due to the ideal structure of I, commutators of I are themselves operators in I.

Let C(I) denote the set of all commutators of I, that is,

$$C(I) = \left\{ AB-BA\colon A,B \in I \right\}.$$

Let [I,I] denote the linear span of C(I) or, equivalently, the set of all finite sums of commutators of I and let $[I,I]_n$ denote the set of all sums of n or fewer commutators of I. Similarly, let $C(I,J) = \{AB-BA: A\in I \ \& \ B\in J\}$ and let $[I,J]$ denote the linear span of $C(I,J)$.

Let C_p (for $0 < p < \infty$) denote the Schatten p-classes of K(H). For $p \geq 1$, these classes are the best known complete normed ideals in K(H), where C_2 is the Hilbert-Schmidt class and C_1 is the trace class. The C_p classes may be defined as follows. For every operator T, let $T = U|T|$ be the polar decomposition of T, where $|T| = (T^*T)^{1/2}$ is the positive part of T and U is a partial isometry. Also $|T| = U^*T$. Since K(H) is an ideal, T is compact if and only if $|T|$ is compact. Furthermore $|T|$ is self-adjoint. By the spectral theorem for compact, self-adjoint operators, $|T|$ is compact if and only if it is diagonalizable (that is, it has an orthonormal basis of eigenvectors) and its eigenvalues, counting multiplicities, tend to 0. Let (λ_n) denote the sequence of eigenvalues, counting multiplicities, of $|T|$, provided T is compact. C_p is defined to be the class of compact operators T for which $(\lambda_n) \in \ell^p$. If $T \in C_p$, then the C_p-norm of T is given by $\|T\|_{C_p} = (\Sigma_n \lambda_n{}^p)^{1/p} = \|(\lambda_n)\|_{\ell^p}$ (for a rigorous development, see [34]).

Calkin and von Neumann [9] observed an important duality between proper ideals I of L(H) and sets of non-negative sequences (λ_n) corresponding to the sequences of eigenvalues, counting multiplicities, of all positive operators in I. So far, we see that C_p corresponds to ℓ^p_+ (the non-negative sequences in ℓ^p) and K(H) corresponds to c_o^+ (the non-negative sequences that tend to 0). The set of non-negative sequences that an ideal I generates in this way is known as the Calkin ideal set of I, and will denoted by Calk(I). Also Salinas [23] defines the characteristic set of I to be the set of non-increasing sequences in the Calkin ideal set of I, and this also defines an important duality. (For Calkin's axioms and the characteristic set axioms, see pp. 57-58.)

The trace of an operator is a central, very important and quite subtle aspect of operator theory. We will exploit several new and peculiar properties of the trace to obtain our main theorem, and we will consider possible generalizations of the trace. Not only here is the trace important. Simultaneously and independently, Berger and Shaw, Helton and Howe, and Brown, Douglas and Fillmore, in [8], [4] and [20], found the trace crucial to major segments of their work. Earlier papers too depend on the trace; for example see Radjavi [28] and Deddens [12]. The trace of an operator is not always well-defined, even in an extended sense. Let $\{e_n\}$ be an orthonormal basis for H. Formally, the trace of an operator T is the sum of the diagonal entries of its matrix with respect to $\{e_n\}$, given by $Tr(T) = \Sigma_n(Te_n, e_n)$. However, this sum is clearly not always defined, and when it is, it seems to depend on the choice of basis $\{e_n\}$. The trace is most useful when applied to operators for which the above sum is convergent (perhaps to $\pm\infty$) and unitarily invariant (that is, independent of the choice of basis). If H is finite-dimensional, the trace is finite and unitarily invariant. If H is infinite-dimensional and separable, there are two important cases when the trace is defined and unitarily invariant, namely, for trace class operators and for positive operators. If T is a trace class operator, then $\Sigma_n(Te_n, e_n)$ converges to a finite number which is unitarily invariant. If T is a positive operator, then the trace of T is well-defined in the extended sense, that is, $\Sigma_n(Te_n, e_n)$ converges to either a finite number, in which case T is in the trace class, or to $+\infty$, in which case T is not in the trace class. In either case the sum is a unitary invariant (see [14, pp. 96-97]). Interesting and useful connections between the trace, the C_p classes and linear functionals may be found in [34]. It should be noted, however, that the trace is not the only property of the diagonals of matrices that is of interest. In this chapter and in Chapter 3, Section 2 we depend heavily on more qualitative properties of diagonals.

The trace is indispensable to the study of commutators. If H is finite-dimensional and A and B are in L(H), then Tr(AB-BA) = 0. If H is infinite-dimensional (not necessarily separable), and if $A \in K(H)$, $B \in L(H)$ and AB and BA are in the trace class, then Tr(AB-BA) = 0 (see [14, p. 99]). From this it is evident that trace class operators with trace 0 are worth special consideration. Thus we are led to the following definition.

Let I be any ideal of L(H) that is contained in the trace class. Denote by I^o the class of operators in I with trace 0, that is,

$$I^{o} = \{T \in I: Tr(T) = 0\}.$$

In particular, C_1^o is the set of all trace class operators with trace 0.

If I is an ideal in L(H) and I^2 is the ideal generated by $\{T^2: T \in I\}$, the following chain of inclusions clearly hold.

(1.1) $C(I) \subset [I,I]_n \subset [I,I] \subset I^2 \subset I$, for each n.

One of the most interesting problems in the structure theory of compact operators is to determine, for a given ideal, which of the above inclusions are proper and which are not (see [26 , Problems 1,2,3,3']). If I^2 is a normed ideal [14], then we also have the inclusion $[I,I]^- \subset I^2$ (where the closure is taken in the norm topology of I^2). The same questions can be asked about two ideals I and J, namely, when are the following inclusions proper and when not.

(1.1a) $C(I,J) \subset [I,J]_n \subset [I,J] \subset IJ \subset I \cap J \subset I$, for each n,

where IJ is the ideal generated by $\{TS: T \in I, S \in J\}$.

If $I \subset C_2$, the situation takes a sudden turn and the trace plays a crucial role. If $A,B \in I$, then $AB, BA \in I^2 \subset C_2^2 = C_1$. By an above remark, $Tr(AB-BA) = 0$. Hence the following two chains of inclusions hold true.

(1.2) $C(I) \subset [I,I]_n \subset [I,I] \subset (I^2)^o \underset{\neq}{\subset} I^2 \subset I$, for each n.

Therefore if $I \subset C_2$, the third inclusion of inclusion chain (1.1) is proper. Furthermore if $IJ \subset C_1$, then

(1.2a) $C(I,J) \subset [I,J]_n \subset [I,J] \subset (IJ)^o \underset{\neq}{\subset} IJ \subset I \cap J \subset I$, for each n.

Note that in inclusion chain (1.2), $(I^2)^o$ only makes sense if $I^2 \subset C_1$, or equivalently $I \subset C_2$, and in (1.2a), $(IJ)^o$ only makes sense if $IJ \subset C_1$.

In this chapter we investigate inclusion chains (1.1), (1.1a), (1.2) and (1.2a) to determine, for some of the major ideals such as K(H), F(H) and C_p $(0 < p < \infty)$, which inclusions are proper and which are not. In particular, we are interested in the structure of C(I) and [I,I].

In the following paragraphs, we shall list the known results on the structure of C(I) and [I,I], the outstanding unsolved problems, and some new questions.

Known Results

1. $C(L(H)) = L(H) \setminus \{\lambda I + K: \lambda \in \mathbb{C}$, I is the identity operator, and $K \in K(H)\}$. [6],[2]
2. $[K(H),K(H)] = K(H)$. [26]
3. $[C_{2p},C_{2p}] = C_p$ for all $p > 1$. [26]
4. $[C_{2p},C_{2p}] \supset C_1$ for all $p \geq 1$. [26]
5. $[\bigcap_{p>1} C_{2p}, \bigcap_{p>1} C_{2p}] = \bigcap_{p>1} C_p$. [31]

Open Questions

1. Is $C(K(H)) = K(H)$? Even more basic: If P is a rank one projection operator, is $P \in C(K(H))$? [26],[24]
2. Is $C(C_{2p}) = C_p$ $(p > 1)$? [26]
3. Is $C(C_2) = C_1^o$? [26]
4. Is $[C_2, C_2] = C_1^o$? [26]
5. For what ideals I is
   a) $[I,I] = I^2$ or
   b) $[I,I] = (I^2)^o$ (in case $I \subset C_2$)?
6. What ideals I containing C_2 have the property that $[I,I] \supset C_1$?
7. Consider 5 and 6 above replacing $[I,I]$ by $C(I)$.
8. Consider 1-7 above replacing $[I,I]$ by $[I,J]$, $C(I)$ by $C(I,J)$, and I^2 by IJ (replace $I \subset C_2$ by $IJ \subset C_1$).

We shall devote this chapter to giving a complete survey of all that is known in this area. We will give a well-known proof that $C(L(H)) \subset L(H) \setminus \{\lambda I + K: \lambda \in \mathbb{C}$, I is the identity operator, and $K \in K(H)\}$, the easier half of Known Result 1. We will a give a new and easier proof of Known Results 2-5, which will then be subsumed under one unified approach. This same approach will also answer an unpublished question of Salinas, and yield results about $[I,J]$. The most significant and intricate part of our work is directed toward Open Question 4. We show that a significant portion of C_1^o is contained in $[C_2, C_2]$ and we obtain other related results which reach in several directions. The results and techniques are then applied to Open Questions 5b, 6 and 1.

The complete proof of Known Result 1 was discovered by A. Brown and C. Pearcy [6] in 1965 and a relatively short and elegant proof was given by J.H. Anderson and J.G. Stampfli [2] in 1971. To date, this is the deepest result on the structure theory of commutator classes.

At present, it appears that the open questions are much more difficult than Known Results 2-5 (perhaps even Known Result 1).

In light of the inclusions (1.1), (1.1a) and (1.2) it is clear that, with respect to the open questions as well as Known Results 2-5, the left-hand side of each equation is contained in the right-hand side. Hence, the crux of these problems is in deciding whether or not the left-hand side contains the right-hand side. Therefore, the questions and results may be (and usually are) viewed as statements about commutator representations. For instance, Open Question 3 is essentially asking whether or not every trace class operator with trace 0 is a Hilbert-Schmidt commutator.

An affirmative answer to any of the Open Questions 1-3 and 7, the <u>single</u> commutator representation questions, obviously yields an affirmative answer to its corresponding question among Open Questions 4-6. It is the intractability of the single commutator problems that has led to consideration of the weaker questions concerning the structure of the linear spans of the commutator classes. For example, for I = K(H), Open Question 5a is solved (in [26]) but Open Question 1 is not.

2. The Non-Commutators of L(H)

Theorem 2.1. (Wielandt [38]) If $\mathcal{A}$ is a normed algebra with identity e, then $e \neq ab - ba$ for any $a,b \in \mathcal{A}$.

Proof. Assume the theorem is false, that is, assume $e = ab - ba$ for some $a,b \in \mathcal{A}$. It follows by induction that $na^{n-1} = a^n b - ba^n$ for all positive integers n. The case n = 1 holds by our initial assumption. The induction hypothesis $na^{n-1} = a^n b - ba^n$ together with $e = ab - ba$ gives us $a^{n+1}b = a(a^n b) = a(na^{n-1} + ba^n) = na^n + (ab)a^n = (n+1)a^n + ba^{n+1}$ and our induction is complete. Taking the norm of both sides of this identity and using its properties we obtain $n\|a^{n-1}\| \leq \|a^n b\| + \|ba^n\| \leq 2\|a\|\|b\|\|a^{n-1}\|$ for all positive integers n. If $a^k \neq 0$ for every positive integer k, then $n \leq 2\|a\|\|b\|$ for every positive integer n, which is impossible. Therefore $a^k = 0$ for some k. But then $ka^{k-1} = a^k b - ba^k = 0$ and therefore $a^{k-1} = 0$. Hence $a = 0$. However, this implies $e = ab - ba = 0$ which is a contradiction since $e \neq 0$. Q.E.D.

Corollary 2.2. The identity operator I in L(H) is not a commutator. Furthermore, for all complex numbers $\lambda \neq 0$, λI is not a commutator.

Proof. L(H), with its operator norm, is a normed algebra with identity I. Therefore, by Theorem 2.1, I is not a commutator of L(H). That λI ($\lambda \neq 0$) is not a commutator is clear. Q.E.D.

Corollary 2.3. (Halmos). Operators of the form $\lambda I + K$, where $\lambda \neq 0$, $\lambda \in \mathbb{C}$, and $K \in K(H)$, are not commutators. That is,

$$C(L(H)) \subset L(H) \setminus \{\lambda I + K: \lambda \neq 0, \lambda \in \mathbb{C}, \text{ and } K \in K(H)\}.$$

Proof. Consider the Calkin algebra L(H)/K(H). It is a well-known B*-algebra with identity I + K(H) and therefore Theorem 2.1 applies. This together with the fact that the projection operator from L(H) onto L(H)/K(H) is a Banach algebra homomorphism leads immediately to a proof by contradiction. Q.E.D.

In 1965, Arlen Brown and Carl Pearcy [6] characterized C(L(H)) by proving that operators of the form $\lambda I + K$, where $\lambda \neq 0$, $\lambda \in \mathbb{C}$, and $K \in K(H)$, are the only non-commutators. That is, the inclusion above is actually equality. In 1971, J.H. Anderson and J.G. Stampfli [2] discovered a much shorter proof of this fact. However, even this proof is somewhat lengthy. Hence we state the theorem without proof.

Theorem 2.4. (A. Brown and C. Pearcy)

$$C(L(H) = L(H) \setminus \{\lambda I + K: \lambda \neq 0, \lambda \in \mathbb{C}, \text{ and } K \in K(H)\}.$$

The next corollary completely settles the problem of the inclusion chain (1.1) for I = L(H) and is of some interest besides.

Corollary 2.5. Every operator in L(H) is the sum of two commutators of L(H).

Proof. Let $\{h_n\}_{n=1}^{\infty}$ be any infinite orthonormal sequence in H. For every compact operator K, $\|Kh_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. If $\lambda \neq 0$ is a complex number and $K \in K(H)$, then $\lim_{n\to\infty} ((\lambda I + K)h_n, h_n) = \lambda + \lim_{n\to\infty} (Kh_n, h_n) = \lambda$. Therefore, if T is an operator for which $\lim_{n\to\infty} (Th_n, h_n) = 0$ for some infinite orthonormal sequence $\{h_n\}$, then T is not of the form $\lambda I + K$ for $\lambda \neq 0$, $\lambda \in ₵$, and $K \in K(H)$.

We claim that every operator T is the sum of two operators that are not of the form $\lambda I + K$, where $\lambda \neq 0$, $\lambda \in ₵$, and $K \in K(H)$. Let $\{e_n\}_{n=1}^{\infty}$ be any basis for H. Let $t_{ij} = (Te_j, e_i)$. Then the matrix representation for T with respect to $\{e_n\}$ is $T = (t_{ij})$. Let D be the diagonal operator with diagonal entries $(t_{11}, 0, t_{33}, 0, t_{55}, 0, \ldots)$. Then $T = D + (T-D)$. It suffices to show that D and T-D are not operators of the form $\lambda I + K$, for $\lambda \neq 0$, $\lambda \in ₵$, and $K \in K(H)$. With respect to the infinite orthonormal sequence $\{e_{2n}\}_{n=1}^{\infty}$, $(De_{2n}, e_{2n}) = 0$ for all n. With respect to the infinite orthonormal sequence $\{e_{2n-1}\}_{n=1}^{\infty}$, $((T-D)e_{2n-1}, e_{2n-1}) = t_{2n-1,2n-1} - t_{2n-1,2n-1} = 0$ for all n. Therefore, by the preceding paragraph, neither D nor T-D is of the form $\lambda I + K$ for $\lambda \neq 0$, $\lambda \in ₵$, and $K \in K(H)$. Hence, by Theorem 2.4, D and T-D are commutators, and therefore T is the sum of two commutators. Q.E.D.

Theorem 2.4 and Corollary 2.5 complete the inclusion chain (1.1) for I = L(H) as follows:

$$(2.6)\ C(L(H)) \underset{\neq}{\subset} [L(H),L(H)]_2 = [L(H),L(H)] = (L(H))^2 = L(H).$$

3. <u>Cases when $[I,I] = I^2$ and $[I,J] = IJ$</u>

This section begins our study of commutators of compact operators. C. Pearcy and D. Topping [26] and N. Salinas [31] appear to have the only results known on the equation $[I,I] = I^2$.

Our approach subsumes the results in [26] and [31] together with our new results on the equations $[I,I] = I^2$ and $[I,J] = IJ$. We formalize the two basic techniques used in [26] and [31]. One technique exploits an important relationship between some ideals in L(H) and the tensor operation between operators. The other technique, and the most difficult, requires a delicate construction. We offer a more general and algebraic approach to the first technique and a shorter and conceptually easier construction for the second. The idea we use for this simpler construction is the simplest application of a new, more general notion which we discovered and will employ more fully in the next section.

We shall make free use of the elementary facts about ideals I in L(H) and their corresponding Calkin ideal sets, which we will denote by Calk(I). A treatment of these facts may be found in [31]. Also, the facts about characteristic sets introduced in [23] will be used.

<u>Tensors and Ideals</u>. It is well-known that there are many isometric isomorphisms (i.e. unitary transformations) between $H \otimes H$ and H, some of which are canonical (e.g. if $\{e_n\}$ is the standard basis for H, then any one-to-one correspondence between $\{e_n \otimes e_m\}_{n,m=1}^{\infty}$ and $\{e_n\}$ extends by linearity to a canonical unitary transformation mapping $H \otimes H$ onto H.). Each unitary transformation between $H \otimes H$ and H induces a *-isometric isomorphism between $L(H \otimes H)$ and $L(H)$ that preserves compact positive operators and their eigenvalues (counting multiplicities). Namely, if $U: H \otimes H \longrightarrow H$ is a unitary transformation, then $A \longrightarrow UAU^{-1}$ is a *-isometric isomorphism mapping $L(H \otimes H) \longrightarrow L(H)$. From this it is clear that each such induced *-isometric isomorphism from $L(H \otimes H)$ onto $L(H)$ takes each ideal in $L(H \otimes H)$ to the unique ideal in $L(H)$ with the same Calkin ideal set. That is, we induce a multiplicative lattice isomorphism between the lattice of ideals in $L(H \otimes H)$ and the lattice of ideals in $L(H)$. Furthermore, this isomorphism is independent of the original choice of isometric isomorphism (i.e. unitary transformation) between $H \otimes H$ and H. Hence we can identify ideals in $L(H \otimes H)$ with ideals in $L(H)$. In light of this identification, an ideal I will be thought of as lying in either $L(H \otimes H)$ or $L(H)$, whichever is convenient at the moment.

Let A and B be operators in $L(H)$. Let $A = (a_{ij})$ be a matrix representation for A with respect to some basis.

Then $A \otimes B = (a_{ij}B)$ is an operator in $L(H \otimes H)$. Let $I \otimes J$ denote the ideal in $L(H \otimes H)$ (hence in $L(H)$) generated by $\{A \otimes B : A \in I \text{ and } B \in J\}$. Clearly then $\mathrm{Calk}(I \otimes J)$ is generated by $\{(a_i b_j)_{i,j=1}^{\infty} : (a_n) \in \mathrm{Calk}(I) \text{ and } (b_n) \in \mathrm{Calk}(J)\}$, where $(a_i b_j)$ is understood to be any sequential ordering of the set $\{a_i b_j\}$, counting multiplicities.

How do I, J, and $I \otimes J$ compare? We claim that $I, J \subset I \otimes J$. An earlier remark that $F(H) \subset J$ implies that the diagonal operator in $L(H)$ with diagonal entries $(1,0,0,\ldots)$ is contained in J and therefore the sequence $(b_n) = (1,0,0,\ldots) \in \mathrm{Calk}(J)$. If $(a_n) \in \mathrm{Calk}(I)$, then the terms of (a_n) appear in $(a_i b_j)$ and therefore $(a_n) \in \mathrm{Calk}(I \otimes J)$. Whence, $\mathrm{Calk}(I) \subset \mathrm{Calk}(I \otimes J)$, and hence $I \subset I \otimes J$ (under our identification). Similarly it can be shown that $J \subset I \otimes J$.

By the last remark, $I \otimes J = I$ if and only if $I \otimes J \subset I$. The case $I = J$ is of particular importance and for this reason we make the following definition.

<u>Definition 3.1</u>. An ideal I has the <u>tensor product closure property</u> (TPCP) if $I \otimes I \subset I$ (equivalently $I \otimes I = I$).

Almost all of the ideals we consider have the tensor product closure property. The best way we know to test an ideal for this property is to use the following criterion.

<u>Proposition 3.1a</u>. The ideal I has the TPCP if and only if $(a_n) \in \mathrm{Calk}(I) \Longrightarrow (a_i a_j) \in \mathrm{Calk}(I)$.

Proof. Assume first that $(a_n) \in \mathrm{Calk}(I) \Longrightarrow (a_i a_j) \in \mathrm{Calk}(I)$. If (a_n) and $(b_n) \in \mathrm{Calk}(I)$, then $(a_n + b_n) \in \mathrm{Calk}(I)$. Whence $((a_i + b_i)(a_j + b_j)) \in \mathrm{Calk}(I)$. However, Calk(I) consists only of non-negative sequences, and so the inequality $a_i b_j \leq (a_i + b_i)(a_j + b_j)$ holds. By Calkin's axioms [9], any non-negative sequence that is bounded by a sequence in Calk(I) is itself contained in Calk(I). Therefore $(a_i b_j) \in \mathrm{Calk}(I)$; that is, I has the TPCP. The converse is clear. Q.E.D.

Considering the size of the lattice of ideals, it seems likely that most ideals do not have the TPCP. We give one general example of an ideal which fails to have the TPCP. Let $a_n \downarrow 0$. Let $(a_n^{(m)}) = (a_1,\ldots,a_1,a_2,\ldots,a_2,\ldots)$ where each a_n is repeated m times. Let $\mathrm{Calk}(I) = \{(x_{\pi(n)}): x_n \downarrow 0,\ x_n = O(a_n^{(m)})$ for some m, and π permutes $Z^+\}$. It is not hard to see that this is indeed a Calkin ideal set for some ideal I. This can be verified directly from Calkin's axioms or may be proved using characteristic sets introduced in [23] ($\{(x_n): x_n \downarrow 0$ and $x_n = O(a_n^{(m)})$ for some $m\}$ is the characteristic set to consider). Now choose $a_n = 2^{-n}$. Hence $a_n^{(m)} = 2^{-[(n+m-1)/m]}$. Other choices would suffice but this one seems to involve the easiest computation. To prove that I does not have the TPCP, it suffices to show that $(a_i a_j) = (1/2^{i+j})_{i,j=1}^{\infty}$, when arranged in decreasing order, is not $O(2^{-[n/m]})$ for any m. Indeed, the rearranged decreasing sequence is precisely the sequence given by $(1/2^2, 1/2^3, 1/2^3, 1/2^4, 1/2^4, 1/2^4, \ldots)$ where $1/2^n$ occurs n-1 times (that is, the number of positive integer solution pairs of the equation $i + j = n$). Call this sequence (x_n). If $\Sigma_{k=0}^{N-1} k < n \le \Sigma_{k=0}^{N} k$ for $N \ge 1$, then $x_n = 1/2^{N+1}$. Therefore $n \longrightarrow \infty$ implies $N \longrightarrow \infty$ and $2^{(n/m)}x_n = 2^{(n/m)-N-1} \ge 2^{(m^{-1}\Sigma_{k=0}^{N-1} k)-N-1} = 2^{(N(N-1)/2m)-N-1} \longrightarrow \infty$ as $n \longrightarrow \infty$. That is, $x_n \ne O(2^{-[n/m]})$ for any m.

What ideals do have the TPCP? We list some ideals with the TPCP in the next lemma, but first we shall consider a special type of ideal.

Let I be an arbitrary two-sided ideal, as usual. Then $I \otimes I$ is the ideal whose Calkin ideal set is the ideal set generated by $\{(a_i a_j) : (a_n) \in \mathrm{Calk}(I)\}$. Let $\overset{n}{\underset{1}{\otimes}} I$ denote $I \otimes I \otimes \ldots \otimes I$ (n times). Its Calkin ideal set is generated by $\{(a_{i_1} a_{i_2} \cdots a_{i_n})_{i_1,\ldots,i_n = 1}^{\infty} : (a_n) \in \mathrm{Calk}(I)\}$. It is clear that the collection of ideals $\left\{\overset{n}{\underset{1}{\otimes}} I\right\}_{n=1}^{\infty}$ is an increasing sequence and that $\bigcup_{n=1}^{\infty} (\overset{n}{\underset{1}{\otimes}} I)$ is the smallest ideal containing I that has the tensor product closure property. If I already has the TPCP, then $\overset{n}{\underset{1}{\otimes}} I = I$ for every n. There are several interesting questions to ask about the collection $\left\{\overset{n}{\underset{1}{\otimes}} I\right\}$.

I. Suppose an ideal $I \subsetneq K(H)$.

a) Can $K(H) = I \otimes I$?

b) Can $K(H) = \overset{n}{\underset{1}{\otimes}} I$?

c) Can $K(H) = \bigcup_{n=1}^{\infty} (\overset{n}{\underset{1}{\otimes}} I)$?

II. Suppose the ideals $I, J \subsetneq K(H)$. Can $K(H) = I \otimes J$?

(Note that the question 'Is $K(H) = I + J$ (i.e. the ideal generated by I and J)?' is a fairly well-known unsolved problem.)

III. If $\overset{n}{\underset{1}{\otimes}} I$ has the TPCP, must I have the TPCP?

We do not know the answer to Ia, Ib, or Ic. They seem to be hard questions. However, it is easy to see that the answer to Ia is the same as the answer to Ib. It is also interesting to note that if the answer to Ia or Ib was no, then the answer to Ic would also be no. This follows directly from a result of Salinas [33, Theorem 7.4]. The answer to II is yes; this follows directly from our recent discovery, mentioned earlier and independent of this thesis, that $K(H) = I + J$ for two ideals $I, J \subsetneq K(H)$ (assuming the continuum hypothesis). Indeed $I + J \subset I \otimes J \subset K(H)$ clearly holds, and our previous equality implies equality throughout this inclusion. Hence $K(H) = I \otimes J$. We do not know the answer to III. However, it is clear that $\overset{n}{\underset{1}{\otimes}} I$ has the TPCP if and only if $(\overset{n}{\underset{1}{\otimes}} I) \otimes I \subset \overset{n}{\underset{1}{\otimes}} I$, because $\left\{\overset{n}{\underset{1}{\otimes}} I\right\}$ is nested. The relations between ideals and tensors are, at present, not very well understood.

Lemma 3.2. The following ideals have the tensor product closure property:

1. $K(H)$.
2. C_p, for all $0 < p < \infty$.
3. $\bigcup_{n=1}^{\infty} (\overset{n}{\underset{1}{\otimes}} I)$
4. $\bigcup_{\alpha} I_\alpha$, for any nested set of ideals $\{I_\alpha\}$ each

having the TPCP.

5. $\bigcap_\alpha I_\alpha$, for any set of ideals $\{I_\alpha\}$ each having the TPCP.

<u>Proof</u>. To show $I \otimes I \subset I$, it suffices to show $Calk(I \otimes I) \subset Calk(I)$, and for this, it suffices to show that if $(a_n) \in Calk(I)$ then $(a_i a_j) \in Calk(I)$.

1. Let $(a_n) \in Calk(K(H)) = c_o^+$. Then $\lim_{n \to \infty} a_n = 0$. Therefore, for every $\epsilon > 0$, all but a finite number in the set $\{a_n\}$ are less than ϵ. Therefore, for every $\epsilon > 0$, all but a finite number in the set $\{a_i a_j\}$ are less than ϵ. Hence $(a_i a_j) \in c_o^+ = Calk(K(H))$.

2. Let $(a_n) \in Calk(C_p) = \ell_+^p$. Then $\Sigma_{n=1}^{\infty} a_n^{\ p} < \infty$. Therefore $\Sigma_{i,j=1}^{\infty} (a_i a_j)^p = \Sigma_{i=1}^{\infty} a_i^{\ p} (\Sigma_{j=1}^{\infty} a_j^{\ p}) = (\Sigma_{i=1}^{\infty} a_i^{\ p})^2 < \infty$, hence $(a_i a_j) \in \ell_+^p = Calk(C_p)$.

3. For all positive integers n and m, we have that $(\otimes_1^n I) \otimes (\otimes_1^m I) = \otimes_1^{n+m} I \subset \bigcup_{n=1}^{\infty} (\otimes_1^n I)$. Therefore $(\otimes_1^n I) \otimes [\bigcup_{n=1}^{\infty} (\otimes_1^n I)] \subset \bigcup_{n=1}^{\infty} (\otimes_1^n I)$ and hence $[\bigcup_{n=1}^{\infty} (\otimes_1^n I)] \otimes [\bigcup_{n=1}^{\infty} (\otimes_1^n I)] \subset \bigcup_{n=1}^{\infty} (\otimes_1^n I)$.

4. For each β, $I_\beta \otimes [\bigcup_\alpha I_\alpha] = \bigcup_\alpha (I_\beta \otimes I_\alpha) \subset \bigcup_\alpha (\max(I_\beta, I_\alpha) \otimes \max(I_\beta, I_\alpha)) = \bigcup_\alpha \max(I_\beta, I_\alpha) \subset \bigcup_\alpha I_\alpha$. Therefore $(\bigcup_\alpha I_\alpha) \otimes (\bigcup_\alpha I_\alpha) \subset \bigcup_\alpha I_\alpha$.

5. $(\bigcap_\alpha I_\alpha) \otimes (\bigcap_\alpha I_\alpha) \subset I_\beta \otimes I_\beta = I_\beta$ for all β. Hence $(\bigcap_\alpha I_\alpha) \otimes (\bigcap_\alpha I_\alpha) \subset \bigcap_\alpha I_\alpha$. Q.E.D.

It is interesting to note that each of the above proofs is quite different from the others and each depends entirely on the defining properties of the particular Calkin ideal set.

We now begin our discussion of the second technique mentioned at the beginning of this section. Let P denote the diagonal matrix with diagonal entries (1,0,0,...). With respect to the standard basis $\{e_n\}$, P is a rank one orthogonal projection operator in L(H) and every rank one projection operator (not necessarily orthogonal) is similar to P.

Open Question 1 of Section 1 states that it is unknown whether or not P is a single commutator of compact operators (i.e. $P \in C(K(H))$). We also do not know if $P \in C(K(H))$ implies $C(K(H)) = K(H)$. However, we shall soon see that for all ideals I with the TPCP, we have that $P \in [I,I]$ if and only if $[I,I] = I^2$. In particular, $P \in [K(H),K(H)]$ implies $[K(H),K(H)] = K(H)$.

For any ideal I, since $P \in F(H) \subset I$, we have that $[I,I] = I^2$ implies $P \in [I,I]$. In general, we do not know when P belongs to $[I,I]$. The question 'Is $P \in C(I)$?' appears to be very difficult, even when $I = K(H)$ (We present some preliminary results on this in Section 6 of this chapter). Each known proof that $P \in [I,I]$, for a particular ideal I, is not easy. In fact, in deciding whether or not $[I,I] = I^2$, the crucial and most difficult question is whether or not $P \in [I,I]$.

The operator P and how it relates to commutators in various ideals is a recurring theme throughout this chapter. Our main results on commutators depend on special diagonal operators which resemble P in certain ways. The techniques and difficulties that arise in regard to P have much in common with the techniques and difficulties that arise in regard to our special diagonal operators.

The main construction of this section provides new proofs for the known cases when $P \in [I,I]$. Before starting the construction, let us introduce some notation which will be useful throughout this chapter.

<u>Notation</u>. For a sequence $x = (x_n)$ of complex numbers (finite or infinite), let $D(x) = D(x_1, x_2, \ldots) = D((x_n))$ denote the diagonal operator with the diagonal entries (x_n). In particular, $P = D(1,0,0,\ldots)$.

Furthermore, we shall use '$A \cong B$' to mean that the two operators A and B are unitarily equivalent. Note that A and B may possibly be acting on different Hilbert spaces.

It is well-known that if (x_n) is a rearrangement of (y_n), then $D((x_n)) \cong D((y_n))$.

<u>The Main Construction</u>. We claim that $P = D(1,0,0,\ldots)$ is the sum of two operators each of which is a commutator of compact operators. It is clear that

$$P = D(1,-1/2,-1/2,1/4,1/4,1/4,1/4,-1/8,\ldots)$$
$$+ D(0,1/2,1/2,-1/4,-1/4,-1/4,-1/4,1/8,\ldots).$$

If two diagonalizable operators A and B have the same eigenvalues, counting multiplicities, then $A \cong B$. Therefore,

$D(1,-1/2,-1/2,1/4,1/4,1/4,1/4,-1/8,\ldots)$

$\cong D(1,-1/2,-1/2,1/4,-1/8,-1/8,1/4,-1/8,-1/8,$
$1/4,-1/8,-1/8,1/4,-1/8,-1/8,1/16,-1/32,-1/32,\ldots)$

$\cong \Sigma_{n=0}^{\infty} \oplus[4^{-n}D(1,-1/2,-1/2) \oplus \cdots \oplus 4^{-n}D(1,-1/2,-1/2)]$,

where the n^{th} bracket contains 4^n summands of the 3x3 matrix $4^{-n}D(1,-1/2,-1/2)$.

Also,

$D(0,1/2,1/2,-1/4,-1/4,-1/4,-1/4,1/8,\ldots)$

$\cong D(0,1/2,-1/4,-1/4,1/2,-1/4,-1/4,$
$1/8,-1/16,-1/16,1/8,-1/16,-1/16,1/8,-1/16,-1/16,$
$1/8,-1/16,-1/16,1/8,-1/16,-1/16,1/8,-1/16,-1/16,$
$1/8,-1/16,-1/16,1/8,-1/16,-1/16,$
$1/32,-1/64,-1/64,\ldots)$

$\cong (0) \oplus \Sigma_{n=0}^{\infty} \oplus[2^{-1}4^{-n}D(1,-1/2,-1/2) \oplus \cdots$
$2^{-1}4^{-n}D(1,-1/2,-1/2)]$,

where the n^{th} bracket contains $2\cdot 4^n$ summands of the 3x3 matrix $2^{-1}4^{-n}D(1,-1/2,-1/2)$.

Let $\alpha = (\alpha(n))$ denote the sequence $(1,1/4,1/4,1/4,1/4,1/16,\ldots)$ in which $1/4^k$ occurs 4^k times, and let $\beta = (\beta(n))$ denote the sequence $(1/2,1/2,1/8,1/8,1/8,1/8,1/8,1/8,1/8,1/8,1/32,\ldots)$ in which $2^{-1}4^{-k}$ occurs $2\cdot 4^k$ times. Obviously $\beta(n) = 2^{-1}\alpha([(n+1)/2])$ (where $[x]$ = the greatest integer less than or equal to x). Therefore

$D(1,-1/2,-1/2,1/4,1/4,1/4,1/4,-1/8,\ldots)$

$$\cong \Sigma_{n=1}^{\infty} \oplus(\alpha(n)D(1,-1/2,-1/2)) \quad \text{and}$$

$D(0,1/2,1/2,-1/4,-1/4,-1/4,-1/4,1/8,\ldots)$

$$\cong (0) \oplus \Sigma_{n=1}^{\infty} \oplus(\beta(n)D(1,-1/2,-1/2)) .$$

Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2^{-1/2} & 0 \end{pmatrix}$.

Clearly $D(1,-1/2,-1/2) = A^*A - AA^*$.

Let (x_n) and (y_n) be any two sequences in c_o^+ for which $x_n y_n = \alpha(n)$ for every n. Then we also have that $2^{-1} x_{[(n+1)/2]} y_{[(n+1)/2]} = \beta(n)$ for every n. Let

$$X_1 = \Sigma_{n=1}^{\infty} \oplus(x_n A^*) , \quad Y_1 = \Sigma_{n=1}^{\infty} \oplus(y_n A) ,$$

$$S_1 = (0) \oplus \Sigma_{n=1}^{\infty} \oplus(2^{-1/2} x_{[(n+1)/2]} A^*), \text{ and}$$

$$T_1 = (0) \oplus \Sigma_{n=1}^{\infty} \oplus(2^{-1/2} y_{[(n+1)/2]} A) .$$

(If $x_n = y_n = (\alpha(n))^{1/2}$, then $X_1 = Y_1^*$ and $S_1 = T_1^*$.)

Computing, we obtain

$$X_1Y_1 - Y_1X_1 = \Sigma_{n=1}^{\infty} \oplus(\alpha(n)D(1,-1/2,-1/2)) \text{ and}$$

$$S_1T_1 - T_1S_1 = (0) \oplus \Sigma_{n=1}^{\infty} \oplus(\beta(n)D(1,-1/2,-1/2)).$$

Therefore, via the underlying unitary operators, there exist operators X,Y,S, and T for which $X \cong X_1$, $Y \cong Y_1$, $S \cong S_1$, and $T \cong T_1$, such that

$$XY - YX = D(1,-1/2,-1/2,1/4,1/4,1/4,1/4,-1/8,\ldots), \text{ and}$$

$$ST - TS = D(0,1/2,1/2,-1/4,-1/4,-1/4,-1/4,1/8,\ldots) .$$

By inspection we obtain the following facts.

(3.3) $P = XY - YX + ST - TS$.

(3.4) $|X| \cong |X_1| \cong D(x_1, 2^{-1/2}x_1, x_2, 2^{-1/2}x_2, \ldots) \oplus 0$

(Recall $|T| = (T^*T)^{1/2}$) ,

$$|Y| \cong |Y_1| \cong D(y_1, 2^{-1/2}y_1, y_2, 2^{-1/2}y_2, \ldots) \oplus 0 ,$$

$$|S| \cong |S_1| \cong D(x_1, x_1, 2^{-1/2}x_1, 2^{-1/2}x_1, x_2, x_2, 2^{-1/2}x_2, 2^{-1/2}x_2, \ldots) \oplus 0, \text{ and}$$

$$|T| \cong |T_1| \cong D(y_1, y_1, 2^{-1/2}y_1, 2^{-1/2}y_1, y_2, y_2, 2^{-1/2}y_2, 2^{-1/2}y_2, \ldots) \oplus 0 .$$

This completes our construction.

Now we are ready to handle the '$[I,I] = I^2$' and '$[I,J] = IJ$' problem for the previously known cases and for several new ideals.

In what follows, one should keep in mind that the hypothesis that a sequence α is contained in Calk(I) is equivalent to the hypothesis that $D(\alpha)$ is contained in I.

<u>Lemma 3.5</u>. If I and J are two ideals in L(H) for which $\alpha = (1,1/4,1/4,1/4,1/4,1/16,\ldots) \in \text{Calk}(IJ)$, then $P \in [I,J]_2$.

<u>Proof</u>. It is well-known and easy to prove that

$$\text{Calk}(IJ) = \{(x_n y_n) : (x_n) \in \text{Calk}(I) \text{ and } (y_n) \in \text{Calk}(J)\}$$

(see [31]). Therefore, for some $(x_n) \in \text{Calk}(I)$ and $(y_n) \in \text{Calk}(J)$, we obtain $x_n y_n = \alpha(n)$ for every n. By our main construction, $P = XY - YX + ST - TS$ where (3.4) holds. It follows easily from the Calkin ideal set axioms that X and $S \in I$, and Y and $T \in J$. Thus $P \in [I,J]_2$.

Q.E.D.

<u>Theorem 3.6</u>. If I and J are two ideals in L(H) for which $\alpha = (1,1/4,1/4,1/4,1/4,1/16,\ldots) \in \text{Calk}(IJ)$ and I and J have the tensor product closure property, then $[I,J]_8 = [I,J] = IJ$.

<u>Proof</u>. Due to the inclusion chain (1.1a), it suffices to show $[I,J]_8 \supset IJ$. Let T be any operator in IJ. The special case when $I = J = L(H)$ was considered in Section 2, so we may assume either I or J is a proper ideal of L(H). Therefore $IJ \subset I \cap J \underset{\neq}{\subset} L(H)$, and thus $IJ \subset K(H)$. Hence T is compact.

Set $T = T' + iT''$ where T' and T" are the real and imaginary parts of T. Since T is compact and $T' = (T+T^*)/2$ and $T'' = (T-T^*)/2i$, T' and T" are compact and self-adjoint. It therefore suffices to show that T' and $T'' \in [I,J]_4$. Indeed, we shall show that if S is any compact, self-adjoint operator contained in IJ, then $S \in [I,J]_4$.

By the spectral theorem for compact, self-adjoint operators, $S \cong D((a_n))$ for some real-valued sequence (a_n) for which $(|a_n|) \in \text{Calk}(IJ)$. Also we have the relation $D((a_n)) = D(a_1,0,a_3,0,\ldots) + D(0,a_2,0,a_4,\ldots)$. Furthermore $D(a_1,0,a_3,0,\ldots) \cong D(a_1,a_3,\ldots) \oplus 0$ and $D(0,a_2,0,a_4,\ldots)$

$\cong D(a_2,a_4,\ldots) \oplus 0$, where in each relation the left-hand operator acts on H and the right-hand operator acts on $H\oplus H$. It follows easily from Calkin's axioms that subsequences of sequences in a Calkin ideal set are themselves in that Calkin ideal set. Therefore $(a_{2n}), (a_{2n-1}) \in \text{Calk}(IJ)$. Hence $D((a_n))$ is the sum of two operators, each of which is unitarily equivalent to an operator acting on $H\oplus H$ of the form $D((d_n)) \oplus 0$ for some $(d_n) \in \text{Calk}(IJ)$. Therefore it suffices to show that every operator of this form is contained in $[I,J]_2$.

If $(d_n) \in \text{Calk}(IJ)$, as in the proof of Lemma 3.5, there exist $(\delta_n) \in \text{Calk}(I)$ and $(\gamma_n) \in \text{Calk}(J)$ such that $\delta_n\gamma_n = d_n$ for every n. Also if 0 is the zero operator on H, then $0 \cong 0 \oplus 0 \oplus \cdots$; that is, 0 is unitarily equivalent to a countable direct sum of copies of itself. Therefore $D((d_n)) \oplus 0 \cong D((d_n)) \oplus 0 \oplus 0 \oplus \cdots \cong P \otimes D((d_n)) = P \otimes [D((\delta_n))\cdot D((\gamma_n))]$ and $D((\delta_n))$ and $D((\gamma_n))$ commute.

By hypothesis $\alpha \in \text{Calk}(IJ)$. Hence Lemma 3.5 provides us with operators X and $S \in I$ and Y and $T \in J$ such that $P = XY - YX + ST - TS$.

Therefore, $D((d_n)) \oplus 0 \cong P \otimes [D((\delta_n))\cdot D((\gamma_n))]$
$= (XY - YX + ST - TS) \otimes [D((\delta_n))\cdot D((\gamma_n))]$
$= [X \otimes D((\delta_n))][Y \otimes D((\gamma_n))] - [Y \otimes D((\gamma_n))][X \otimes D((\delta_n))]$
$+ [S \otimes D((\delta_n))][T \otimes D((\gamma_n))] - [T \otimes D((\gamma_n))][S \otimes D((\delta_n))]$.

If I has the tensor product closure property, then $X \otimes D((\delta_n))$ and $S \otimes D((\delta_n))$ are in I. If J has the tensor product closure property, then $Y \otimes D((\gamma_n))$ and $T \otimes D((\gamma_n))$ are in J. Whence $D((d_n)) \oplus 0 \in [I,J]_2$. Q.E.D.

If we examine the proofs of Theorem 3.6, Lemma 3.5 and the main construction, we see that we can actually obtain a more general theorem by a requiring a weaker condition than the TPCP. We state the theorem without proof. Its proof is almost identical to that of Theorem 3.6.

<u>Theorem 3.6a</u>. Let I and J be two ideals in L(H). Suppose that for every real-valued sequence $(d_n) \in \mathrm{Calk}(IJ)$, there exist sequences (δ_n) and $(x_n) \in \mathrm{Calk}(I)$ and (γ_n) and $(y_n) \in \mathrm{Calk}(J)$ such that

(1) $\delta_n \gamma_n = d_n$ and $x_n y_n = \alpha(n)$ for every n, and

(2) $(\delta_i x_j) \in \mathrm{Calk}(I)$ and $(\gamma_i y_j) \in \mathrm{Calk}(J)$.

Then $[I,J]_8 = [I,J] = IJ$.

The structure of $[I,J]$ has been studied mainly in the case when I = J. For this reason we state the following obvious corollary.

<u>Corollary 3.7</u>. If I is an ideal in L(H) with the tensor product closure property such that $\alpha \in \mathrm{Calk}(I^2)$, then $[I,I]_8 = [I,I] = I^2$.

The sequence α plays a crucial role in all our results. It is closely related to the harmonic sequence $(1/n)$. Indeed, $(4n)^{-1} < \alpha(n) \leq 4(3n)^{-1}$. Therefore an ideal contains $D(\alpha)$ if and only if it contains $D((1/n))$ or, in other words, $D(\alpha)$ and $D((1/n))$ are contained in exactly the same ideals. Similarly $\beta = (1/2, 1/2, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/32, \ldots)$ is equivalent to $(1/n)$ (we shall say (δ_n) and (γ_n) are equivalent if δ_n/γ_n is bounded above and below) and $D(\beta)$ is in the same ideals as $D((1/n))$.

Let (x_n) be a real-valued sequence. Let $I((x_n))$ denote the ideal generated by the diagonal operator $D((x_n))$. Then it is easy to see that $I((\alpha(n)^{1/2})) = I((n^{-1/2}))$ and that $I((\alpha(n)^{1/2}))^2 = I(\alpha) = I((1/n))$. It is well-known that every decreasing summable sequence is $o(1/n)$ ($a_n = o(b_n)$ means $a_n/b_n \longrightarrow 0$ as $n \longrightarrow \infty$), and this implies that $C_1 \subset I((1/n)) = I(\alpha)$ (equivalently $C_2 \subset I((\alpha(n)^{1/2}))$). Furthermore $I(\alpha)$ is the smallest ideal in $L(H)$ containing α.

Let us consider some well-known facts.

1. $C_pC_q = C_r$ if $p,q > 0$ and $p^{-1} + q^{-1} = r^{-1}$, and $C_p{}^2 = C_{p/2}$ if $p > 0$.

2. $(1/n) \in \ell^r$ for $r > 1$.

These facts together with Lemma 3.2, Theorem 3.6 and Corollary 3.7 prove the next theorem. All that needs to be verified is that for each pair of ideals I,J listed $\alpha(n) = x_ny_n$ for some pair of sequences $(x_n) \in \mathrm{Calk}(I)$ and $(y_n) \in \mathrm{Calk}(J)$.

<u>Theorem 3.8</u>. In the following table, the ideals I, J and IJ have the property that $[I,J]_8 = [I,J] = IJ$.

	I	J	IJ
1.	K(H)	K(H)	K(H) ,

since $(K(H))^2 = K(H)$. (Pearcy and Topping [26])

	I	J	IJ
2.	K(H)	L(H)	K(H)

$\underline{I}$	$\underline{J}$	$\underline{IJ}$
3. C_p	C_q	C_r ,
provided $p,q > 0$, $p^{-1}+q^{-1}=r^{-1}$, and $r > 1$.		
4. C_p	C_p	$C_{p/2}$,
provided $p > 2$. (Set $p = q$ in 3) (Pearcy-Topping [26]).		
5. C_p	$K(H)$	C_p ,
provided $p > 1$.		
6. $\bigcap_{t>p} C_t$	$\bigcap_{t>q} C_t$	$\bigcap_{t>r} C_r$,
provided $p,q > 0$, $p^{-1}+q^{-1}=r^{-1}$, and $r \geq 1$.		
7. $\bigcap_{t>2} C_t$	$\bigcap_{t>2} C_t$	$\bigcap_{t>1} C_t$,
setting $p = q = 2$ in 7. (Salinas [31])		
8. $\bigcap_{t>1} C_t$	$K(H)$	$\bigcap_{t>1} C_t$
9. $\bigcup_{t<p} C_t$	$\bigcup_{t<q} C_t$	$\bigcup_{t<r} C_t$,
provided $0 < p,q \leq \infty$, $p^{-1}+q^{-1}=r^{-1}$, and $1 < r \leq \infty$.		
10. $\bigcup_{t<p} C_t$	$\bigcup_{t<p} C_t$	$\bigcup_{t<p/2} C_t$,
provided $p > 2$. (Set $p = q$ in 9).		
11. $\bigcup_{n=1}^{\infty} (\bigotimes_1^n I((\alpha(n))^{1/2})))$	$\bigcup_{n=1}^{\infty} (\bigotimes_1^n I((\alpha(n))^{1/2})))$	$\bigcup_{n=1}^{\infty} (\bigotimes_1^n I(\alpha))$
12. $\bigcup_{n=1}^{\infty} (\bigotimes_1^n I(\alpha))$	$K(H)$	$\bigcup_{n=1}^{\infty} (\bigotimes_1^n I(\alpha))$.

We promised a shorter proof for the Known Results 2-5, but arriving at them via Theorem 3.8-1,4,7 is not shorter than the proof in [26]. We needed to proceed along this route in order to obtain our more general results. To obtain the shorter and more lucid proof we promised, one should extract the main ideas from the main construction and the proof of Theorem 3.6 and, in regard to the C_p-classes, use the fact that α and β are equivalent to $(1/n)$. Indeed, the first three paragraphs in the proof of Theorem 3.6 show that it suffices to show that if $S \in IJ$ and is self-adjoint, then $S \oplus 0 \cong P \boxtimes S \in [I,J]_2$. The last paragraph in Theorem 3.6 gives the essential techniques to prove this. It is elementary to show that the solution operators are in the proper ideals. Also, the main construction can be shortened if one were willing to show less computation. Our use of α instead of $(1/n)$, which is employed in [26] and [31], eliminates the need for Riemann's theorem to obtain results on the rearrangements of certain conditionally convergent series which were needed in [26] and [31].

<u>On a Problem of Salinas</u>. Salinas asked the following question (unpublished). Does there exist an ideal I for which $C_2 \underset{\neq}{\subset} I \underset{\neq}{\subset} \bigcap_{p>2} C_p$, such that $[I,I] = I^2$? Answer: yes. Theorem 3.8-11 settles this. That is, the ideal $I = \bigcup_{n=1}^{\infty} (\boxtimes_1^n I((\alpha(n)^{1/2})))$ satisfies $[I,I] = I^2$. All that we need to show is that $C_2 \underset{\neq}{\subset} I \underset{\neq}{\subset} \bigcap_{p>2} C_p$.

By an earlier remark, $C_2 \subset I((n^{-1/2})) = I((\alpha(n)^{1/2}))$. It is clear that $D((n^{-1/2})) \in I((n^{-1/2})) \setminus C_2$ and also that $I((n^{-1/2})) = I((\alpha(n)^{1/2})) \subset I$. Therefore $C_2 \underset{\neq}{\subset} I$.

Also, $I((n^{-1/2})) \subset \bigcap_{p>2} C_p$ and $\bigcap_{p>2} C_p$ has the TPCP. Therefore, since I is the smallest ideal containing

$I((n^{-1/2})) = I((\alpha(n)^{1/2}))$ that has the TPCP, we have that $I \subset \bigcap_{p>2} C_p$. To show that $I \neq \bigcap_{p>2} C_p$ we must do quite a bit of technical work with the sequence α. First note that it suffices to show that $I^2 \neq \bigcap_{p>1} C_p$. We know that $I^2 = \bigcup_{n=1}^{\infty} (\boxtimes_1^n I(\alpha))$. It suffices to produce a sequence σ such that $D(\sigma) \in \bigcap_{p>1} C_p \setminus \bigcup_{n=1}^{\infty} (\boxtimes_1^n I(\alpha))$. For this, it suffices that $\sigma \in \ell^p_+$ for every $p > 1$ and $\sigma \notin \mathrm{Calk}(\bigcup_{n=1}^{\infty} (\boxtimes_1^n I(\alpha))) = \bigcup_{n=1}^{\infty} \mathrm{Calk}(\boxtimes_1^n I(\alpha))$ (i.e. $\sigma \notin \mathrm{Calk}(\boxtimes_1^n I(\alpha))$ for every n). The best procedure to determine when $\sigma \in \mathrm{Calk}(\boxtimes^n I(\alpha))$ is based on characteristic sets (see [23] and what follows).

To complete this argument, we need two results. First, for each fixed positive integer m, let $(\alpha^{(m)}(n))$ denote the sequence $(\alpha(i_1)\cdots\alpha(i_m))_{i_1,\ldots,i_m=1}^{\infty}$ arranged in decreasing order. We need to prove that if σ is any non-negative, non-increasing sequence, then $\sigma \in \mathrm{Calk}(\boxtimes_1^n I(\alpha))$ if and only if $\sigma_n = O(\alpha^{(m)}(n))$. Secondly, we need to construct such a sequence σ so that $\sigma \in \ell^r$ for every $r > 1$, but $\sigma_n \neq O(\alpha^{(m)}(n))$ for every m. From this it follows that $I^2 \neq \bigcap_{p>1} C_p$, which would complete the solution.

The characteristic set of an ideal is the set of all non-negative, non-increasing sequences contained in the corresponding Calkin ideal set. We first claim that the characteristic set of $I(\alpha)$ is the set of all non-negative, non-increasing sequences (x_n) for which $x_n = O(\alpha(n))$ (see p. 29). Let us indicate the proof. Since α is equivalent to $(1/n)$ (i.e. $\alpha(n)/n^{-1}$ is bounded above and below), we see that $\alpha(n)/\alpha(2n)$ is bounded above.

It is easy to verify that if $(a_n) \in c_o^+$ and $a_n\downarrow$, then a_n/a_{2n} is bounded above if and only if the set $\{(x_n) \in c_o^+ : x_n\downarrow \text{ and } x_n = O(a_n)\}$ is a characteristic set (we omit the details). But then the set of all rearrangements of the sequences (x_n) in the characteristic set $\{(x_n) \in c_o^+ : x_n\downarrow \text{ and } x_n = O(\alpha(n))\}$ forms a Calkin ideal set [23] that is clearly contained in $\mathrm{Calk}(I(\alpha))$. However $I(\alpha)$ is the smallest ideal containing $D(\alpha)$. This implies that the Calkin ideal set of all such rearrangements is precisely $\mathrm{Calk}(I(\alpha))$, and therefore that the characteristic set of $I(\alpha)$ is precisely the set $\{(x_n) \in c_o^+ : x_n\downarrow \text{ and } x_n = O(\alpha(n))\}$.

In a similar manner, we need to represent the characteristic set of $\overset{m}{\underset{1}{\boxtimes}} I(\alpha)$. We claim that the characteristic set of $\overset{m}{\underset{1}{\boxtimes}} I(\alpha)$ is the set of non-negative, non-increasing sequences (x_n) that are $O(\alpha^{(m)}(n))$. It is clear that for this, it suffices to prove the following two facts. First, $\overset{m}{\underset{1}{\boxtimes}} I(\alpha) = I(\alpha^{(m)}(n))$, and second, the characteristic set of $I(\alpha^{(m)}(n))$ is precisely the set $\{(x_n) \in c_o^+ : x_n\downarrow \text{ and } x_n = O(\alpha^{(m)}(n))\}$.

To prove the first fact, note that by earlier remarks, $\mathrm{Calk}(\overset{m}{\underset{1}{\boxtimes}} I(\alpha))$ is generated by $\{(x^{(1)}_{i_1} \cdot x^{(2)}_{i_2} \cdots x^{(m)}_{i_m})^{\infty}_{i_1,\ldots,i_m=1} : (x^{(k)}_n) \in \mathrm{Calk}(I(\alpha)) \text{ for } 1 \leq k \leq m\}$. Therefore $(\alpha(i_1)\cdot\alpha(i_2)\cdots\alpha(i_m)) \in \mathrm{Calk}(\overset{m}{\underset{1}{\boxtimes}} I(\alpha))$. Hence $(\alpha^{(m)}(n)) \in \mathrm{Calk}(\overset{m}{\underset{1}{\boxtimes}} I(\alpha))$, and so $D((\alpha^{(m)}(n))) \in \overset{m}{\underset{1}{\boxtimes}} I(\alpha)$ (which is an ideal). But $I((\alpha^{(m)}(n)))$ is the smallest ideal containing $D((\alpha^{(m)}(n)))$. Therefore $\overset{m}{\underset{1}{\boxtimes}} I(\alpha) \supset I((\alpha^{(m)}(n)))$. On the other hand, every sequence in

$\text{Calk}(\bigotimes_1^m I(\alpha))$ is dominated by a $(x^{(1)}_{i_1}\cdots x^{(m)}_{i_m})$ where $(x_n^{(k)}) \in \text{Calk}(I(\alpha))$ for each $1 \le k \le m$. But by what we have previously shown, the sequence $(x_n^{(k)}) \in \text{Calk}(I(\alpha))$ if and only if its decreasing rearrangement $(\bar{x}_n^{(k)})$ is $O(\alpha(n))$. However, this implies that the decreasing rearrangement of $(\bar{x}^{(1)}_{i_1}\cdots\bar{x}^{(m)}_{i_m})$ is $O(\alpha^{(m)}(n))$. Hence $(\bar{x}^{(1)}_{i_1}\cdots\bar{x}^{(m)}_{i_m}) \in \text{Calk}(I(\alpha^{(m)}(n)))$, and so it is clear that $(x^{(1)}_{i_1}\cdots x^{(m)}_{i_m}) \in \text{Calk}(I(\alpha^{(m)}(n)))$ (since they have precisely the same unordered entries, counting multiplicities). Therefore $\bigotimes_1^m I(\alpha) \subset I((\alpha^{(m)}(n)))$, and hence $\bigotimes_1^m I(\alpha) = I((\alpha^{(m)}(n)))$.

To prove the second fact, as we remarked earlier, it is sufficient to show that $\alpha^{(m)}(n)/\alpha^{(m)}(2n)$ is bounded above as a function of n. To do this, we need to find $(\alpha^{(m)}(n))$ explicitly. To describe $(\alpha^{(m)}(n))$ explicitly, first recall that α is the non-increasing sequence with entries 4^{-k} occurring 4^k times, for every non-negative integer k, and that $(\alpha^{(m)}(n))$ is the decreasing rearrangement of $(\alpha(i_1)\cdots\alpha(i_m))$. Clearly then, for each n, $\alpha^{(m)}(n)$ is of the form 4^{-k} for some non-negative integer k. Therefore all we need to know to describe $(\alpha^{(m)}(n))$ is how many times 4^{-k} occurs in the sequence, for each k.

Let $(i_1,\ldots,i_m)$ be a non-negative integer m-tuple where $i_1 + \cdots + i_m = k$. Hence, $4^{-i_1}\cdots4^{-i_m} = 4^{-k}$, and since 4^{-i} occurs in α exactly 4^i times, $4^{-i_1}\cdots4^{-i_m} = 4^{-k}$ occurs in a 4^{-k} block $4^{i_1}\cdots4^{i_m} = 4^k$ times. Thus every such m-tuple contributes 4^k terms, each of size 4^{-k}. The number of such 4^{-k} blocks is exactly the number of m-tuples of non-negative integers $(i_1,\ldots,i_m)$ such that $i_1+\cdots+i_m=k$. Suppose we let $f_m(k)$ denote the number of distinct m-tuples of non-negative integers $(i_1,\ldots,i_m)$ whose sum is k. It can be shown, using induction, that $f_m(k)$ is an increasing function in both k and m, and that $f_m(k)$ is an $(m-1)^{st}$ degree polynomial in k with a positive leading coefficient. It is clear, therefore, that $(\alpha^{(m)}(n))$ is just the decreasing sequence with entries 4^{-k} repeated $4^k f_m(k)$ times, for each k. Therefore, it is clear that if n and N satisfy

$$(*) \qquad \Sigma_{k=0}^{N-1} 4^k f_m(k) < n \leq \Sigma_{k=0}^{N} 4^k f_m(k) \quad ,$$

then $\alpha^{(m)}(n) = 4^{-N}$. We now use this fact to prove that $\alpha^{(m)}(n)/\alpha^{(m)}(2n)$ is bounded above. Indeed, we claim that for all but a finite number of values of n, we have $\alpha^{(m)}(2n) \geq 4^{-(N+1)}$ (whence $\alpha^{(m)}(n)/\alpha^{(m)}(2n) \leq 4$). For this it suffices to show that if n is sufficiently large, then

$$2n \leq \Sigma_{k=0}^{N+1} 4^k f_m(k) \quad .$$

But by (*),

$$2n \leq 2\,\Sigma_{k=0}^{N} 4^k f_m(k) \quad .$$

It therefore suffices to show that if n is sufficiently large, then

$$2\,\Sigma_{k=0}^{N} 4^k f_m(k) \leq \Sigma_{k=0}^{N+1} 4^k f_m(k) \quad .$$

The last inequality is equivalent to the inequality

$$\Sigma_{k=0}^{N} 4^k f_m(k) \leq 4^{N+1} f_m(N+1) \quad .$$

Note that since $f_m(k)$ is an increasing function of k, we have

$$\Sigma_{k=0}^{N} 4^k f_m(k) \le (\Sigma_{k=0}^{N} 4^k) f_m(N) = \left(\frac{4^{N+1}-1}{3}\right) f_m(N) \le \left(\frac{4^{N+1}}{3}\right) f_m(N) .$$

Hence, it suffices to show that if n is sufficiently large, then

$$\left(\frac{4^{N+1}}{3}\right) f_m(N) \le 4^{N+1} f_m(N+1) ;$$

that is,

$$(**) \qquad 1/3 \le f_m(N+1)/f_m(N) .$$

However, since $n \le \Sigma_{k=0}^{N+1} 4^k f_m(k)$, we have that $N \to \infty$ as $n \to \infty$. But as we remarked earlier, $f_m(N)$ is an $(m-1)^{st}$ degree polynomial in N, and hence

$$\lim_{N \to \infty} \frac{f_m(N+1)}{f_m(N)} = 1 > 1/3 .$$

Therefore if n is sufficiently large, then (**) holds, and we have proved that $\alpha^{(m)}(n)/\alpha^{(m)}(2n)$ is bounded above.

This provides us with a criterion that determines when a non-negative, non-increasing sequence $\sigma = (\sigma_n)$ is contained in $\mathrm{Calk}(\Sigma_1^n I(\alpha)) = \mathrm{Calk}(I((\alpha^{(m)}(n))))$. In fact, for such a σ , containment holds if and only if $\sigma_n = O(\alpha^{(m)}(n))$. This proves our first result.

We are now ready to obtain the second and final result, namely, we shall choose σ so that it is contained in every ℓ_+^r for every $r > 1$, but so that it fails to be $O(\alpha^{(m)}(n))$, for every fixed m. Indeed, define $\sigma = (\sigma_n)$ to be the non-negative, non-increasing sequence with entries 4^{-k} occurring $4^k \cdot [4^{k^{1/2}}]$ times.

If $r > 1$, then

$$\|\sigma\|_{\ell^r}^r = \Sigma_k 4^{-rk} \cdot 4^k [4^{k^{1/2}}] \le \Sigma_k 4^{-((r-1)k - k^{1/2})} < \infty$$

and so $\sigma \in \ell^r_+$.

To show $\sigma_n \neq O(\alpha^{(m)}(n))$ for every m, it suffices to show that for every m we have $\alpha^{(m)}(n) = o(\sigma_n)$ (since it is impossible to have two sequences (x_n) and (y_n) that satisfy $x_n = O(y_n)$ and $y_n = o(x_n)$). Furthermore, it follows easily from the description of $(\alpha^{(m)}(n))$ and σ that for every m, if n is sufficiently large, then $\alpha^{(m)}(n) \leq \alpha^{(m+1)}(n) \leq \sigma_n$. Hence, it is enough to show that $\alpha^{(m)}(n) = o(\alpha^{(m+1)}(n))$.

For each positive integer n, let N(n) and M(n) be the unique positive integers for which $\Sigma_{k=0}^{N(n)} 4^k f_{m+1}(k) < n \leq \Sigma_{k=0}^{N(n)+1} 4^k f_{m+1}(k)$ and $\Sigma_{k=0}^{M(n)} 4^k f_m(k) < n \leq \Sigma_{k=0}^{M(n)+1} 4^k f_m(k)$.

Then $\alpha^{(m)}(n) = 4^{-(M(n)+1)}$ and $\alpha^{(m+1)}(n) = 4^{-(N(n)+1)}$, and so $\alpha^{(m)}(n)/\alpha^{(m+1)}(n) = 4^{-(M(n) - N(n))}$, where $M(n) \geq N(n)$ for every n (since $f_m(k)$ is an increasing function in m), and M(n) and N(n) tend to ∞ as n tends to ∞. Therefore, to show $\alpha^{(m)}(n) = o(\alpha^{(m+1)}(n))$, it is enough to show that $M(n) - N(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. Suppose not; that is, suppose that $M(n) - N(n) \leq N_o$ for some positive integer N_o and infinitely many n. The definitions of N(n) and M(n) imply that

$$\Sigma_{k=0}^{N(n)} 4^k f_{m+1}(k) < \Sigma_{k=0}^{M(n)+1} 4^k f_m(k) ,$$

and since $f_m(k)$ is an increasing function in both k and m, we obtain

$$4^{N(n)} f_{m+1}(N(n)) < \Sigma_{k=N(n)}^{M(n)+1} 4^k f_m(k)$$

$$< (M(n) - N(n) + 2) 4^{M(n)+1} f_m(M(n)+1).$$

If $0 \leq M(n)-N(n) \leq N_o$, then

$$f_{m+1}(N(n))/f_m(M(n)+1) \leq (M(n) - N(n) + 2)4^{M(n)-N(n)+1}$$

$$\leq (N_o+2)4^{N_o+1} \quad \text{and}$$

$f_m(M(n)+1) \leq f_m(N(n)+N_o+1)$. Therefore, for infinitely many n, we have

$$f_{m+1}(N(n))/f_m(N(n)+N_o+1) \leq f_{m+1}(N(n))/f_m(M(n)+1)$$

$$\leq (N_o+2)4^{N_o+1};$$

that is, as $n \longrightarrow \infty$, we have $N(n) \longrightarrow \infty$, and so for infinitely many positive integers N, we have that $f_{m+1}(N)/f_m(N+N_o+1)$ remains bounded. But this is impossible due to the fact that for each m, $f_m(k)$ is an $(m-1)^{st}$ degree polynomial in k with a positive leading coefficient, and this implies that $\lim_{N \longrightarrow \infty} f_{m+1}(N)/f_m(N+N_o+1) = \infty$. This proves that $M(n)-N(n) \longrightarrow \infty$ as $n \longrightarrow \infty$, and so $\alpha^{(m)}(n) = o(\alpha^{(m+1)}(n))$ for each m, $\alpha^{(m)}(n) = o(\sigma_n)$, $\sigma_n \neq O(\alpha^{(m)}(n))$ for every m, $\sigma \in \bigcap_{r>1} \ell^r \setminus \bigcup_{n=1}^{\infty} \text{Calk}(\oplus_1^n I(\alpha))$ and $D(\sigma) \in \bigcap_{p>1} C_p \setminus \bigcup_{n=1}^{\infty} (\oplus_1^n I(\alpha))$. This completes our solution of Salinas' problem.

<u>An Important Theme</u>. In light of Theorem 3.8 and our preceding remarks 'On a Problem of Salinas', an important theme of this section becomes evident. Let I be an ideal for which $C_2 \subsetneq I \subset K(H)$. Then the smaller I is, the more difficult it is for us to decide whether or not $[I,I] = I^2$. There is an analogous phenomenon in the next section. That is, if $F(H) \subset I \subset C_2$, then the larger I is, the more difficult it is for us to decide whether or not $[I,I]=(I^2)^o$.

New Questions. Lemma 3.5, Theorem 3.6, Corollary 3.7 and Theorem 3.8 give rise to questions which seem difficult. The smallest ideal I for which we can show that $P \in [I,I]$ is $I((\alpha(n)^{1/2}))$ (see Lemma 3.5) and the smallest ideal I that we can show satisfies $[I,I] = I^2$ is the ideal $I = \bigcup_{n=1}^{\infty} (\bigotimes_1^n I((\alpha(n)^{1/2})))$ (see Theorem 3.8-11). The latter ideal is the smallest ideal with the TPCP such that $D(\alpha) \in I^2$.

Question 3.9. Is the converse of Lemma 3.5 true? In particular, if an ideal I satisfies $P \in [I,I]$, must $D(\alpha) \in I^2$? In other words, is $I((\alpha(n)^{1/2}))$ the smallest ideal I for which $P \in [I,I]$? Furthermore, does there exist an ideal $I \supsetneq C_2$ such that $P \notin [I,I]$?

Question 3.10. If an ideal I satisfies $[I,I] = I^2$, must I have the TPCP, or must $D(\alpha) \in I^2$? Furthermore, is the converse of Theorem 3.6 or Theorem 3.6a true?

Question 3.11. Are Theorems 3.6 and 3.6a equivalent?

Question 3.12. (The Main Question). Is $I = \bigcup_{n=1}^{\infty} (\bigotimes_1^n I((\alpha(n)^{1/2})))$ the smallest ideal I for which $[I,I] = I^2$?

In regard to the last question, note that $C_2 \subsetneq I((1/n^{1/2})) = I((\alpha(n)^{1/2})) \subset \bigcup_{n=1}^{\infty} (\bigotimes_1^n I((\alpha(n)^{1/2})))$ and that $[C_2,C_2] \subset C_1^o \subsetneq C_1$. In this setting, the phenomenon of most interest is the structure of $[I,I]$ for ideals I for which

$C_2 \subsetneq I \subsetneq \bigcup_{n=1}^{\infty} (\otimes_1^n I((\alpha(n))^{1/2})))$. In all the results we know on [I,I], the use of diagonal operators and sequence rearrangements was indispensable. It is not clear whether or not diagonal operators are central to the questions, though they are central to our techniques. Some of our later results suggest both possibilities. However, we conjecture (though weakly) that the answer to questions 3.9, 3.10, and 3.12 is yes, but that the proofs lie outside the domain of present operator theory techniques. The following remarks give some impetus for our conjecture.

Remarks. We have shown (unpublished) that the techniques of our main construction do not generalize in the obvious way. Furthermore, in joint work with Edward Azoff, we have shown that the Pearcy and Topping techniques [26] do not generalize in the obvious way. Finally, Salinas [31] attempted to generalize the techniques and results in [26], but his work contains an error (of which he is aware) and the best result that his proof establishes is what we call Theorem 3.8-7.

We close this section with a generalization of Theorems 3.6 and 3.6a, whose proof is almost identical to that of Theorem 3.6. It will be useful in the event that there turn out to be ideals I for which $D(\alpha) \not\in I^2$ but yet $P \in [I,I]$. It indicates the basic strategies of the proofs in [26] and [31]. We omit the proof.

Theorem 3.13. If the ideal I has the TPCP, then $P \in [I,I]$ if and only if $[I,I] = I^2$.

4. Cases when $[I,I] = (I^2)^o$ (The Main Section)

This section begins our study of commutators (of compact operators) which are in the trace class and have trace 0. Inclusion chain 1.2 states that if $I^2 \subset C_1$ (equivalently, $I \subset C_2$), then for each n,

$$C(I) \subset [I,I]_n \subset [I,I] \subset (I^2)^o \underset{\neq}{\subset} I^2 \subset I.$$

So $[I,I] \subset (I^2)^o$, and therefore the only operators in I^2 which can be commutators or finite sums of commutators of I are those with trace 0. The main problem then is to find out which operators in I^2 with trace 0 are commutators of I and which are finite sums of commutators of I. The structure of C(I) is unknown for every ideal I for which $F(H) \underset{\neq}{\subset} I \underset{\neq}{\subset} L(H)$. Open Question 4 asks whether or not $[C_2,C_2] = C_1^o$ (equivalently, $C_1^o \subset [C_2,C_2]$). This question provided the general impetus for our work with the equation '$[I,I] = (I^2)^o$'. We still do not know whether or not $C_1^o \subset [C_2,C_2]$, but we have some indications that the answer is no. We can show, however, that $(\bigcup_{p<1} C_p)^o$ is contained in $[C_2,C_2]$. In fact, we shall prove that for $I = \bigcup_{p<2} C_p$, we obtain $[I,I] = (I^2)^o = (\bigcup_{p<1} C_p)^o$. Our main construction will yield proofs that $[I,I] = (I^2)^o$ for several other ideals also. Furthermore, an analogy will become evident between many of our results and Known Results 2-5.

In Section 1 we used a fact about the trace to prove the inclusion chain 1.2. For the sake of completeness we will state and prove that fact in what follows.

Proposition 4.1. If $A \in K(H)$, $B \in L(H)$ and AB and $BA \in C_1$, then $Tr(AB - BA) = 0$.

Proof. First we claim that if $T \in F(H)$ then $Tr(TB) = Tr(BT)$. Let $(Ran\ T) + (Ran\ T^*)$ denote the linear span of $(Ran\ T) \cup (Ran\ T^*)$. Then $N = \dim((Ran\ T) + (Ran\ T^*)) \leq \dim(Ran\ T) + \dim(Ran\ T^*) < \infty$. It is elementary to show that $H = ((Ran\ T) + (Ran\ T^*)) \oplus ((\ker T) \cap (\ker T^*))$. Let $\{e_n\}$ be a basis for H in which $\{e_n\}_{n=1}^N$ is a basis for $(Ran\ T) + (Ran\ T^*)$ and $\{e_n\}_{n=N+1}^\infty$ is a basis for $(\ker T) \cap (\ker T^*)$. Let $T = (t_{ij})$ denote the matrix representation for T with respect to the basis $\{e_n\}$. Then $t_{ij} = (Te_j, e_i) = (e_j, T^*e_i) = 0$ if either $i > N$ or $j > N$. But $T \in F(H) \subset C_1$ implies TB and $BT \in C_1$, and hence $Tr(TB)$ and $Tr(BT)$ exist and are independent of basis. However, with respect to $\{e_n\}$, $(TB-BT)_{ij} = \Sigma_k t_{ik}b_{kj} - b_{ik}t_{kj}$ and $Tr(TB - BT) = \Sigma_i \Sigma_k t_{ik}b_{ki} - b_{ik}t_{ki} = \Sigma_{i,k=1}^N t_{ik}b_{ki} - \Sigma_{i,k=1}^N b_{ik}t_{ki} = 0$ (where, for each operator A, $(A)_{ij}$ denotes (Ae_j, e_i); that is, the (i,j)-entry of the matrix of A with respect to $\{e_n\}$).

If we assume $AB \in C_1$, letting $A = U|A|$ be the polar decomposition of A, we obtain $|A|B = U^*U|A|B = U^*AB \in C_1$. Since A is compact, $|A|$ is diagonalizable and there clearly exists a sequence of projection operators $P_m \longrightarrow I$ (SOT) for which $P_m |A| = |A| P_m$ (where if $\{X_n\}$ and X are operators, then $X_n \longrightarrow X$ (SOT) means that $\|X_n f - Xf\| \longrightarrow 0$ as $n \longrightarrow \infty$, for every $f \in H$). It is a fact about C_1 that if $X \in C_1$ then $P_m X \longrightarrow X$ and $XP_m \longrightarrow X$ as $m \longrightarrow \infty$, where the convergence is in the trace norm. Hence $AP_m B = U|A|P_m B = UP_m |A| B \longrightarrow U|A|B = AB$. Similarly $BAP_m \longrightarrow BA$. In both cases the convergence is in the trace norm. But $AP_m \in F(H)$ and $Tr(AP_m B - BAP_m) = 0$. However, the trace is a linear functional on C_1 which is continuous in the trace norm. Therefore, $Tr(AB - BA) = \lim_{m \to \infty} Tr(AP_m B - BAP_m) = 0$. Q.E.D.

The first complete work on commutators was done, as expected, on finite dimensional Hilbert spaces (for references see [18, p. 126]). The next theorem characterizes $C(L(H))$ when $\dim H < \infty$ as follows:

$$C(L(H)) = [L(H), L(H)] = ((L(H))^2)^o = (L(H))^o.$$

<u>Theorem 4.2</u>. If $\dim H < \infty$, then an operator T is a commutator if and only if $Tr\, T = 0$.

<u>Proof</u>. Suppose T is a commutator. Since $\dim H < \infty$ implies that $F(H) = C_1 = L(H)$, Proposition 4.1 gives us that $Tr\, T = 0$.

Conversely, suppose $Tr\, T = 0$. It is well-known that the numerical range of T is convex [18]. These facts imply the existence of a basis $\{e_n\}_{n=1}^N$ for H with respect to which the diagonal entries of $T = (t_{ij})$ are all 0. (Also, it is not hard to give a direct and constructive proof of this using matrix representations of T.) Choose any sequence $d = (d_n)_{n=1}^N$ of distinct numbers (real or complex). Then by computation, for any NxN matrix $X = (x_{ij})$, $(D(d)X - XD(d))_{ij} = (d_i - d_j)x_{ij}$. Since $d_i - d_j \neq 0$ if $i \neq j$, we can solve $(d_i - d_j)x_{ij} = t_{ij}$ for $X = (x_{ij})$, namely let $x_{ij} = t_{ij}/(d_i - d_j)$ if $i \neq j$, and $x_{ij} = 0$ if $i = j$. Then $T = D(d)X - XD(d)$. Q.E.D.

F(H) is the only non-trivial ideal, when H is separable and infinite-dimensional, whose inclusion chains 1.1 and 1.2 are fully understood. Since $F(H) \subset C_1$, inclusion chain 1.2 applies. The next theorem shows that

$$C(F(H)) = [F(H), F(H)] = ((F(H))^2)^o = (F(H))^o.$$

It is actually a corollary to Theorem 4.2, and the techniques used do not extend far beyond $I = F(H)$. For $F(H) \subset I \subset C_2$, we find that the larger I becomes, the more difficult it is for us to show $[I,I] = (I^2)^o$. The techniques employed for the case $I = F(H)$ are only useful to us in one other circumstance, namely, in a partial result regarding $C_{1/2 - \epsilon}$, and this we can also obtain without these techniques.

Theorem 4.3. $C(F(H)) = (F(H))^o$. That is, a finite rank operator is a commutator of finite rank operators if and only if Tr T = 0.

Proof. If T is a commutator of finite rank operators then Proposition 4.1 applies to the commutator and so Tr T = 0.

If $T \in F(H)$ and $\operatorname{Tr} T = 0$, as in Proposition 4.1, $H = ((\operatorname{Ran} T) + (\operatorname{Ran} T^*)) \oplus ((\ker T) \cap (\ker T^*))$, where $\dim((\operatorname{Ran} T) + (\operatorname{Ran} T^*)) < \infty$. Clearly $T = T_1 \oplus 0$, where $T_1 = T|_{(\operatorname{Ran} T)+(\operatorname{Ran} T^*)}$ and 0 acts on $(\ker T)\cap(\ker T^*)$. But $\operatorname{Tr} T_1 = \operatorname{Tr} T = 0$ and T_1 is an operator on the finite-dimensional Hilbert space $(\operatorname{Ran} T)+(\operatorname{Ran} T^*)$. Hence, by Theorem 4.2, $T_1 = AB - BA$ for some operators acting on $(\operatorname{Ran} T)+(\operatorname{Ran} T^*)$. Therefore $T = T_1 \oplus 0 = (AB - BA) \oplus 0 = (A\oplus 0)(B\oplus 0) - (B\oplus 0)(A\oplus 0)$ and $A\oplus 0$ and $B\oplus 0 \in F(H)$. Q.E.D.

From here on, we will make much use of Calkin's ideal set axioms [9]. Therefore we shall now list them together with some important facts.

Calkin's Ideal Representation Theorem. Let $S \subset c_o^+$. There exists an ideal $I \subset L(H)$ such that $S = \operatorname{Calk}(I)$ if and only if S satisfies

1. $(x_n) \in S$, and $0 \le y_n \le x_n$ for every $n \Longrightarrow (y_n) \in S$.
2. If $(x_n), (y_n) \in S$, then $(x_n + y_n) \in S$.
3. $(x_n) \in S \Longrightarrow (x_{\pi(n)}) \in S$ for every π that permutes Z^+.

We shall make free use of several obvious facts. Let $(x_n) \in S = \operatorname{Calk}(I)$ for some ideal I, and let $(y_n) \in c_o^+$. If $y_n \le x_n$ for all but a finite number of values of n, then $(y_n) \in S$. If the set of entries of (y_n) are the same counting multiplicities, as the set of entries of (x_n), except possibly for a finite number of values of n, then $(y_n) \in S$. Furthermore every subsequence of (x_n) is also contained in S.

Let us also give the characteristic set axioms [23] together with some important facts about them.

The Characteristic Set Representation Theorem for Ideals. Let C be a set of non-negative, non-increasing sequences that tend to 0. Let C_I (the characteristic set of I) denote the set of all non-increasing sequences in Calk(I). Then $C = C_I$ for some ideal I (i.e. C is a characteristic set) if and only if C satisfies

1. $(x_n) \in C$, $y_n \downarrow 0$, and $0 \leq y_n \leq x_n$ for every $n \Longrightarrow (y_n) \in C$.
2. If $(x_n), (y_n) \in C$, then $(x_n + y_n) \in C$.
3. $(x_n) \in C \Longrightarrow (x_1, x_1, x_2, x_2, x_3, \ldots) \in C$.

It is easy to see that characteristic sets possess properties quite similar to the properties of Calkin ideal sets mentioned earlier. In addition, if $x = (x_n) \in c_o^+$, $x_n \downarrow 0$, and $(x_n^{(m)}) = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots)$, where each x_n is repeated m times, then the characteristic set of I(x) is precisely the set $\{(y_n) \in c_o^+ : y_n \downarrow 0 \text{ and } y_n = O(x_n^{(m)}) \text{ for some } m\}$. In case x_n/x_{2n} is bounded, then x and $(x_n^{(m)})$ are equivalent for each m, and so the characteristic set of I(x) is $\{(y_n) \in c_o^+ : y_n \downarrow 0 \text{ and } y_n = O(x_n)\}$.

We now begin the development of our strategy for studying [I,I] when $I \subset C_2$. There are corresponding questions about the validity of the equation $[I,J] = (IJ)^o$, but we shall not emphasize them.

The first result is a lemma which introduces a class of diagonal operators that are crucial to our techniques and relates them to the structure of $[I,I]$.

Lemma 4.4. Let I be an ideal such that $I \subseteq C_2$. Then the following are equivalent.

(a) $[I,I] = (I^2)^o$.

(b) If $(d_n) \in \text{Calk}(I^2)$, $d_n \downarrow 0$ and $\Sigma_{n=1}^{\infty} d_n = d$, then $D(-d,d_1,d_2,\ldots) \in [I,I]$.

(c) If $(d_n) \in \text{Calk}(I^2)$, $d_n \downarrow 0$ and $\Sigma d_n = d$, then $D(-d,d_1,d_2,\ldots) \oplus 0 \in [I,I]$, where 0 denotes the zero operator on H and I is considered as an ideal in $L(H \oplus H)$.

In fact, $[C_2,C_2] = C_1^o$ if and only if, for every sequence $(d_n) \in \ell^1$ such that $d_n \downarrow 0$ and $\Sigma d_n = d$, we obtain $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_2,C_2]$.

Proof. Clearly (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). It therefore suffices to show that (c) $\Longrightarrow$ (a). Since $I \subseteq C_2$ implies $[I,I] \subseteq (I^2)^o$, it is enough to show (c) $\Longrightarrow$ $[I,I] \supseteq (I^2)^o$.

Let $T \in (I^2)^o$. Then $T = T' + iT''$ where $T' = (T+T^*)/2$ and $T'' = (T-T^*)/2i$ are the real and imaginary parts of T. Both T' and T'' are therefore compact and self-adjoint. Furthermore T' and $T'' \in I^2$. We assert that T' and $T'' \in (I^2)^o$, that is, $\text{Tr}\, T' = \text{Tr}\, T'' = 0$. To see this, note that for every self-adjoint operator S, the diagonal entries of $S = (s_{ij})$ with respect to a basis $\{e_n\}$ are $s_{ii} = (Se_i,e_i) = (e_i,Se_i) = \overline{(Se_i,e_i)}$ and so s_{ii} is real for each i. Therefore

$\text{Tr } S = \Sigma_i s_{ii}$ is real and hence Tr T' and Tr T" are real. But then $0 = \text{Tr } T = \text{Tr } T' + i(\text{Tr } T")$ implies Tr T'= Tr T"= 0. So T is a linear combination of two self-adjoint operators in I^2 with trace 0. Let us consider any operator S for which S is self-adjoint and $S \in (I^2)^o$.

If $S \in (I^2)^o$ and is self-adjoint, then $S \cong D((a_n))$ for some real-valued infinite sequence (a_n) where $(|a_n|) \in \text{Calk}(I^2)$ and $\Sigma_n a_n = 0$. This follows from the spectral theorem for compact, self-adjoint operators and the unitary invariance of the trace. Let (a_n^+) and (a_n^-) denote the subsequences of the non-negative and of the negative entries, respectively, of the sequence (a_n), counting multiplicities. Then $\text{Tr } S = \Sigma_n a_n^+ + \Sigma_n a_n^- = 0$, and so $\Sigma a_n^+ = -\Sigma a_n^-$. Clearly $D((a_n)) \cong D((a_n^+)) \oplus D((a_n^-))$. We assert that $D((a_n^+)) \oplus D((a_n^-))$ is the sum of at most 4 operators each of which is unitarily equivalent to an operator of the form $\pm D(-d, d_1, d_2, \ldots) \oplus 0$ where $d_n \downarrow 0$, $\Sigma d_n = d$, $(d_n) \in \text{Calk}(I^2)$, and 0 is infinite dimensional. To see this, we must consider three cases. In the first case, suppose (a_n^+) and (a_n^-) are both infinite sequences. Letting $d = \Sigma a_n^+ = -\Sigma a_n^-$, we obtain

$$\begin{aligned} D((a_n^+)) \oplus D((a_n^-)) &= [D((a_n^+)) \oplus 0] + [0 \oplus D((a_n^-))] \\ &= [D((a_n^+)) \oplus (-dP)] + [(-dP) \oplus (dP)] \\ &\quad + [(dP) \oplus D((a_n^-))] , \end{aligned}$$

where each 0 operator is infinite-dimensional, which proves our assertion for this case.

In the second case, suppose (a_n^+) is a finite sequence, say of length N. Since (a_n) is an infinite sequence, (a_n^-) must be an infinite sequence. Let P_N denote the NxN diagonal matrix $D(1,0,\ldots,0)$. Then

$$D((a_n^+)) \oplus D((a_n^-)) = [D((a_n^+)) \oplus (-dP)] + [(-dP_N) \oplus (dP)]$$
$$+ [(dP_N) \oplus D((a_n^-))] ,$$

and as before, $D((a_n^+)) \oplus (-dP) \cong D(-d,a_1^+,a_2^+,\ldots) \oplus 0$ and $(-dP_N) \oplus (dP) \cong D(-d,d) \oplus 0$, where the 0 operator in each case is infinite-dimensional. However $(dP_N) \oplus D((a_n^-))$ is not unitarily equivalent to an operator of the desired form since its kernel is of dimension N-1. But

$$(dP_N) \oplus D((a_n^-)) = [(-\Sigma a_{2n-1}^-)P \oplus D(a_1^-,0,a_3^-,0,\ldots)]$$
$$+ [(-\Sigma a_{2n}^-)P \oplus D(0,a_2^-,0,a_4^-,\ldots)] .$$

Letting $b = -\Sigma a_{2n-1}^-$ and $c = -\Sigma a_{2n}^-$, we obtain $(-\Sigma a_{2n-1}^-)P \oplus D(a_1^-,0,a_3^-,0,\ldots) \cong -D(-b,-a_1^-,-a_3^-,\ldots) \oplus 0$ and $(-\Sigma a_{2n}^-)P \oplus D(0,a_2^-,0,a_4^-,\ldots) \cong -D(-c,-a_2^-,-a_4^-,\ldots) \oplus 0$, where each operator 0 is infinite-dimensional, which proves our assertion for this case.

In the third case, suppose (a_n^-) is a finite sequence of length N. Then, as in the second case, (a_n^+) must be an infinite sequence. This is the only remaining case since (a_n^+) and (a_n^-) cannot both be finite sequences because (a_n) is an infinite sequence. We omit the proof for this case as it is very similar to our proof for the second case.

The one essentially different aspect of this case is that $D((a_n^+))$ can have an infinite-dimensional kernel, in which case the proof could be shorter. However, if we ignore this possibility, the analog of the second case proof suffices. Thus we have proved that every self-adjoint operator $S \in (I^2)^o$ is the linear combination of 4 or fewer operators each of which is unitarily equivalent to an operator of the form $D(-d,d_1,d_2,\ldots) \oplus 0$, where $d_n \downarrow 0$, $\Sigma d_n = d$, $(d_n) \in \mathrm{Calk}(I^2)$, and 0 is infinite-dimensional. Hence T is the sum of 8 or fewer such operators. Finally, since we are assuming (c) holds, we see that $T \in [I,I]$. Q.E.D.

Note that in the proceeding proof, the operator $D(-d,d)$ is easily seen to be a commutator of finite rank operators, namely, $D(-d,d) = \begin{pmatrix} 0 & 0 \\ d^{1/2} & 0 \end{pmatrix}\begin{pmatrix} 0 & d^{1/2} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & d^{1/2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ d^{1/2} & 0 \end{pmatrix}$. It is the cases when (d_n) is an infinite sequence that present the difficulties.

In the beginning, Halmos pointed out that if U is the unilateral shift of multiplicity 1 and $P = D(1,0,0,\ldots)$ then $P = U^*U - UU^*$. This phenomenon and the computation it involves permeates all the work on commutator structures since then. In particular, it is central to the work on commutators of compact operators. Pearcy and Topping [26], and Salinas [31] made use of its properties. Our techniques of Section 3 are also related to the phenomenon, although on the surface they appear to be independent of it. In the techniques that follow, we shall depend heavily on that phenomenon. The next several remarks illustrate our dependence on it.

The operator U is not compact. In order to use Halmos's idea in our context, we need compact substitutes for U that behave similarly. Hence we introduce the following notation.

Notation. If (w_n) is a (finite or infinite) sequence of complex numbers, let $U((w_n)) = U(w_1, w_2, \ldots)$ denote the weighted shift of multiplicity 1 with weights w_n (that is, in some basis $\{e_n\}$, $U((w_n))e_n = w_n e_{n+1}$). Also, let $U^*((w_n))$ denote $[U((w_n))]^*$.

The operators $U((w_n))$ are our building blocks for the solution operators in our commutator equations. We now consider their self-commutators. The self-commutator of $U((w_n))$ is a diagonal operator whose diagonal entries are functions of (w_n). We need to know exactly what these functions are and so we proceed to the following commutator equations which follow by computation.

Special Commutator Equations.

(4.5) $U^*((w_n))U((w_n)) - U((w_n))U^*((w_n))$

$$= D(|w_1|^2, |w_2|^2 - |w_1|^2, |w_3|^2 - |w_2|^2, \ldots, |w_{n+1}|^2 - |w_n|^2, \ldots) .$$

Let (d_n) be a non-negative, non-increasing sequence for which $\Sigma d_n = d$. Setting $w_n = (\Sigma_{i=n}^{\infty} d_i)^{1/2}$ in equation (4.5) we obtain

$$(4.6)\quad A^*A - AA^* = -D(-d, d_1, d_2, \ldots) \quad \text{for}$$

$$A = U(\ (\Sigma_{i=1}^{\infty} d_i)^{1/2},\ (\Sigma_{i=2}^{\infty} d_i)^{1/2},\ \ldots\ ) .$$

For the purpose of generalizing our results to the $[I,J]$ cases, the following equation is useful. If $x = (x_n)$ and $y = (y_n)$ are such that $x_n y_n = \Sigma_{i=n}^{\infty} d_i$ for each n, then

$$(4.6a)\quad U^*(x)U(y) - U(y)U^*(x) = -D(-d, d_1, d_2, \ldots) .$$

Using equation (4.6) we obtain the following basic theorem.

<u>Theorem 4.7</u>. Let $d_n \downarrow 0$ and $\Sigma d_n = d$. If I is an ideal such that $((\Sigma_{i=n}^{\infty} d_i)^{1/2})_{i=1}^{\infty} \in \text{Calk}(I)$, then there exists $A \in I$ for which $|A| = D(((\Sigma_{i=n}^{\infty} d_i)^{1/2}))$ such that $D(-d, d_1, d_2, \ldots) = -(A^*A - AA^*)$, and so $D(-d, d_1, d_2, \ldots) \in C(I)$.

<u>Proof</u>. Let $A = U(((\Sigma_{i=n}^{\infty} d_i)^{1/2}))$ in equation (4.6). The rest follows by inspection. Q.E.D.

With respect to the $[I,J]$ cases, we give a generalization of Theorem 4.7. The proof uses equation (4.6a).

<u>Theorem 4.7a</u>. Let $d_n \downarrow 0$ and $\Sigma d_n = d$. If I and J are ideals such that $((\Sigma_{i=n}^{\infty} d_i)) \in \text{Calk}(IJ)$ (equivalently, $\Sigma_{i=n}^{\infty} d_i = x_n y_n$ for each n, for some $x = (x_n) \in \text{Calk}(I)$ and $y = (y_n) \in \text{Calk}(J)$), then there exist $A \in I$ and $B \in J$ for which $|A| = D(x) \in I$ and $|B| = D(y) \in J$ such that $D(-d, d_1, d_2, \ldots) = AB - BA$, and so $D(-d, d_1, d_2, \ldots) \in C(I,J)$.

<u>Proof</u>. Let $A = U(y)$ and $B = U^*(x)$ in equation (4.6a). The rest follows by inspection. Q.E.D.

Theorem 4.7 immediately yields the first known non-trivial results on the inclusion relation $C_1 \subset [C_2, C_2]$, and we state them in the next two corollaries. (By 'non-trivial results' we mean results on ideals I for which $F(H) \underset{\neq}{\subset} I \subset C_2$.)

Corollary 4.8. If $d_n \downarrow 0$, $\Sigma d_n = d$ and $\Sigma n d_n < \infty$, then $D(-d, d_1, d_2, \ldots) \in C(C_2)$. In fact, $D(-d, d_1, d_2, \ldots) = -(A^*A - AA^*)$ where $A \in C_2$ and $\|A\|_{C_2}^2 = \Sigma n d_n$.

Remarks. Later in this section (Theorem 4.36), a better result is proved. Namely, if $d_n \downarrow 0$, $\Sigma d_n = d$, and $\Sigma(\log n) d_n < \infty$, then $D(-d, d_1, d_2, \ldots) \oplus 0 \in [C_2, C_2]$.

Also, in a later section we show that there exists such an operator A with $\|A\|_{C_2}^2 = \Sigma[(n+1)/2] d_n$.

Proof. Let $I = C_2$ and apply Theorem 4.7 and its proof. We claim $((\Sigma_{i=n}^{\infty} d_i)^{1/2}) \in \mathrm{Calk}(C_2) = \ell_+^2$. Indeed, $\|((\Sigma_{i=n}^{\infty} d_i)^{1/2})\|_{C_2}^2 = \Sigma_{i=1}^{\infty} d_i + \Sigma_{i=2}^{\infty} d_i + \cdots = \Sigma n d_n < \infty$, by hypothesis. Whence $D(-d, d_1, d_2, \ldots) = -(A^*A - AA^*)$ for some $A \in C_2$ for which $\|A\|_{C_2}^2 = \Sigma n d_n$. Q.E.D.

Remark 4.9. As in the proof of Lemma 4.4, if $T \in (I^2)^o$ then T is the sum of 8 or fewer operators each of which is unitarily equivalent to an operator of the form $D(-d, d_1, d_2, \ldots) \oplus 0$, where $d_n \downarrow 0$, $\Sigma d_n = d$, $(d_n) \in \mathrm{Calk}(I^2)$ and 0 is infinite-dimensional. Therefore if each operator of this form is in $[J, J]$ for some ideal J, then $T \in [J, J]$.

<u>Corollary 4.10</u>. $C^o_{(1/2)-\epsilon} \subset [C_2, C_2]_8$ for all $0<\epsilon<1/2$ (equivalently, $\bigcup_{p<1/2} C^o_p \subset [C_2, C_2]_8$).

<u>Proof</u>. By Remark 4.9, it suffices to show that if $d_n \downarrow 0$, $\Sigma d_n = d$, $(d_n) \in \ell^{(1/2)-\epsilon}$, and 0 is infinite-dimensional, then $D(-d, d_1, d_2, \ldots) \oplus 0 \in C(C_2)$. (A better result is obtained later. We shall be able to replace $\ell^{(1/2)-\epsilon}$ by ℓ^p for any $p < 1$.)

If $d_n \downarrow$ and $(d_n) \in \ell^{(1/2)-\epsilon}$, then $d_n^{(1/2)-\epsilon} \downarrow$ and $\Sigma d_n^{(1/2)-\epsilon} < \infty$. Therefore, since every decreasing summable sequence is $o(1/n)$, we obtain $d_n^{(1/2)-\epsilon} = o(1/n)$ and thus $d_n = o(n^{-2/(1-2\epsilon)})$. But then $nd_n = o(n^{-(1+2\epsilon)/(1-2\epsilon)}) \in \ell^1$. So $\Sigma nd_n < \infty$. By corollary 4.8, $D(-d, d_1, d_2, \ldots) = -(A^*A - AA^*)$ for some $A \in C_2$. Hence

$D(-d, d_1, d_2, \ldots) \oplus 0 = -[(A\oplus 0)^*(A\oplus 0) - (A\oplus 0)(A\oplus 0)^*]$

and $A\oplus 0 \in C_2$. Therefore $D(-d, d_1, d_2, \ldots) \oplus 0 \in C(C_2)$. Q.E.D.

<u>Corollary 4.11</u>. $C^o_{(p/4)-\epsilon} \subset [C_p, C_p]_8$ for all $0<p\leq 2$ and $0 < \epsilon < p/4$ (i.e. $\bigcup_{t<p/4} C^o_t = (\bigcup_{t<p/4} C_t)^o \subset [C_p, C_p]_8$).

<u>Proof</u>. Using the proof of Corollary 4.10 it suffices to show that if $d_n \downarrow 0$, $\Sigma d_n = d$ and $(d_n) \in \ell^{(p/4)-\epsilon}$, then $((\Sigma^\infty_{i=n} d_i)^{1/2}) \in \ell^p$.

It is well-known that if $r \leq 1$ and (x_n) is a non-negative sequence, then $(\Sigma x_n)^r \leq \Sigma x_n^r$. Therefore $\|((\Sigma^\infty_{i=n} d_i)^{1/2})\|^p_{\ell^p} = \Sigma_n (\Sigma^\infty_{i=n} d_i)^{p/2} \leq \Sigma_n \Sigma^\infty_{i=n} d_i^{p/2} = \Sigma n d_n^{p/2}$. However, if $d_n \downarrow$ and $(d_n) \in \ell^{(p/4)-\epsilon}$, then $d_n^{(p/4)-\epsilon} = o(1/n)$, and so $\Sigma n d_n^{p/2} = o(n^{-(p+4\epsilon)/(p-4\epsilon)}) \in \ell^1$, thus $\Sigma n d_n^{p/2} < \infty$. Hence $((\Sigma^\infty_{i=n} d_i)^{1/2}) \in \ell^p$. Q.E.D.

Two Themes. The preceding two corollaries illustrate two themes that motivate our strategies in this section.

Firstly, for ideals I for which $F(H) \subset I \subset C_2$, we try to obtain $(I^2)^o \subset [C_2,C_2]$. We will be taking larger and larger such ideals in the hope of either obtaining $I = C_2$ and $(I^2)^o = C_1^o \subset [C_2,C_2]$, or characterizing the largest ideal I for which $(I^2)^o \subset [C_2,C_2]$. As I gets larger, our techniques and constructions become much more difficult. It is not surprising then that we cannot presently obtain either goal. In later sections we shall obtain results that give weak indications that $C_1^o \not\subset [C_2,C_2]$. If this were the case, then the linear span of all ideals satisfying $(I^2)^o \subset [C_2,C_2]$ would be the unique largest ideal that satisfies this same inclusion relation. (The existence of such a largest ideal follows from the fact that the linear span of a collection of ideals is again an ideal (since if I and J are ideals, then I+J is also an ideal), and $(I+J)^o = I^o+J^o$.) Then our results would yield a candidate for this largest ideal.

Secondly, for each ideal I which we discover satisfies $F(H) \subset I \subset C_2$ and $(I^2)^o \subset [C_2,C_2]$, we will try to prove the stronger inclusion relation $(I^2)^o \subset [I,I]$, equivalently $(I^2)^o = [I,I]$.

Theorem 4.7 is important to us, but in itself, it is not strong enough for our purposes. The condition that '$\Sigma nd_n < \infty$' is quite restrictive on the sequences (d_n) that need to be considered. In a sense, the theorem works best on those ideals for which the sequences in their Calkin ideal sets decrease fastest to 0. With more sophisticated techniques we shall improve Theorem 4.7, Corollaries 4.8, 4.10 and 4.11 later in this section.

Theorem 4.7 may be used to give a trivial proof that $[F(H),F(H)] = (F(H))^o$ but this has little interest for us since Theorem 4.3 yields the stronger result $C(F(H)) = (F(H))^o$. However, there is one well-known class of ideals I for which Theorem 4.7 suffices in order to show $[I,I]=(I^2)^o$. These ideals are such that the sequences in their Calkin ideal sets decrease very rapidly to 0. We define these ideals in what follows.

Rapidly Decreasing Sequences and their Associated Ideals.

Let $a_n \uparrow \infty$, $a_{2n}/a_n \leq M$ for every n and some $M > 0$ (equivalently, (a_n) has polynomial growth; i.e. there exist $d,c,k>0$ so that $dn^k \leq a_n \leq cn^k$ for all n), and $\Sigma R^{-a_n} < \infty$ for some $R > 1$. Define

$$L((a_n)) = \{(x_n): x_n \downarrow 0 \text{ and } \Sigma x_n R^{a_n} < \infty \text{ for every } R > 0\}.$$

It is easy to verify via the axioms of characteristic sets (see p. 58) that $L((a_n))$ is a characteristic set. Let $I(L((a_n)))$ denote the ideal whose characteristic set is $L((a_n))$. Also if, for each (x_n) in the definition of $L((a_n))$, we set $R = 1$, we obtain $(x_n) \in \ell^1$. Hence $I(L((a_n))) \subset C_1$ (see pp.58-59 and [29],[23] and [9] for the relevant facts).

We assert that $I = I(L((a_n)))$ is idempotent, that is,

$I^2 = I$. To prove this it clearly suffices to show $I \subset I^2$, and this holds if for each $(x_n) \in L((a_n))$ we have $(x_n^{1/2}) \in L((a_n))$. Assume then that $(x_n) \in L((a_n))$. By hypothesis there exists $R_o > 1$ such that $\Sigma R_o^{-a_n} < \infty$. Let $R > 0$ be arbitrary and choose $r = (RR_o)^2$, whence $R_o^{-1} = R/r^{1/2}$. By hypothesis $\Sigma x_n r^{a_n} < \infty$ and therefore $x_n r^{a_n} \leq M$, equivalently $x_n \leq M/r^{a_n}$, for some $M > 0$ and every n. Hence $x_n^{1/2} R^{a_n} \leq (M^{1/2}/r^{a_n/2}) R^{a_n} = M^{1/2} R_o^{-a_n}$ and therefore $\Sigma x_n^{1/2} R^{a_n} \leq M^{1/2} \Sigma R_o^{-a_n} < \infty$. Thus we have proved our assertion.

Note that if we set $a_n = n$, then $L((a_n))$ consists of all the non-increasing, non-negative sequences that are the Fourier coefficients of the entire functions. Also, if we set $a_n = \log (n+1)$, then $L((a_n))$ consists of all the non-increasing, non-negative sequences in what is usually called "the sequence space of all rapidly decreasing sequences". An example of a sequence that does not satisfy our conditions because it increases too slowly is $a_n = \log \log (n+1)$. Here there does not exist R_o for which $\Sigma R_o^{-a_n} < \infty$, and so what we have previously developed does not hold for this sequence.

The next theorem solves our commutator problem for this newly defined class of ideals.

<u>Theorem 4.12</u>. Suppose $I = I(L((a_n))$ where (a_n) satisfies the conditions of the preceding paragraphs. If $(d_n) \in L((a_n))$ and $\Sigma d_n = d$, then $D(-d, d_1, d_2, \ldots) \in C(I)$. Furthermore we obtain the inclusion chain

$$C(I) \subset [I,I]_8 = [I,I] = (I^2)^o = I^o \subsetneq I = I^2 .$$

Proof. The preceding remarks together with inclusion chain 1.2 yield this inclusion chain except for the inclusion relation $(I^2)^o \subset [I,I]_8$. To prove this, in light of Remark 4.9 and the fact that $L((a_n))$ consists of all sequences of $Calk(I) = Calk(I^2)$ arranged in decreasing order, it suffices to show that if $(d_n) \in L((a_n))$ and $\Sigma d_n = d$, then $D(-d,d_1,d_2,\ldots) \in C(I)$ (for then $D(-d,d_1,d_2,\ldots) \oplus 0 \in C(I)$).

By Theorem 4.7, $D(-d,d_1,d_2,\ldots) = -(A^*A-AA^*)$ for some $A \in I$ provided $((\Sigma_{i=n} d_i)^{1/2}) \in Calk(I)$. But this sequence is decreasing , and also $I = I^2$. This implies that it is enough to show that $((\Sigma_{i=n}^{\infty} d_i)) \in L((a_n))$. Let $R > 0$ be arbitrary and recall that $a_n\uparrow$. Then $\Sigma(\Sigma_{i=n}^{\infty} d_i)R^{a_n} = \Sigma_k d_k(\Sigma_{n=1}^{k} R^{a_n}) \leq \Sigma_k kR^{a_k} d_k$. Choose $R_o > 1$ such that $\Sigma R_o^{-a_n} < \infty$. Since $a_n\uparrow$, $(R_o^{-a_n})$ is decreasing. Therefore $R_o^{-a_n} = o(1/n) = O(1/n)$, that is, $R_o^{-a_n} \leq M/n$ for all n and some $M > 0$, and so $n \leq MR_o^{a_n}$. Finally, $\Sigma kR^{a_k}d_k \leq M\,\Sigma R_o^{a_k} R^{a_k} d_k = M\,\Sigma\,(R_oR)^{a_k} d_k < \infty$ since $(d_k) \in L((a_n))$. So $((\Sigma_{i=n}^{\infty} d_i)) \in L((a_n))$. Q.E.D.

The ideals $I(L((a_n)))$ fit into our theory more easily than any other ideals. Therefore, in the search for an ideal I for which $F(H) \subsetneq I \subsetneq L(H)$ such that $C(I) = (I^2)^o$, it appears that $I(L((a_n)))$ is the candidate that provides us with the most hope.

We now begin the development of our main techniques. They consist of one construction and several calculations. They depend heavily on Theorem 4.7 and in particular on the commutator equation (4.6). We will use these techniques to show that $(I^2)^o \subset [C_2,C_2]$ for certain ideals $I \subset C_2$ and also to show that $[I,I] = (I^2)^o$ for most of them.

THE MAIN CONSTRUCTION

Let $d_n \downarrow 0$, $\Sigma d_n = d$, $(d_n) \in \mathrm{Calk}(I^2)$ for some ideal $I \subset C_2$, and let 0 be infinite-dimensional. We shall consider $D(-d,d_1,d_2,\ldots) \oplus 0$ and try to express it as a finite sum of commutators of compact operators of special types. We shall define these operators explicitly in such a way as to be able to identify some of the ideals that contain them and, in particular, to decide if they are in I. That 0 is infinite-dimensional is crucial to our technique. In a sense, it provides us with 'room to work'. Note that Theorem 4.7 implies that

$$D(-d,d_1,d_2,\ldots) \oplus 0 = (A\oplus 0)^*(A\oplus 0) - (A\oplus 0)(A\oplus 0)^* \quad \text{for which}$$

$$|A\oplus 0| \cong D((\Sigma_{i=1} d_i)^{1/2}, (\Sigma_{i=2} d_i)^{1/2},\ldots) \oplus 0 .$$

But compared to $(d_n^{1/2}) \in \mathrm{Calk}(I)$, the sequence $((\Sigma^\infty_{i=n} d_i)^{1/2})$ may be so large as to fail to be contained in Calk(I). Indeed, $(\Sigma_{i=n} d_i)^{1/2}$ may (and usually does) tend to 0 much slower than $d_n^{1/2}$. This has the effect of putting A in far fewer ideals than $D((d_n^{1/2}))$.

Let us first describe one of the main ideas in our construction. One of the keys to the main construction of Section 3 was 'adding and subtracting positive real numbers'. We used the fact that $(1,0,0,\ldots) = (1,-1/2,-1/2,1/4,1/4,1/4,1/4,-1/8,\ldots) + (0,1/2,1/2,-1/4,-1/4,-1/4,-1/4,\ldots)$. Then we rearranged each of the sequences on the right-hand side to obtain matrices that were realizable as the right kinds of commutators. The same strategy serves us here in regard to $D(-d,d_1,d_2,\ldots) \oplus 0$ which, interestingly enough, strongly resembles $-P$ (recall $P = D(1,0,0,\ldots)$). However, this path presents quite a few pitfalls. For example, $P = D(1,0,0,\ldots)$ is contained in every ideal but $D(1,-1/2,-1/2,1/4,1/4,1/4,1/4,\ldots)$ is not. In fact $D(1,-1/2,-1/2,1/4,1/4,1/4,1/4,\ldots)$ is contained in exactly those ideals that contain the operator $D((1/n))$, which is not in the trace class. In our present situation, one difficulty is that we must stay within the trace class, in fact, we must stay within I^2 if we want $(I^2)^o \subset [I,I]$. For this reason, we cannot add and subtract too many 'large' numbers. On the other hand, if we add and subtract too few 'large' numbers, then we are not able to realize our matrices as the right kinds of commutators. Another difficulty is that, whereas the sequence $(1,0,\ldots)$ has no variables, $(-d,d_1,d_2,\ldots)$ has many variables and the choice of what we add and subtract must depend on (d_n). Furthermore, adding and subtracting is counter-intuitive in that it increases the trace norms of the operators involved, which is not what one would ordinarily want to do.

In joint work with John Conway, we have shown that the real numbers that we shall choose to add and subtract in the following construction are the best possible choices, relative to that construction. This is especially striking in light of our joint work with Edward Azoff, mentioned earlier, in which we show that our choice of what we added and subtracted in the construction of Section 3 was the best possible choice, relative to that construction.

We begin the forthcoming construction with some notation.

Notation. Let $(i_1, i_2, \ldots, i_m)$ denote an m-tuple of positive integers. Let $\mathcal{S} = \{(i_1, \ldots, i_m): i_k = 1, 2, \ldots$ for $1 \leq k \leq m$, and $m = 1, 2, \ldots\}$ be the set of all finite-tuples of positive integers. Then $\mathcal{S}$ is countable. Therefore $\mathcal{S}$ can be used to reindex (d_n) in many ways. Let $\pi : Z^+ \longrightarrow \mathcal{S}$ be any one-to-one correspondence between the set of positive integers Z^+ and $\mathcal{S}$. Define a new symbol $d(i_1, \ldots i_m)$ in terms of (d_n) and π by the equation $d(i_1, \ldots i_m) = d_{\pi^{-1}(i_1, \ldots, i_m)}$. In other words, for each $(i_1, \ldots i_m) \in \mathcal{S}$, $d(i_1, \ldots i_m)$ is the non-negative real number d_n for $n = \pi^{-1}(i_1, \ldots i_m)$ and $\{d(i_1, \ldots i_m): (i_1, \ldots i_m) \in \mathcal{S}\} = \{d_n\}$ counting multiplicities.

It may at first seem to the reader that nothing is gained by this reindexing procedure. Indeed the well-ordering of Z^+ reflects the monotonicity of (d_n) and this is lost by reindexing using $\mathcal{S}$. However more is going on than meets the eye. This notation and the next are essential to our construction and calculations. In fact, our joint work with John Conway shows that in some sense our notation is central to this construction.

Let ( ) denote the empty-tuple and let $\mathcal{S}_o = \mathcal{S} \cup \{( )\}$. For $(i_1,\ldots i_m) \in \mathcal{S}_o$, let $(k;i_1,\ldots i_m)$ denote an (m+1)-tuple of positive integers for which the first position is 'distinguished' and we allow m = 0 in the sense that $(k;\cdot)$ is a 1-tuple with k in the distinguished position and no positive integer entries in the 'non-distinguished' positions. The role of the distinguished position will be quite different from the role of the non-distinguished positions. Define a new symbol $d(k;i_1,\ldots i_m)$ to be the non-negative real number defined by the equation

$$d(k;i_1,\ldots,i_m) = \Sigma^{\infty}_{i_{m+1},\ldots,i_{m+k}=1} d(i_1,\ldots,i_m,i_{m+1},\ldots,i_{m+k}).$$

This series converges since $\Sigma d_n = d$. By inspection we also have

$$d(k;i_1,\ldots,i_m) = \Sigma^{\infty}_{i_{m+1}=1} d(k-1;i_1,\ldots,i_m,i_{m+1})$$

and in particular $d(k;\cdot) = \Sigma_{i_1} d(k-1;i_1)$.

We must be careful not to confuse the symbol $d(i_1,\ldots,i_m)$ with the symbol $d(k;i_1,\ldots i_m)$. The first has no distinguished positions but the second does. The first is used to define the second in that $d(k;i_1,\ldots,i_m)$ is a specially chosen partial sum of $\{d(i_1,\ldots,i_m): (i_1,\ldots,i_m) \in \mathcal{S}\}$.

We summarize the above relations.

<u>Notation Summary 4.13</u>.

1. The sequence $(d_n) \in \ell^1$ satisfies $d_n \downarrow 0$ and $\Sigma d_n = d$.
2. If $\pi\colon Z^+ \longrightarrow \mathcal{S}$ is one-to-one and onto, then

   $d(i_1,\ldots,i_m) = d_{\pi^{-1}(i_1,\ldots,i_m)}$ and

   $\{d(i_1,\ldots,i_m)\} = \{d_n\}$, counting multiplicities.
3. For each positive integer k,

   $d(k;i_1,\ldots,i_m) = \Sigma_{i_{m+1},\ldots,i_{m+k}} d(i_1,\ldots,i_m,i_{m+1},\ldots,i_{m+k})$.

   In particular $d(k;\cdot) = \Sigma_{i_1,\ldots,i_k} d(i_1,\ldots,i_k)$.
4. For each positive integer $k > 1$,

   $d(k;i_1,\ldots,i_m) = \Sigma_{i_{m+1}} d(k-1;i_1,\ldots,i_m,i_{m+1})$.

   In particular $d(k;\cdot) = \Sigma_{i_1} d(k-1;i_1)$.

<u>The Construction</u>. The set $\{d(k;i_1,\ldots,i_m)\colon k \in Z^+$ and $(i_1,\ldots,i_m) \in \mathcal{S}_o\}$, counting multiplicities, is countable. Let σ denote the sequence whose entries are precisely the real numbers contained in the set

$$\{d(k;i_1,\ldots,i_m)\} \cup \{-d(k;i_1,\ldots,i_m)\},$$ counting multiplicities and arranged in any sequential order. Then

$$D(-d,d_1,d_2,\ldots) \oplus 0 = \tag{4.14}$$
$$[D(-d,d_1,d_2,\ldots) \oplus D(\sigma)] - [0 \oplus D(\sigma)] .$$

Clearly,

$$(4.15) \qquad D(\sigma)$$

$$\cong \sum_{k\in Z^+,(i_1,\ldots,i_m)\in\mathcal{S}_o} \oplus \; D(-d(k;i_1,\ldots,i_m),d(k;i_1,\ldots,i_m))$$

$$= \sum_{k\in Z^+,(i_1,\ldots,i_m)\in\mathcal{S}_o} \oplus \; d(k;i_1,\ldots,i_m)D(-1,1) \quad .$$

The operator $D(-d,d_1,d_2,\ldots) \oplus D(\sigma)$ will undergo an intricate unitary transformation. We shall rearrange its diagonal into 3 types of blocks each of which is of the form $D(-x,x_1,x_2,\ldots)$ where $x_n \downarrow 0$ and $\Sigma x_n = x$. The first block is $D(-d,d(1;\cdot),d(2;\cdot),\ldots)$. The second type of block arises from an m-tuple $(i_1,\ldots,i_m) \in \mathcal{S}_o$ as an operator of the form

$D(-d(1;i_1,\ldots,i_m),\ d(i_1,\ldots,i_m,1),\ d(i_1,\ldots,i_m,2),\ \ldots)$.

Note that the first block contains the entry -d and the blocks of the second type contain the entries $d(i_1,\ldots,i_{m+1})$, where $m \geq 0$, which are precisely the entries d_n. The third type of block is of the form

$D(-d(k;i_1,\ldots,i_m),d(k-1;i_1,\ldots,i_m,1),d(k-1;i_1,\ldots,i_m,2),\ldots)$

where $1 < k \in Z^+$ and $(i_1,\ldots,i_m) \in \mathcal{S}_o$. Hence, by careful inspection, we see that each of the symbols $-d,d_1,d_2,\ldots$ and $\pm d(k;i_1,\ldots,i_m)$ appear exactly once in one of the three types of blocks. Also, by Notation Summary 4.13, each type of diagonal block is of the form $D(-x,x_1,x_2,\ldots)$ where $x_n \downarrow 0$ and $\Sigma x_n = x$. Therefore

$$(4.16)\quad D(-d,d_1,d_2,\ldots) \oplus D(\sigma) \cong$$

$$D(-d,d(1;\cdot),d(2;\cdot),d(3;\cdot),\ldots)$$

$$\oplus \quad \Sigma \oplus_{(i_1,\ldots,i_m)\in \mathcal{S}_o} D(-d(1;i_1,\ldots,i_m),d(i_1,\ldots,i_m,1), d(i_1,\ldots,i_m,2),\ldots)$$

$$\oplus \quad \Sigma \oplus_{1<k\in Z^+,(i_1,\ldots,i_m)\in \mathcal{S}_o} D(-d(k;i_1,\ldots,i_m), d(k-1;i_1,\ldots,i_m,1), d(k-1;i_1,\ldots,i_m,2),\ldots)$$

where if H_1 denotes the countable direct sum of copies of H then the left-hand operator acts on $H \oplus H$ but the right-hand operator acts on $H \oplus H_1 \oplus H_1$. The direct sums in (4.16) may be taken in any order.

We proceed to express the right-hand operators in (4.15) and (4.16) as commutators, keeping careful track of the character of the approximation numbers of our solution operators. (The approximation numbers of a compact operator A are the eigenvalues, counting multiplicities, of $|A|$.) We express each diagonal block as a commutator using equation (4.6) and Theorem 4.7 (equation (4.6a) and Theorem 4.7a for the [I,J] case).

<u>To express $D(\sigma)$ as a commutator (hence also $0 \oplus D(\sigma)$)</u> we use (4.15). Recall that $U((w_n))$ is the weighted shift operator of multiplicity 1 with weight sequence (w_n). In particular, $U(1,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Therefore if we let

(4.17) $A = \Sigma \oplus_{k\in Z^+, (i_1,\ldots,i_m)\in \mathcal{S}_o} d(k;i_1,\ldots,i_m)^{1/2} U(1,0)$

then

(4.18) $|A| \cong D((d(k;i_1,\ldots,i_m)^{1/2})_{k\in Z^+, (i_1,\ldots,i_m)\in \mathcal{S}_o}) \oplus 0$

and

(4.19) $D(\sigma) = \Sigma \oplus_{k\in Z^+, (i_1,\ldots,i_m)\in \mathcal{S}_o} d(k;i_1,\ldots,i_m) D(-1,1)$

$= -(A^*A - AA^*)$.

Furthermore if $x = (x(k;i_1,\ldots,i_m))$ and $y = (y(k;i_1,\ldots,i_m))$ such that $d(k;i_1,\ldots,i_m) = x(k;i_1,\ldots,i_m)\cdot y(k;i_1,\ldots,i_m)$, then letting

(4.17a) $A = \Sigma \oplus_{k\in Z^+, (i_1,\ldots,i_m)\in \mathcal{S}_o} y(k;i_1,\ldots,i_m) U(1,0)$ and

$B = \Sigma \oplus_{k\in Z^+, (i_1,\ldots,i_m)\in \mathcal{S}_o} x(k;i_1,\ldots,i_m) U^*(1,0)$

we obtain

(4.18a) $|A| \cong D(y) \oplus 0$ and $|B| \cong D(x) \oplus 0$ and

(4.19a) $D(\sigma) = \Sigma \oplus_{k\in Z^+, (i_1,\ldots,i_m)\in \mathcal{S}_o} d(k;i_1,\ldots,i_m) D(-1,1)$

$= AB - BA$.

To express $D(-d,d_1,d_2,\ldots) \oplus D(\sigma)$ as a commutator

we use (4.16).

For $(i_1,\ldots,i_m) \in \mathcal{S}_o$ and $1 < k \in Z^+$, let

$$(4.20)\quad A_1 = U(\ ((\Sigma_{i=n}^{\infty} d(i;\cdot))^{1/2})_{n=1}^{\infty}\ )\ ,$$

$$A_2(1;i_1,\ldots,i_m) = U(\ ((\Sigma_{i=n}^{\infty} d(i_1,\ldots,i_m,i))^{1/2}\ )_{n=1}^{\infty}\ ),$$

$$A_3(k;i_1,\ldots,i_m) = U(\ ((\Sigma_{i=n}^{\infty} d(k-1;i_1,\ldots,i_m,i))^{1/2})\ ),$$

and

$$A_2 = \Sigma \oplus_{(i_1,\ldots,i_m)\in\mathcal{S}_o} A_2(1;i_1,\ldots,i_m)\ ,$$

$$A_3 = \Sigma \oplus_{1<k\in Z^+,(i_1,\ldots,i_m)\in\mathcal{S}_o} A_3(k;i_1,\ldots,i_m)\ .$$

Then

$$(4.21)\qquad |A_1| = D(((\Sigma_{i=n}^{\infty} d(i;\cdot))^{1/2}))\ ,$$

$$|A_2(1;i_1,\ldots,i_m)| = D(((\Sigma_{i=n}^{\infty} d(i_1,\ldots,i_m,i))^{1/2}))\ ,$$

$$|A_3(k;i_1,\ldots,i_m)| = D(((\Sigma_{i=n}^{\infty} d(k-1;i_1,\ldots,i_m,i))^{1/2}))\ ,$$

and

$$(4.22)\quad D(-d,d(1;\cdot),d(2;\cdot),\ldots) = -(A_1^*A_1 - A_1A_1^*)\ ,$$

$$\Sigma \oplus_{(i_1,\ldots,i_m)\in\mathcal{S}_o} D(-d(1;i_1,\ldots,i_m),d(i_1,\ldots,i_m,1),d(i_1,\ldots,i_m,2),\ldots)$$

$$= -(A_2^*A_2 - A_2A_2^*)\ ,\ \text{and}$$

$$\Sigma \oplus_{1<k\in Z^+,(i_1,\ldots,i_m)\in\mathcal{S}_o} D(-d(k;(i_1,\ldots,i_m),d(k-1;i_1,\ldots,i_m,1),d(k-1;i_1,\ldots,i_m,2),\ldots)$$

$$= -(A_3^*A_3 - A_3A_3^*)\ .$$

Also

(4.23) $|A_2| = \Sigma \oplus_{(i_1,\ldots,i_m)\in\mathcal{S}_o} D(((\Sigma_{i=n}^{\infty} d(i_1,\ldots,i_m,i))^{1/2}))$

and

$|A_3| = \Sigma \oplus_{1<k\in Z^+,(i_1,\ldots,i_m)\in\mathcal{S}_o} D(((\Sigma_{i=n}^{\infty} d(k-1;i_1,\ldots,i_m,i))^{1/2}))$.

Hence if $A = A_1 \oplus A_2 \oplus A_3$ then $|A| = |A_1| \oplus |A_2| \oplus |A_3|$ and $D(-d,d_1,d_2,\ldots) \oplus D(\sigma) \cong -(A^*A - AA^*)$. Furthermore, the eigenvalues of $|A|$ are given by equations (4.21) and (4.23).

Similarly, if $x(n)y(n) = \Sigma_{i=n}^{\infty} d(i;\cdot)$,
$x(n,(k;i_1,\ldots,i_m))y(n(k;i_1,\ldots,i_m))$
$= \Sigma_{i=n}^{\infty} d(k-1;i_1,\ldots,i_m,i)$ for $1 < k \in Z^+$, and
$x(n,(1;i_1,\ldots,i_m))y(n,(1;i_1,\ldots,i_m)) = \Sigma_{i=n}^{\infty} d(i_1,\ldots,i_m,i)$,
then letting x and y be the corresponding sequences (arranged in any order) we obtain
$D(-d,d_1,d_2,\ldots) \oplus D(\sigma) \cong AB - BA$ where $|A| \cong D(y)$ and $|B| \cong D(x)$.

<u>Summary 4.24</u>. Let s denote the sequence (arranged in any order) whose entries are precisely

$$(\Sigma_{i=n}^{\infty} d(i;\cdot))^{1/2} \text{ for each } n \in Z^+,$$

$(\Sigma_{i=n}^{\infty} d(i_1,\ldots,i_m,i))^{1/2}$ for each $n \in Z^+$ and $(i_1,\ldots,i_m) \in \mathcal{S}_o$,

and

$$(\Sigma_{i=n}^{\infty} d(k-1;i_1,\ldots,i_m,i))^{1/2}$$

for each $n \in Z^+$, $1 < k \in Z^+$ and $(i_1,\ldots,i_m) \in \mathcal{S}_o$.
Then there exists an operator A such that

1. $D(-d,d_1,d_2,\ldots) \oplus D(\sigma) \cong -(A^*A - AA^*)$ and
2. $|A| \cong D(s)$.

Let s_1 denote the sequence (arranged in any order) whose entries are precisely $d(k;i_1,\dots,i_m)^{1/2}$ for each $k \in Z^+$ and $(i_1,\dots,i_m) \in \mathcal{S}_o$. Then by equations (4.17)-(4.19) there exists an operator A_1 such that

3. $0 \oplus D(\sigma) = -[(0\oplus A_1)^*(0\oplus A_1) - (0\oplus A_1)(0\oplus A_1)^*]$ and

4. $|0\oplus A_1| \cong D(s_1) \oplus 0$.

Thus if we choose $X \cong A$ via the unitary transformation in 1, and $Y \cong 0\oplus A_1$ via the unitary transformation in 3 , then

5. $D(-d,d_1,d_2,\dots) \oplus 0 = -(X^*X-XX^*) + (Y^*Y-YY^*)$ and

6. $|X| \cong D(s)$ and $|Y| \cong D(s_1) \oplus 0$.

Similarly, if $x = (x_n)$ and $y = (y_n)$ are such that $(x_n y_n) = s$, and if $x_1 = (x_n')$ and $y_1 = (y_n')$ are such that $(x_n' y_n') = s_1$, then there exist operators X, Y, X_1, and Y_1 such that

7. $D(-d,d_1,d_2,\dots) \oplus 0 \cong -(XY-YX) + (X_1Y_1-Y_1X_1)$ and

8. $|X| \cong D(x)$, $|Y| \cong D(y)\oplus 0$, $|X_1| \cong D(x_1)$, and $|Y_1| \cong D(y_1)\oplus 0$.

The sequence s above is extremely important to our theory. Using Notation Summary 4.13 we can express s in terms of the symbols '$d(i_1,\dots,i_m)$' as follows.

<u>s</u>. The sequence s is any fixed sequential ordering of the following non-negative real numbers.

1. $(\Sigma_{i=n}^{\infty} \Sigma_{i_1,\dots,i_i=1}^{\infty} d(i_1,\dots,i_i))^{1/2}$ for each $n \in Z^+$;

2. $(\Sigma_{i=n}^{\infty} d(i_1,\dots,i_m,i))^{1/2}$, for each $n \in Z^+$ and $(i_1,\dots,i_m) \in \mathcal{S}_o$; and

3. $(\Sigma_{i=n}^{\infty} \Sigma_{j_1,\ldots,j_{k-1}=1}^{\infty} d(i_1,\ldots,i_m,i,j_1,\ldots,j_{k-1}))^{1/2}$

for each $n \in Z^+$, $(i_1,\ldots,i_m) \in \mathcal{S}_o$, and $1 < k \in Z^+$.

Our main construction is now complete.

The next theorem is the first of two main theorems and follows from the main construction. Let us first make the following definition.

<u>Definition</u>. Let I be an ideal in L(H), and let $d_n \downarrow 0$ and $\Sigma d_n = d < \infty$. Then we shall say that (d_n) is <u>s-closed with respect to Calk(I)</u> provided that there exists a map $\pi: Z^+ \longrightarrow \mathcal{S}$ that is one-to-one and onto such that if we define $d(i_1,\ldots,i_m) = d_{\pi^{-1}(i_1,\ldots,i_m)}$ for each $(i_1,\ldots,i_m) \in \mathcal{S}$, then the corresponding sequence $s \in$ Calk(I).

<u>Note</u>. Let $d_n \downarrow 0$, $\Sigma d_n = d < \infty$, and suppose also that (d_n) is s-closed with respect to Calk(I). That is, for some such map π, the corresponding sequence $s \in$ Calk(I). We claim that $(d_n) \in$ Calk(I^2). Indeed, s contains all the entries $(\Sigma_{i=n}^{\infty} d(i_1,\ldots,i_m,i))^{1/2}$ for each $n \in Z^+$ and $(i_1,\ldots,i_m) \in \mathcal{S}_o$; we also have that $d(i_1,\ldots,i_m,n)^{1/2} \leq (\Sigma_{i=n}^{\infty} d(i_1,\ldots,i_m,i))^{1/2}$; and $\{d_n\} = \{d(i_1,\ldots,i_m,i) : (i_1,\ldots,i_m,i) \in \mathcal{S}\}$, counting multiplicities. Hence we have that $(d_n^{1/2}) \in$ Calk(I), and so $(d_n) \in$ Calk(I^2).

<u>Theorem 4.25</u>. (Main Theorem I).

Let I be an ideal in L(H).

If $d_n \downarrow 0$, $\Sigma d_n = d < \infty$, and (d_n) is s-closed with respect to Calk(I), then $D(-d,d_1,d_2,\ldots) \oplus 0 \in [I,I]_2$, provided 0 is infinite-dimensional.

Furthermore, if $I \subset C_2$ and if (d_n) is s-closed with respect to Calk(I) for every $(d_n) \in \text{Calk}(I^2)$ for which $d_n \downarrow 0$, then $[I,I] = (I^2)^o$.

<u>Proof</u>. Consider Summary 4.24-5 and 4.24-6.

That is, $D(-d,d_1,d_2,\ldots) \oplus 0 = -(X^*X-XX^*) + (Y^*Y-YY^*)$

where $|X| \cong D(s)$ and $|Y| \cong D(s_1) \oplus 0$. Our hypothesis that $s \in \text{Calk}(I)$ implies that $X \in I$. We further claim that $Y \in I$, indeed, that $s_1 \in \text{Calk}(I)$. In fact, we assert that $s \in \text{Calk}(I)$ implies that $s_1 \in \text{Calk}(I)$. This follows from the facts that the entries of s_1 are precisely the real numbers $d(k;i_1,\ldots,i_m)^{1/2}$ where $k \in Z^+$ and $(i_1,\ldots,i_m) \in \mathcal{S}_o$; and that some of the entries of s are $(\Sigma_{i=1}^{\infty} d(k-1;i_1,\ldots,i_m,i))^{1/2}$ for $1 < k \in Z^+$ and $(i_1,\ldots,i_m) \in \mathcal{S}_o$, and $(\Sigma_{i=n}^{\infty} d(i;\cdot))^{1/2}$ for each $n \in Z^+$. For then, if $m \leq 1$, $d(k;i_1,\ldots,i_m)^{1/2} \leq (\Sigma_{i=1}^{\infty} d(k;i_1,\ldots,i_m,i))^{1/2}$, and if $m = 0$, $d(k;\cdot)^{1/2} \leq (\Sigma_{i=k}^{\infty} d(i;\cdot))^{1/2}$. Hence $s_1 \in \text{Calk}(I)$. Therefore X, X^*, Y and $Y^* \in I$, and so $D(-d,d_1,d_2,\ldots) \oplus 0 \in [I,I]_2$. This proves the first paragraph of the theorem.

From this and Lemma 4.4, it follows that if the hypothesis of the theorem holds for every such sequence $(d_n) \in \mathrm{Calk}(I^2)$, then $[I,I] = (I^2)^o$. Q.E.D.

As usual, we state without proof the analog for the $[I,J]$ case.

<u>Theorem 4.25a</u>. Let I and J be two ideals in L(H).

If $d_n \downarrow 0$, $\Sigma d_n = d < \infty$, and (d_n) is s-closed with respect to $\mathrm{Calk}((IJ)^{1/2})$, then $D(-d,d_1,d_2,\ldots) \oplus 0 \in [I,J]_2$, provided 0 is infinite-dimensional.

Furthermore, if $IJ \subset C_1$ and if (d_n) is s-closed with respect to $\mathrm{Calk}((IJ)^{1/2})$ for every $(d_n) \in \mathrm{Calk}(IJ)$ for which $d_n \downarrow 0$, then $[I,J] = (IJ)^o$.

We shall now begin to study the structure of $[I,I]$ for certain normed ideals (e.g. C_p where $1 \leq p \leq 2$) and certain other ideals that depend, for their definition, on particular quantitative finiteness conditions (e.g. C_p where $0 < p < 1$). Theorem 4.25 gives qualitative information. However, to decide whether or not its hypothesis is satisfied, that is, whether or not $s \in \mathrm{Calk}(I)$, is quite hard and in every case we know, the proof involves difficult calculations.

On first sight, the condition that $s \in \mathrm{Calk}(I)$ appears to lead nowhere. It seems unrelated to any of the standard ideals and appears to yield very little more than does the condition $((\Sigma_{i=n}^{\infty} d_i)^{1/2}) \in \mathrm{Calk}(I)$ of Theorem 4.7. Oddly enough, that $s \in \mathrm{Calk}(I)$ will sometimes follow from considering expressions like $(\Sigma_{i=n}^{\infty} d_i)^{1/2}$. in case $I = C_2$, Corollary 4.8 requires that $\Sigma\, nd_n < \infty$. What sort of quantitative condition Theorem 4.25 imposes on (d_n) is hard to see. In this vein, the power of our method will become evident with our second main theorem. But first we require some preliminaries which develop the computational techniques we need.

Computational Methods. There are three categories of questions which we need to consider. They are as follows.

I a) How many elements $(i_1,\dots,i_m) \in \mathcal{S}$ satisfy $i_1 + \dots + i_m = n$?

b) If $\mathcal{S}^+$ is the set of all finite-tuples $(i_1,\dots,i_m) \in \mathcal{S}$ for which $i_k \geq 2$ for $1 \leq k \leq m$, how many elements $(i_1,\dots,i_m) \in \mathcal{S}^+$ satisfy $i_1 + \dots + i_m = n$?

c) Describe the sequence obtained when the set $\{i_1 + \dots + i_m : (i_1,\dots,i_m) \in \mathcal{S}$, counting multiplicities$\}$ is arranged in non-decreasing order.

d) Describe the sequence obtained when the set $\{i_1 + \dots + i_m + m : (i_1,\dots,i_m) \in \mathcal{S}$, counting multiplicities$\}$ is arranged in non-decreasing order.

e) Compare the sequences in c and d.

II If $w_n \uparrow \infty$, $d_n \downarrow 0$ and $S((w_n),(d_n)) = \{\Sigma\, w_{\sigma(n)} d_n : \sigma$ is any permutation of $Z^+\}$, then clearly $S((w_n),(d_n)) \subset (0,\infty]$.

Does $S((w_n),(d_n))$ have a minimum, and if so, what is it?

III What is the nature of $\|s\|_{\ell^p}$ $(0 < p \leq 2)$?

We shall solve all these problems.

Problem Ia. We solve problem Ia in the following proposition.

Proposition 4.26. If $f(n)$, for $n \in Z^+$, denotes the number of finite-tuples $(i_1,\ldots,i_m) \in \mathcal{S}$ which satisfy $i_1 + \cdots + i_m = n$, then $f(n) = 2^{n-1}$.

Proof. For each $n \in Z^+$, let $\mathcal{S}_n$ denote the set of elements $(i_1,\ldots,i_m) \in \mathcal{S}$ for which $i_1 + \cdots + i_m = n$. Let $\mathcal{S}_n^{(1)}$ denote the elements $(i_1,\ldots,i_m) \in \mathcal{S}_n$ for which $i_1 = 1$ and let $\mathcal{S}_n^{(1+)}$ denote the elements $(i_1,\ldots,i_m) \in \mathcal{S}_n$ for which $i_1 > 1$. Clearly $\mathcal{S}_n^{(1)} \cap \mathcal{S}_n^{(1+)} = \emptyset$ and $\mathcal{S}_n = \mathcal{S}_n^{(1)} \cup \mathcal{S}_n^{(1+)}$. Therefore $f(n) = |\mathcal{S}_n| = |\mathcal{S}_n^{(1)}| + |\mathcal{S}_n^{(1+)}|$. The map $(1,i_2,\ldots,i_m) \longrightarrow (i_2,\ldots,i_m)$ clearly maps $\mathcal{S}_n^{(1)} \longrightarrow \mathcal{S}_{n-1}$ one-to-one and onto if $n \geq 2$, and so $|\mathcal{S}_n^{(1)}| = |\mathcal{S}_{n-1}| = f(n-1)$. The map $(i_1,i_2,\ldots,i_m) \longrightarrow (i_1-1,i_2,\ldots,i_m)$ clearly maps $\mathcal{S}_n^{(1+)} \longrightarrow \mathcal{S}_{n-1}$ one-to one and onto if $n \geq 2$, and so $|\mathcal{S}_n^{(1+)}| = |\mathcal{S}_{n-1}| = f(n-1)$. Therefore $f(n) = |\mathcal{S}_n^{(1)}| + |\mathcal{S}_n^{(1+)}| = |\mathcal{S}_{n-1}| + |\mathcal{S}_{n-1}| = 2f(n-1)$. Also, by inspection, $f(1) = 1$. But $f(n) = 2f(n-1)$ for $n \geq 2$ and $f(1) = 1$ implies, by induction, that $f(n) = 2^{n-1}$. Q.E.D.

Note. This is a well-known partition problem for which there are several alternate proofs.

Problem Ib. Let $g(n)$ denote the number of solutions $(i_1,\ldots,i_m) \in \mathcal{S}^+$ for which $i_1 + \cdots + i_m = n$.

By inspection $g(2) = g(3) = 1$. We assert that $g(n)$ satisfies the difference equation

$$g(n) = g(n-1) + g(n-2)\ , \text{ for } n \geq 4.$$

To prove this, let $\mathcal{S}_n^+$ denote the set of $(i_1,\ldots,i_m) \in \mathcal{S}^+$ for which $i_1 + \cdots + i_m = n$. Let $\mathcal{S}_n^{(2)}$ denote the set of $(i_1,\ldots,i_m) \in \mathcal{S}_n^+$ for which $i_1 = 2$ and let $\mathcal{S}_n^{(2+)}$ denote the set of $(i_1,\ldots,i_m) \in \mathcal{S}_n^+$ for which $i_1 > 2$. Clearly $\mathcal{S}_n^{(2)} \cap \mathcal{S}_n^{(2+)} = \emptyset$ and $\mathcal{S}_n^+ = \mathcal{S}_n^{(2)} \cup \mathcal{S}_n^{(2+)}$. Therefore $g(n) = |\mathcal{S}_n^+| = |\mathcal{S}_n^{(2)}| + |\mathcal{S}_n^{(2+)}|$. The map $(2,i_2,\ldots,i_m) \longrightarrow (i_2,\ldots,i_m)$ clearly maps $\mathcal{S}_n^{(2)} \longrightarrow \mathcal{S}_{n-2}^+$ one-to-one and onto if $n \geq 4$, and so $|\mathcal{S}_n^{(2)}| = |\mathcal{S}_{n-2}^+| = g(n-2)$. The map $(i_1,i_2,\ldots,i_m) \longrightarrow (i_1 - 1,i_2,\ldots,i_m)$ maps $\mathcal{S}_n^{(2+)} \longrightarrow \mathcal{S}_{n-1}^+$ one-to-one and onto if $n \geq 3$, and so $|\mathcal{S}_n^{(2+)}| = |\mathcal{S}_{n-1}^+| = g(n-1)$. Therefore if $n \geq 4$, $g(n) = |\mathcal{S}_{n-2}^+| + |\mathcal{S}_{n-1}^+| = g(n-2) + g(n-1)$. Hence our difference equation is proved.

If we let $G(n) = g(n+1)$, then $G(1) = G(2) = 1$, and for $n \geq 3$, $G(n) = G(n-2) + G(n-1)$. In this setting, however, it is well-known that $G(n)$ is precisely the <u>Fibonacci numbers</u>. The first few Fibonacci numbers are 1,1,2,3,5,8,13,21,34. It is also well-known that $G(n) = 5^{-1/2}[((1+5^{1/2})/2)^n - ((1-5^{1/2})/2)^n]$. Hence $G(n)$ is asymptotic to $5^{-1/2}((1+5^{1/2})/2)^n$.

Therefore $g(n) = 5^{-1/2}[((1+5^{1/2})/2)^{n-1} - ((1-5^{1/2})/2)^{n-1}]$ for $n \geq 2$ and $g(n)$ is asymptotic to $2(5+5^{1/2})^{-1}((1+5^{1/2})/2)^n$. Note that one can prove that $g(n) \leq 2^{n-1}$ without the above equations. It is obvious that $\mathcal{S}_n^+ \subset \mathcal{S}_n$ and therefore $g(n) = |\mathcal{S}_n^+| \leq |\mathcal{S}_n| = f(n) = 2^{n-1}$.

Problem Ib is solved.

<u>Problem Ic</u>. The sequence obtained from the set $\{i_1 + \cdots + i_m : (i_1,\ldots,i_m) \in \mathcal{S}$, counting multiplicities$\}$ arranged in decreasing order consists of all positive integers, with each integer n occurring precisely once for each $(i_1,\ldots,i_m) \in \mathcal{S}$ for which $i_1 + \cdots + i_m = n$. By Proposition 4.26, each n occurs precisely 2^{n-1} times.

Let us obtain a more analytic description of our sequence. Denote the above sequence by (a_k). By inspection, if $2^N \leq k < 2^{N+1}$ then $a_k = N$. If we let $[x]$ = the greatest integer less than or equal to x, then $N = \lfloor \log_2 k \rfloor$. Therefore $a_k = \lfloor \log_2 k \rfloor$ and Problem Ic is solved.

<u>Problem Id</u>. Note that $i_1 + \cdots + i_m + m = (i_1+1) + (i_2+1) + \cdots + (i_m+1)$. From this it is clear that the map $(i_1,\ldots,i_m) \longrightarrow (i_1+1,i_2+1,\ldots,i_m+1)$ maps the set of $(i_1,\ldots,i_m) \in \mathcal{S}$ which satisfy $i_1 + \cdots + i_m + m = n$ one-to-one and onto $\mathcal{S}_n^+$ if $n \geq 2$. But $|\mathcal{S}_n^+| = g(n)$. Therefore the increasing sequence with the entries $i_1 + \cdots + i_m + m$ for each $(i_1,\ldots,i_m) \in \mathcal{S}$ is the sequence of all positive integers $n \geq 2$, with each n occurring precisely once for each $(i_1,\ldots,i_m) \in \mathcal{S}$ such that $i_1 + \cdots + i_m + m = n$, equivalently, g(n) times. In other words, we have that $\{i_1 + \cdots + i_m + m : (i_1,\ldots,i_m) \in \mathcal{S}$, counting multiplicities$\} = \{i_1 + \cdots + i_m : (i_1,\ldots,i_m) \in \mathcal{S}^+$, counting multiplicities$\}$.

A more analytic description can be obtained but it is not as nice as the description of the sequence of Problem Ic. Proceeding as in Ic, if we denote the above sequence by (b_k), then if $(\Sigma_{i=1}^{N-1} g(i)) + 1 \leq k < (\Sigma_{i=1}^{N} g(i)) + 1$, we obtain $b_k = N$. It is rather complicated to express b_k as a function of k, and we shall not need this. Thus, we shall consider Problem Id to be solved.

Problem Ie. Let (b_k) denote the sequence of Problem Id. Let (a_k) denote the sequence of Problem Ic, which we have shown is given by $a_k = [\log_2 k]$. We give the solution to Problem Ie in the next Lemma. This our most important computational tool.

Lemma 4.26a. For every $k \in Z^+$,

$$a_k = [\log_2 k] \leq b_k \leq 2[\log_2 k] = 2a_k .$$

Proof. Recall that (a_k) is the increasing rearrangement of the set $\{i_1 + \cdots + i_m : (i_1,\ldots,i_m) \in \mathcal{S}$, counting multiplicities$\}$ and (b_k) is the increasing rearrangement of the set $\{i_1 + \cdots + i_m : (i_1,\ldots,i_m) \in \mathcal{S}^+$, counting multiplicities$\}$. Furthermore the map $(i_1,\ldots,i_m) \longrightarrow (i_1-1, i_2-1,\ldots,i_m-1)$ is a one-to-one correspondence from $\mathcal{S}^+ \longrightarrow \mathcal{S}$. For each entry a_k of (a_k), let $(i_1,\ldots,i_m) \in \mathcal{S}^+$ be such that the finite-tuple associated with a_k is precisely $(i_1-1, i_2-1,\ldots,i_m-1)$. Let $\pi(k)$ denote that positive integer for which the entry $b_{\pi(k)}$ of (b_k) is associated with the finite-tuple $(i_1,\ldots,i_m)$.

Then $\pi: Z^+ \longrightarrow Z^+$ one-to-one and onto. Also

$(i_1-1)+\cdots+(i_m-1) < i_1+\cdots+i_m$

$\leq 2((i_1-1)+\cdots+(i_m-1))$,

since $i_k-1 < i_k \leq 2(i_k-1)$ for every $1 \leq k \leq m$. But

$a_k = (i_1-1)+\cdots+(i_m-1)$ and $b_{\pi(k)} = i_1+\cdots+i_m$.

Therefore $a_k \leq b_{\pi(k)} \leq 2a_k$.

We claim that $a_k \leq b_k \leq 2a_k$ for every $k \in Z^+$. Indeed, suppose not. That is, suppose there exists $k \in Z^+$ such that either $b_k < a_k$ or $2a_k < b_k$. We shall show that both cases are impossible. If $b_k < a_k$ then there are at least k entries b_i for which $b_i < a_k$, namely when $1 \leq i \leq k$. But $a_k \leq a_n \leq b_{\pi(n)}$ for every $n \geq k$. This implies that at most k-1 entries b_i satisfy $b_i < a_k$, which is a contradiction. On the other hand, if $2a_k < b_k$ then $2a_k < b_i$ for all $i \geq k$ and so $b_i \leq 2a_k$ for at most the k-1 entries where $1 \leq i \leq k-1$. But $b_{\pi(i)} \leq 2a_i \leq 2a_k$ for at least the k entries b_i where $1 \leq i \leq k$, which is a contradiction. Therefore $a_k \leq b_k \leq 2a_k$ for every $k \in Z^+$.

Q.E.D.

Summary 4.27. If (a_n) is the sequence whose entries are the positive integers generated by $i_1+\cdots+i_m$ as $(i_1,\ldots,i_m)$ ranges over $\mathcal{S}$, and arranged in increasing order, then each $n \in Z^+$ occurs 2^{n-1} times and $a_n = \lfloor \log_2 n \rfloor$.

If (b_n) is the sequence whose entries are the positive integers generated by $i_1+\cdots+i_m+m$ as $(i_1,\ldots,i_m)$ ranges over $\mathcal{S}$ (equivalently, $i_1+\cdots+i_m$ as $(i_1,\ldots,i_m)$ ranges over $\mathcal{S}^+$), and arranged in increasing order, then each positive integer $n \geq 2$ occurs that number of times equal to the $(n-1)^{st}$ Fibonacci number and

$$\lfloor \log_2 n \rfloor \leq b_n \leq 2\lfloor \log_2 n \rfloor .$$

This completes our solutions to the questions of category I.

Problem II. This problem leads to many other interesting ones of the same type and provides the impetus for Chapter 4 : Infinite Series Rearrangements of a New Type.

Let $w = (w_n)$ and $d = (d_n)$ be strictly positive sequences for which $w_n \uparrow \infty$ and $d_n \underset{\neq}{\downarrow} 0$. Define $S(w,d) = \{\Sigma_n w_{\pi(n)} d_n : \pi \text{ is any permutation of } Z^+\}$ (we use the set notation here in the standard sense, that is, we ignore multiplicities). We wish to say something about the structure of $S(w,d)$. This is the theme of Chapter 4. For the present, we require only a few facts about $S(w,d)$.

It is well-known that $S(w,d) \subset (0,\infty]$.

The next lemma solves Problem II.

Lemma 4.28. The minimum in $S(w,d)$ exists (it may be ∞) and is given by $\Sigma\, w_n d_n$.

Proof. Let π be any permutation of Z^+. It suffices to prove that $\Sigma\, w_n d_n \leq \Sigma_n w_{\pi(n)} d_n$. Define π_1 in terms of π as follows. Set $\pi_1(1) = 1$, $\pi_1(\pi^{-1}(1)) = \pi(1)$, and for $n \neq 1$ or $\pi^{-1}(1)$, set $\pi_1(n) = \pi(n)$. It is straightforward to verify that $\pi_1 : Z^+ \longrightarrow Z^+$ one-to-one and onto and fixes 1. We assert that $\Sigma_n\, w_{\pi_1(n)} d_n \leq \Sigma_n w_{\pi(n)} d_n$. To see this, note that $\pi(1) \geq 1$ and $\pi^{-1}(1) \geq 1$.

Hence $w_{\pi(1)} - w_1 \geq 0$ and $d_1 - d_{\pi^{-1}(1)} \geq 0$. Therefore

$$\Sigma_n (w_{\pi(n)} - w_{\pi_1(n)})d_n = (w_{\pi(1)} - w_{\pi_1(1)})d_1 + (w_{\pi(\pi^{-1}(1))} - w_{\pi_1(\pi^{-1}(1))})d_{\pi^{-1}(1)}$$

$$= (w_{\pi(1)} - w_1)(d_1 - d_{\pi^{-1}(1)})$$

$$\geq 0 .$$

Similarly we obtain a permutation π_2 that fixes 1 and 2 for which $\Sigma\, w_{\pi_2(n)}d_n \leq \Sigma\, w_{\pi_1(n)}d_n$. Thus we obtain, inductively for each k, a permutation π_k that fixes $1,\ldots,k$ for which

$$\Sigma_{n=1}^{k} w_n d_n = \Sigma_{n=1}^{k} w_{\pi_k(n)}d_n \leq \Sigma_{n=1}^{\infty} w_{\pi_k(n)}d_n \leq \Sigma_{n=1}^{\infty} w_{\pi(n)}d_n$$

and the result follows. Q.E.D.

Problem III. The sequence s consists of precisely the entries

$(\Sigma_{i=n}^{\infty} \Sigma_{i_1,\ldots,i_i=1}^{\infty} d(i_1,\ldots,i_i))^{1/2}$ as n ranges over Z^+,

$(\Sigma_{i=n}^{\infty} d(i_1,\ldots,i_m,i))^{1/2}$ as n ranges over Z^+ and $(i_1,\ldots,i_m)$ ranges over $\mathcal{S}_o$, and

$(\Sigma_{i=n}^{\infty} \Sigma_{j_1,\ldots,j_k=1}^{\infty} d(i_1,\ldots,i_m,i,j_1,\ldots,j_k))^{1/2}$

as n ranges over Z^+, k ranges over Z^+ and $(i_1,\ldots,i_m)$ ranges over $\mathcal{S}_o$ (see paragraph s on page 81 and recall that $\mathcal{S}_o = \mathcal{S} \cup \{(\ )\}$). With this we can compute the ℓ^p-norm of the sequence s.

Therefore, for p > 0, we have

(4.29) $$\|s\|^p_{\ell^p} =$$

$$\Sigma^\infty_{m=1}(\Sigma^\infty_{n=m}\ \Sigma^\infty_{i_1,\ldots,i_n=1}\ d(i_1,\ldots,i_n))^{p/2}$$

$$+\ \Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}_o}\ \Sigma^\infty_{i=1}(\Sigma^\infty_{n=i}\ d(i_1,\ldots,i_m,n))^{p/2}$$

$$+\ \Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}_o}\ \Sigma^\infty_{k=1}\Sigma^\infty_{i=1}(\Sigma^\infty_{n=i}\ \Sigma^\infty_{j_1,\ldots,j_k=1}\ d(i_1,\ldots,i_m,n,j_1,\ldots,j_k,))^{p/2}.$$

Furthermore, if $p \le 2$ then $p/2 \le 1$ and therefore we claim

$$\|s\|^p_{\ell^p} \le$$

$$\Sigma^\infty_{m=1}\ \Sigma^\infty_{n=m}\ \Sigma^\infty_{i_1,\ldots,i_n=1}\ d(i_1,\ldots,i_n)^{p/2}$$

$$+\ \Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}_o}\ \Sigma^\infty_{i=1}\Sigma^\infty_{n=i}\ d(i_1,\ldots,i_m,n)^{p/2}$$

$$+\ \Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}_o}\ \Sigma^\infty_{k=1}\Sigma^\infty_{i=1}\Sigma^\infty_{n=i}\Sigma^\infty_{j_1,\ldots j_k=1}\ d(i_1,\ldots,i_m,n,j_1,\ldots,j_k)^{p/2}$$

$$=$$

$$\Sigma^\infty_{m=1}\ m\,\Sigma^\infty_{i_1,\ldots,i_m=1}\ d(i_1,\ldots,i_m)^{p/2}$$

$$+\ \Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}_o}\ \Sigma^\infty_{i=1}\ i\,d(i_1,\ldots,i_m,i)^{p/2}$$

$$+\ \Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}_o}\ \Sigma^\infty_{k=1}\Sigma^\infty_{j_1,\ldots,j_k=1}\Sigma^\infty_{i=1}\ i\,d(i_1,\ldots,i_m,i,j_1,\ldots,j_k)^{p/2}$$

$$=$$

$$\Sigma_{m=1}^{\infty} \Sigma_{i_1,\dots,i_m=1}^{\infty} m\, d(i_1,\dots,i_m)^{p/2}$$

$$+ \Sigma_{(i_1,\dots,i_m)\in\mathcal{S}} i_m\, d(i_1,\dots,i_m)^{p/2}$$

$$+ \Sigma_{(i_1,\dots,i_m)\in\mathcal{S}} \Sigma_{1\le k\le m} i_k\, d(i_1,\dots,i_k,\dots,i_m)^{p/2}$$

$$=$$

$$\Sigma_{(i_1,\dots,i_m)\in\mathcal{S}} (i_1 + \cdots + i_m + m)\, d(i_1,\dots,i_m)^{p/2} .$$

The first inequality holds since $(\Sigma x_n)^r \le \Sigma x_n{}^r$ if $r \le 1$ and $x_n \ge 0$ for every n. The first equality follows by interchanging the orders of summation and using the identity $(\Sigma_{i=1}^{\infty}(\Sigma_{n=i}^{\infty} x_n)) = \Sigma_{i=1}^{\infty} i\, x_i$, if $x_n \ge 0$ for all n. In the second equality, we substitute i_m for i and $(i_1,\dots,i_m) \in \mathcal{S}$ for $(i_1,\dots,i_m) \in \mathcal{S}_o$, in the second sum; and in the third sum, for $1 \le k \le m$, we substitute i_k for i, and $(i_1,\dots,i_m) \in \mathcal{S}$ for $(i_1,\dots,i_m) \in \mathcal{S}_o$, $(j_1,\dots,j_k) \in \mathcal{S}$, and $n \in Z^+$. The last equality is obvious.

In short,

$$(4.30)\quad \|s\|_{\ell^p}^p \le \Sigma_{(i_1,\dots,i_m)\in\mathcal{S}} (i_1+\cdots+i_m+m)\, d(i_1,\dots,i_m)^{p/2},$$

where if p = 2 then equality holds.

Inequality (4.30) is the key estimate that makes our construction worthwhile. This will now become evident.

For any map $\pi: Z^+ \longrightarrow \mathcal{S}$, we have defined $d(i_1,\ldots,i_m) = d_{\pi^{-1}(i_1,\ldots,i_m)}$. Hence when $n = \pi^{-1}(i_1,\ldots,i_m)$ we have $d_{\pi^{-1}(i_1,\ldots,i_m)} = d_n$.

Let (b_n) denote the increasing sequence of Problem Id which is generated by the positive integers $i_1+\cdots+i_m+m$ as $(i_1,\ldots,i_m)$ ranges over $\mathcal{S}$. From the definition of (b_n), we may choose a map $\sigma: \mathcal{S} \longrightarrow Z^+$ one-to-one and onto such that if $n = \sigma(i_1,\ldots,i_m)$, then $b_n = i_1+\cdots+i_m+m$. Therefore

$$\Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}} (i_1+\cdots+i_m+m)\, d(i_1,\ldots,i_m)^{p/2}$$

$$= \Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}}\, b_{\sigma(i_1,\ldots,i_m)}\, d_{\pi^{-1}(i_1,\ldots,i_m)}^{p/2}$$

$$= \Sigma_{n=1}^{\infty}\, b_{\sigma(\pi(n))}\, d_n^{p/2}.$$

Hence

$$(4.31)\quad \|s\|_{\ell^p}^p \le \Sigma_{(i_1,\ldots,i_m)\in\mathcal{S}} (i_1+\cdots+i_m+m)\, d(i_1,\ldots,i_m)^{p/2}$$

$$= \Sigma_{n=1}^{\infty}\, b_{\sigma(\pi(n))}\, d_n^{p/2}.$$

Note that σ is fixed but π is variable. We need inequality (4.31) to decide when $s \in \ell^p$. Indeed, we need to know when $\Sigma\, b_{\sigma(\pi(n))}\, d_n^{p/2}$ is finite. Therefore we wish to know the minimum of $\Sigma\, b_{\sigma(\pi(n))}\, d_n^{p/2}$ as π ranges over all one-to-one and onto maps $\pi: Z^+ \longrightarrow \mathcal{S}$. This is answered by Lemma 4.28, alias Problem II. It is minimum when $\pi = \sigma^{-1}$

and that minimum is given by $\Sigma\, b_n d_n^{p/2}$. In other words, the choice $\pi = \sigma^{-1}$ in our construction and results yields the absolute minimum for the sum $\Sigma\, b_{\sigma(\pi(n))}\, d_n^{p/2}$, relative to the construction and the notations. In the case p = 2, this choice yields the absolute minimum for $\|s\|_{\ell^2}^2$, relative to these techniques.

For $\pi = \sigma^{-1}$, we obtain

$$(4.32)\quad \|s\|_p^p \leq \Sigma_{(i_1,\ldots,i_m)\in \mathcal{S}} (i_1+\cdots+i_m+m)\, d(i_1,\ldots,i_m)^{p/2}$$

$$= \Sigma\, b_n d_n^{p/2}$$

where if p = 2 we have equality throughout. Hence by Lemma 4.26a, alias Problem Ie, we have

$$(4.33)\quad \Sigma\,[\log_2 n] d_n \leq \|s\|_{\ell^2}^2 = \Sigma\, b_n d_n \leq \Sigma\, 2[\log_2 n] d_n\,, \quad \text{and}$$

$$(4.34)\quad \|s\|_{\ell^p}^p \leq \Sigma\, b_n d_n^{p/2} \leq 2\Sigma\,[\log_2 n] d_n^{p/2}\,, \text{ for } p \leq 2.$$

This concludes our answer to Problem III.

We are now ready to state and prove our second main theorem.

<u>Theorem 4.35</u>. (Main Theorem II). If $d_n \downarrow 0$, $\Sigma d_n = d < \infty$ and $p \leq 2$, then there exist $X, Y \in K(H)$ for which

$$D(-d, d_1, d_2, \ldots) \oplus 0 = -(X^*X - XX^*) + (Y^*Y - YY^*) \quad \text{such that}$$

$$\|X\|_{C_p}^p \leq 2\Sigma\,[\log_2 n] d_n^{p/2} \quad \text{and}$$

$$\|Y\|_{C_p}^p \leq 2\Sigma\,[\log_2 n] d_n^{p/2}\,.$$

If $p \leq 2$ and $\Sigma (\log n)d_n^{p/2} < \infty$, then $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_p,C_p]_2$.

Proof. By Summary 4.24-5 and 4.24-6 we have that

$$D(-d,d_1,d_2,\ldots) \oplus 0 = -(X^*X-XX^*) + (Y^*Y-YY^*)$$

where $|X| \cong D(s)$ and $|Y| \cong D(s_1)\oplus 0$. By equation (4.34) and the fact that s_1 is pointwise bounded above by a subsequence of s (see the proof of Theorem 4.25), we have

$$\|X\|_{C_p}^p = \||X|\|_{C_p}^p = \|D(s)\|_{C_p}^p = \|s\|_{\ell^p}^p \leq 2\Sigma[\log_2 n]d_n^{p/2}$$

and

$$\|Y\|_{C_p}^p = \||Y|\|_{C_p}^p = \|D(s_1)\|_{C_p}^p = \|s_1\|_{\ell^p}^p \leq \|s\|_{\ell^p}^p$$
$$\leq 2\Sigma[\log_2 n]d_n^{p/2} .$$

The sum $\Sigma[\log_2 n]d_n^{p/2}$ is finite if and only if $\Sigma(\log n)d_n^{p/2}$ is finite. This is clear since $\log n = (\log_2 n)(\log_e 2)$ for every n, and $x-1 \leq [x] \leq x$ for every real number x. The rest of the proof is obvious. Q.E.D.

As usual we give an [I,J] version of Theorem 4.35 without proof.

Theorem 4.35a. If $d_n \downarrow 0$, $\Sigma d_n = d < \infty$, and $0 < r \leq 1$ where for $0 < p,q < \infty$, $p^{-1} + q^{-1} = r^{-1}$, then there exist $A,B,X,Y \in K(H)$ for which

$$D(-d,d_1,d_2,\ldots) \oplus 0 = -(AB-BA) + (XY-YX)$$

such that

$$\|A\|_{C_p}^p \leq 2\Sigma[\log_2 n]d_n^{\,r} \;, \quad \|X\|_{C_p}^p \leq 2\Sigma[\log_2 n]d_n^{\,r} \;,$$

$$\|B\|_{C_p}^p \leq 2\Sigma[\log_2 n]d_n^{\,r} \text{ and } \|Y\|_{C_p}^p \leq 2\Sigma[\log_2 n]d_n^{\,r} \;.$$

Furthermore, if r,p,q are as above, and $\Sigma(\log n)d_n^{\,r} < \infty$, then $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_p,C_q]_2$.

The next theorem is an obvious consequence of Theorem 4.35 but deserves special attention as it is our best theorem on the $[C_2,C_2]$ problem.

<u>Theorem 4.36</u>. If $d_n \downarrow 0$, $\Sigma d_n = d < \infty$, and

$$\Sigma\ (\log n)\ d_n < \infty \;,$$

then there exist $A,B,X,Y \in C_2$ such that

$$D(-d,d_1,d_2,\ldots) \oplus 0 = AB-BA + XY-YX \in [C_2,C_2]$$

and the Hilbert-Schmidt norms of A,B,X and Y are all less than or equal to $(2\Sigma[\log_2 n]d_n)^{1/2}$.

<u>Note</u>. A slightly more complicated version of the main construction produces Theorem 4.36 with slightly better bounds on the Hilbert-Schmidt norms (and the same for Theorems 4.35 and 4.35a in regard to the related norms). That is, we could insure that $\|A\|_{C_2} = \|B\|_{C_2}$ and $\|X\|_{C_2} = \|Y\|_{C_2}$ and

$$\Sigma[\log_2 n]d_n \leq \|A\|_{C_2}^2 + \|X\|_{C_2}^2 = \Sigma\, b_n d_n \leq 2\Sigma[\log_2 n]d_n.$$

<u>Note</u>. We proved earlier (Corollary 4.8) that $\Sigma\, nd_n < \infty$ is a sufficient condition that $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_2,C_2]$.

Theorem 4.36 yields that $\Sigma(\log n)d_n < \infty$ is a sufficient condition that $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_2,C_2]$. This indicates a strategy for showing $C_1^o \subset [C_2,C_2]$, thereby solving Open Question 4 in the affirmative. That is, try to obtain similar sufficient conditions with improved weight factors on the d_n's. In other words, if we new that for every sequence (w_n) for which $w_n \uparrow \infty$, we had that $\Sigma w_n d_n < \infty$ implies $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_2,C_2]$ for every such sequence (d_n) where $d_n \downarrow 0$ and $\Sigma d_n = d < \infty$, then it would follow that $C_1^o \subset [C_2,C_2]$.

Our results on the structure of $[I,I]$ and $[I,J]$ follow easily from Theorems 4.25(Main Theorem I), 4.25a, 4.35(Main Theorem II), 4.35a, and 4.36.

<u>A Special Ideal</u>. It is easy to verify that the set $\{(d_n^{1/2}): d_n \downarrow 0 \text{ and } \Sigma(\log n)d_n < \infty\}$ is a characteristic set (see p. 58) and hence generates an ideal which we denote by I_o. Clearly $I_o^2 \subset C_1$, $I_o \subset C_2$ and $I_o^2 \neq I_o$. We do not know whether or not $[I_o,I_o] = (I_o^2)^o$. Theorem 4.36 and Remark 4.9 immediately gives $(I_o^2)^o \subset [C_2,C_2]$. This gives our main result on the structure of $[C_2,C_2]$ which shows that a significant portion of C_1^o is contained in $[C_2,C_2]$, and greatly improves Corollary 4.10.

<u>Theorem 4.37</u>. For each $0 < q < 1$,

$$C_q^o \subset (\bigcup_{p<1} C_p)^o = \bigcup_{p<1} C_p^o \underset{\neq}{\subset} (I_o^2)^o \subset [C_2,C_2].$$

<u>Proof</u>. The last inclusion follows by the above remark.

The first inclusion and the first equality are clear. For the second inclusion, it suffices to show $\underset{p<1}{\cup} C_p \subset I_o^2$. Indeed, it suffices to show $\underset{p<1}{\cup} \ell_+^p = \text{Calk}(\underset{p<1}{\cup} C_p) \subset \text{Calk}(I_o^2)$. Hence it is enough to show that if $d_n \downarrow 0$ and $(d_n) \in \ell^p$ for some $p < 1$, then $\Sigma (\log n) d_n < \infty$. But $d_n \downarrow 0$ and $(d_n) \in \ell^p$ implies that $d_n = o(n^{-1/p})$. From this it follows easily that $\Sigma (\log n) d_n < \infty$. To prove the inequality, simply note that if $d_n = (n \log^3 n)^{-1}$, then $\Sigma(\log n) d_n < \infty$ but $\Sigma d_n{}^p = \infty$ for every $p < 1$. The rest is clear. Q.E.D.

We mentioned earlier that we would obtain a candidate for the largest ideal $I \subset C_2$ such that $(I^2)^o \subset [C_2, C_2]$. The ideal I_o is that candidate.

<u>Question 4.38</u>. If I_o is the ideal whose characteristic set is $\{(d_n{}^{1/2}): d_n \downarrow 0 \text{ and } \Sigma (\log n) d_n < \infty\}$, is I_o the largest ideal I for which $(I^2)^o \subset [C_2, C_2]$? Furthermore is $(I_o^2)^o = [C_2, C_2]$?

Several other questions arise concerning I_o.

<u>Question 4.38a</u>. Is $(I_o^2)^o \subset C(C_2)$? Is $(I_o^2)^o = C(C_2)$?

A prime motivation for our main construction and its related techniques is as follows. By Corollary 4.8 it is easy to show that if $d_n = n^{-(2+\epsilon)}$ for some $\epsilon > 0$, and if $d = \Sigma d_n$, then $D(-d, d_1, d_2, \ldots) \oplus 0 \in [C_2, C_2]$. The techniques of the main construction give us Theorem 4.36 which easily yields that if $d_n = (n \log^{2+\epsilon} n)^{-1}$ for some $\epsilon > 0$, and if $d = \Sigma d_n$, then $D(-d, d_1, d_2, \ldots) \oplus 0 \in [C_2, C_2]$;

the same holds for $d_n = n^{-p}$ when $p > 1$. The next question to ask, and which we cannot answer, is as follows.

Question 4.39. If $d_n = (n \log^2 n)^{-1}$ and $d = \Sigma d_n$, is $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_2,C_2]$?

Note that very few deep negative results are known on the structure of commutator classes.

There is an analogue to Theorem 4.37 for the case $p \leq 2$. Let I_p be the ideal whose characteristic set is $\{(d_n^{1/2}): d_n \downarrow 0 \text{ and } \Sigma(\log n)\, d_n^{p/2} < \infty\}$. That this is a characteristic set depends on the fact that $p \leq 2$. Clearly $I_p^2 \subset C_{p/2}$ and I_2 is just the ideal I_o defined above. Theorem 4.35 and Remark 4.9 give us $(I_p^2)^o \subset [C_p,C_p]$. We give the following analogue to Theorem 4.37 without proof.

Theorem 4.40. For each $0 < q < p/2$ where $p \leq 2$,

$$C_q^o \subset (\bigcup_{q<p/2} C_q)^o = \bigcup_{q<p/2} C_q^o \subsetneq (I_p^2)^o \subset [C_p,C_p].$$

Note that we could ask questions about I_p analogous to those we asked about I_o.

We now give our best results on the equation $[I,I] = (I^2)^o$.

Theorem 4.41. If $q \leq 2$ then for $I = \bigcup_{p<q} C_p$, $[I,I] = (I^2)^o$. In particular

$$(\bigcup_{p<1} C_p)^o = [\bigcup_{p<2} C_p, \bigcup_{p<2} C_p].$$

<u>Proof</u>. By Theorem 4.4 it suffices to show that if $d_n \downarrow 0$, $(d_n) \in \bigcup_{p<q/2} \ell^p$ and $\Sigma d_n = d$, then $D(-d, d_1, d_2, \ldots) \oplus 0 \in [\bigcup_{p<q} C_p, \bigcup_{p<q} C_p]$.

Theorem 4.35 states that

$$D(-d, d_1, d_2, \ldots) \oplus 0 = -(X^*X - XX^*) + (Y^*Y - YY^*)$$

where if $p \leq 2$ and $\Sigma (\log n)\, d_n^{p/2} < \infty$, then $X, Y \in C_p$.

It therefore is enough to show that $(d_n) \in \bigcup_{p<q/2} \ell^p$ and $d_n \downarrow 0$ implies $\Sigma (\log n)\, d_n^{p/2} < \infty$ for some $p < q$. But $(d_n) \in \bigcup_{p<q/2} \ell^p$ implies $(d_n) \in \ell^{p_1}$ for some $p_1 < q/2$ and $d_n \downarrow 0$ further implies that $d_n^{p_1} = o(1/n)$, equivalently, $d_n = o(n^{-1/p_1})$. Choose any $p < q$ such that $p_1 < p/2 < q/2$. Then

$$\Sigma (\log n)\, d_n^{p/2} \leq \Sigma (\log n)(M\, n^{-1/p_1})^{p/2} < \infty ,$$

since $(1/p_1)(p/2) > 1$. Q.E.D.

A similar approach can be used to prove the following theorem.

<u>Theorem 4.42</u>. If $0 \leq q < 2$ then for $I = \bigcap_{p>q} C_p$, we have $[I, I] = (I^2)^o$. In particular

$$\left(\bigcap_{p>0} C_p\right)^o = \left[\bigcap_{p>0} C_p, \bigcap_{p>0} C_p\right].$$

<u>Note</u>. This last ideal in Theorem 4.42 satisfies $I^o = (I^2)^o = [I, I]$.

As usual we give the corresponding $[I,J]$ results without proof.

Theorem 4.40a. If $0 < r \leq 1$ and $p,q > 0$ where $p^{-1} + q^{-1} = r^{-1}$, then

$$(\bigcup_{t<r} C_t)^o \subset (I_{2r}^2)^o \subset [C_p, C_q].$$

Theorem 4.41a. If $0 < r \leq 1$ and $p,q > 0$ where $p^{-1} + q^{-1} = r^{-1}$, then

$$(\bigcup_{t<r} C_t)^o = [\bigcup_{t<p} C_t, \bigcup_{t<q} C_t].$$

Theorem 4.42a. If $0 < r < 1$ and $p,q > 0$ where $p^{-1} + q^{-1} = r^{-1}$, then

$$(\bigcap_{t>r} C_t)^o = [\bigcap_{t>p} C_t, \bigcap_{t>q} C_t].$$

Let us now summarize our results on the equation $[I,I] = (I^2)^o$.

Summary 4.43. The following ideals $I \subset C_2$ have the property that $[I,I] = (I^2)^o$.

1. $F(H)$.
2. $I(L((a_n)))$ where (a_n) satisfies the requisite conditions stated on page 68.
3. $\bigcup_{p<q} C_p$ where $0 < q \leq 2$.
4. $\bigcap_{p>q} C_p$ where $0 \leq q < 2$.

We conclude this section with a discussion of $[C_1, K(H)]$.

It is easy to see by Proposition 4.1 that $[C_1, K(H)] \subset C_1^o$. We believe the following natural question is strongly related to Open Question 4 (Is $C_1^o = [C_2, C_2]$?).

Question 4.44. Is $[C_1, K(H)] = C_1^o$?

Is $[C_1, L(H)] = C_1^o$?

Is $[C_1, K(H)] = [C_1, L(H)]$?

The main construction yields partial results on this question. Using Theorem 4.25a and the estimates that we developed on s together with Remark 4.9, it is not hard to obtain the following theorem, in which the essential part is the last inclusion.

Theorem 4.45. For each $0 < q < 1$,

$$C_q^o \subset (\bigcup_{p<1} C_p)^o \subset (I_o^2)^o \subset [C_1, K(H)].$$

This concludes Section 4.

5. Cases when $C_1 \subset [I, I]$

In this section we concern ourselves with Open Question 6 which asks whether or not an ideal $I \supset C_2$ satisfies $C_1 \subset [I, I]$. We have seen that determining the precise structure of $[I, I]$ can be a difficult problem. If the precise structure of $[I, I]$ is not determined, one may ask what important classes of operators are contained in $[I, I]$. In other words, how 'full' is $[I, I]$? If a particular class of operators is contained in $[I, I]$, then this also becomes a fact about commutator representations. For example, $C_1 \subset [I,I]$ simply means that every trace class operator has a representation as a finite linear combination of commutators of I.

Known Result 4 states that $C_1 \subset [C_p, C_p]$ for every $p > 2$, and follows from Known Result 3 since $C_1 \subset C_{p/2} = [C_p, C_p]$ for every $p > 2$. Similarly Known Result 5 implies that $C_1 \subset [\bigcap_{p>2} C_p, \bigcap_{p>2} C_p]$.

Note that if $C_1 \subset [I,I]$, it follows that $C_1 \subset I^2$ since $[I,I] \subset I^2$, and hence $C_2 \subset I$. This suggests that our previous work may play a role here. Indeed, Sections 3 and 4 do play a strong role in the development of this section.

The rank one projection operator P which played an essential role in Section 3 (recall $P = D(1,0,0,\ldots)$) also plays an essential role here. Since $P \in C_1$, it follows that if $C_1 \subset [I,I]$ then $P \in [I,I]$ and $C_2 \subsetneq I$ (the inequality holds since $Tr(P) = 1 \neq 0$ which implies $P \notin [C_2, C_2]$). It is an important theme of this section to find conditions on ideals $I \supset C_2$ for which we have that $C_1 \subset [I,I]$ if and only if $P \in [I,I]$.

The following theorem uses the main results of Section 4 to relate the operator P to the 'fullness' of $[I,I]$. It is based on the ideal I_o introduced on page 99.

<u>Theorem 5.1</u>. Let I be an ideal in L(H) for which $I \supset C_2$. Let J be an ideal in L(H) for which $J \subset I_o^2$.

Then the following are equivalent:

(a) $J \subset [I,I]$,

(b) there exists $T \in [I,I] \cap J$ such that $Tr(T) \neq 0$,

(c) $P \in [I,I]$.

Proof. (a) $\Longrightarrow$ (b): The operator $P \in J \subset [I,I]$ and $Tr(P) = 1 \neq 0$.

(b) $\Longrightarrow$ (c): Let $a = Tr(T) \neq 0$ where $T \in [I,I] \cap J \subset J \subset I_o^2 \subset C_1$. Then $P = a^{-1}(aP - T) + a^{-1}T$. But $Tr(aP - T) = a-a = 0$. Hence if $T \in J$, then $aP - T \in J^o \subset (I_o^2)^o \subset [C_2,C_2] \subset [I,I]$, by Theorem 4.37. Therefore $P \in [I,I]$.

(c) $\Longrightarrow$ (a): Let $T \in J \subset I_o^2$. Then $T = (T - (Tr(T))P + (Tr(T))P$. By Theorem 4.37, we can obtain $T - (Tr(T))P \in J^o \subset (I_o^2)^o \subset [C_2,C_2] \subset [I,I]$. But by (c), $P \in [I,I]$. Therefore $T \in [I,I]$. Q.E.D.

The next corollary follows immediately from Theorem 5.1.

Corollary 5.2. Let ideal $I \supset C_2$. If J is any of the ideals a) C_p, for $p < 1$; b) $\bigcup_{p<q} C_p$, for $q \leq 1$; c) $\bigcap_{p>q} C_p$, for $0 \leq q < 1$; d) I_o^2, then $J \subset [I,I]$ if and only if $P \in [I,I]$.

By Theorem 3.13 and previous remarks we know that

$$I \otimes I \subset I \text{ and } P \in [I,I] \Longrightarrow [I,I] = I^2 \Longrightarrow P \in [I,I] \text{ and } I \not\subset C_2 .$$

Therefore if $C_2 \subset I$, then $C_1 \subset I^2$ and we have the following theorem.

Theorem 5.3. If an ideal $I \supset C_2$ has the TPCP, then $P \in [I,I]$ if and only if $C_1 \subset [I,I]$.

A somewhat weaker condition than the TPCP would suffice as we see in the next theorem.

Theorem 5.4. If an ideal I satisfies $I \boxtimes C_2 \subset I$, then $P \in [I,I]$ if and only if $C_1 \subset [I,I]$.

Proof. If $C_1 \subset [I,I]$ then clearly $P \in [I,I]$. Suppose then that $I \boxtimes C_2 \subset I$ and $P \in [I,I]$. If $T \in C_1$, then $T = T_1 + iT_2$ where T_1 and T_2 are the self-adjoint real and imaginary parts of T. Hence it suffices to show that every self-adjoint operator $S \in C_1$ is contained in $[I,I]$.

By the spectral theorem for compact, self-adjoint operators, $S \cong D((a_n))$ for some real sequence $(a_n) \in \ell^1$. Clearly $D((a_n)) = D(a_1,0,a_3,0,\ldots) + D(0,a_2,0,a_4,\ldots)$. Furthermore $D(a_1,0,a_3,0,\ldots) \cong D((a_{2n-1})) \oplus 0$ $\cong D((a_{2n-1})) \oplus 0 \oplus 0 \oplus \cdots \cong P \boxtimes D((a_{2n-1}))$ and similarly $D(0,a_2,0,a_4,\ldots) \cong P \boxtimes D((a_{2n}))$, where (a_{2n-1}) and (a_{2n}) are both contained in ℓ^1. By hypothesis $P = \Sigma_{i=1}^m (X_iY_i - Y_iX_i)$ where $\{X_i\}_{i=1}^m$ and $\{Y_i\}_{i=1}^m$ are contained in I. Then

$P \boxtimes D((a_{2n-1})) = [\Sigma_{i=1}^m (X_iY_i - Y_iX_i)] \boxtimes D((a_{2n-1}))$

$= \Sigma_{i=1}^m [(X_i \boxtimes D((a_{2n-1}))^{1/2})(Y_i \boxtimes D((a_{2n-1}))^{1/2})$

$- (Y_i \boxtimes D((a_{2n-1}))^{1/2})(X_i \boxtimes D((a_{2n-1}))^{1/2})]$

and $I \boxtimes C_2 \subset I$. Thus $X_i \boxtimes D((a_{2n-1}))^{1/2}$ and $Y_i \boxtimes D((a_{2n-1}))^{1/2}$ are contained in I for $1 \leq i \leq m$. The same holds for $P \boxtimes D((a_{2n}))$. Hence $P \boxtimes D((a_{2n-1}))$ and $P \boxtimes D((a_{2n})) \in [I,I]$.

Therefore, via the underlying unitary transformations, $D((a_n)) \in [I,I]$ and so $S \in [I,I]$. Q.E.D.

Note. If the answer to Open Question 4 was yes, that is, if $C_1^o = [C_2,C_2]$, then for every ideal $I \supset C_2$ we would have that $P \in [I,I]$ if and only if $C_1 \subset [I,I]$. This follows from a fact we proved in the proof of Theorem 5.1, namely that C_1 is the linear span of C_1^o and P. However, since we cannot count on this being true and indeed, we have some evidence that it is false, we shall try to find other conditions on I so that '$P \in [I,I]$ if and only if $C_1 \subset [I,I]$'.

The next lemma shows that it is not unusual that an ideal I satisfy $I \otimes C_2 \subset I$.

Lemma 5.5. Let I^2 be a complete normed ideal with norm $\|\cdot\|$. Suppose

(1) there exists a basis $\{e_n\}$ of H and $M > 0$ such that if P_n is the orthogonal projection operator whose range is the one-dimensional subspace spanned by e_n, then $\|P_n\| \leq M$ for every n ; and

(2) for every sequence $\{T_n\} \subset I^2$, $\|T_n\| \longrightarrow 0$ implies $T_n \longrightarrow 0$ (WOT).

Then $C_2 \subset I$.

Furthermore if in addition $\|\cdot\|$ is unitarily invariant, then $I \otimes C_2 \subset I$.

Proof. It is clear that $C_2 \subset I$ if and only if $C_1 \subset I^2$, and that $I \boxtimes C_2 \subset I$ if and only if $I^2 \boxtimes C_1 \subset I^2$ (consider the Calkin ideal sets). It therefore suffices to show that $C_1 \subset I^2$, and when the norm is unitarily invariant, that $I^2 \boxtimes C_1 \subset I^2$.

If $T \in C_1$, then $T = T_1 + iT_2$ where T_1 and $T_2 \in C_1$ and are self-adjoint. As in the last proof, to show $T \in I^2$, it suffices to show that every self-adjoint operator $S \in C_1$ also satisfies $S \in I^2$. As in the preceding proof, $S \cong D((a_n))$ for some $(a_n) \in \text{Calk}(C_1) = \ell^1_+$ and $D((a_n)) = \Sigma\, a_n P_n$ where the convergence is taken in the trace norm. We assert that $D((a_n)) = \Sigma\, a_n P_n$ where the convergence is taken in the I^2 norm.

Consider $S_N = \Sigma_{n=1}^{N} a_n P_n$. If $N < N_1$ then $\|S_{N_1} - S_N\| = \|\Sigma_{n=N+1}^{N_1} a_n P_n\| \leq \Sigma_{n=N+1}^{N_1} |a_n|\, \|P_n\| \leq M\, \Sigma_{n=N+1}^{N_1} |a_n|$ $\longrightarrow 0$ as $N, N_1 \longrightarrow \infty$, since $(a_n) \in \ell^1$. In other words $\{S_N\}$ is a Cauchy sequence in I^2. By the completeness of I^2, $S_N - X \longrightarrow 0$ in the I^2-norm as $N \longrightarrow \infty$, for some $X \in I^2$. Then by hypothesis (2), $S_N \longrightarrow X$ (WOT). However, $S_N \longrightarrow \Sigma\, a_n P_n = D((a_n))$ in the trace norm, and hence in the weak operator topology. Therefore $D((a_n)) = X \in I^2$. Via the underlying unitary transformation, we then obtain $S \in I^2$. Hence $C_1 \subset I^2$.

To show that $I^2 \boxtimes C_1 \subset I^2$ we proceed in a similar manner. If we could show that $T \in I^2$ and $S \in C_1$ imply that $T \boxtimes S \in I^2$, then by the way in which ideals are generated we would have that $I^2 \boxtimes C_1 \subset I^2$. Also

$T \boxtimes S = (T_1 + iT_2) \boxtimes (S_1 + iS_2) = T_1 \boxtimes S_1 - T_2 \boxtimes S_2 + i(T_2 \boxtimes S_1 + T_1 \boxtimes S_2)$, where T_1, T_2, S_1, S_2 are the usual real and imaginary parts of T and S respectively. Therefore it suffices to show if $T \in I^2$ and $S \in C_1$ are both self-adjoint, then $T \otimes S \in I^2$.

It is clear, by considering Calkin ideal sets, that $I^2 \otimes C_1 = C_1 \otimes I^2$. Therefore it is enough to show that $S \in C_1$, $T \in I^2$ and S,T are self-adjoint imply $S \otimes T \in I^2$. As before, $S \cong D((a_n))$ for some real sequence $(a_n) \in \ell^1$, and $D((a_n)) = \Sigma\, a_n P_n$ where the convergence is taken in the trace norm. That is, $S_N \longrightarrow D((a_n))$ in the trace norm as $N \longrightarrow \infty$. If $N < N_1$, then

$$\|S_{N_1} \otimes T - S_N \otimes T\| = \|\Sigma_{n=N+1}^{N_1} a_n P_n \otimes T\| \leq \Sigma_{n=N+1}^{N_1} |a_n|\, \|P_n \otimes T\|$$

$$= (\Sigma_{n=N+1}^{N_1} |a_n|)\, \|P_1 \otimes T\| \longrightarrow 0$$

as $N, N_1 \longrightarrow \infty$, since $(a_n) \in \ell^1$ and $\|P_n \otimes T\| = \|P_1 \otimes T\|$. That $\|P_n \otimes T\| = \|P_1 \otimes T\|$ follows from the fact that $P_n \otimes T \cong P_1 \otimes T$ and from the unitary invariance of the I^2-norm in our hypothesis (which invariance carries over under any fixed unitary transformation from L(H) onto $L(H \otimes H)$). By considering the eigenvalues of $S_N \otimes T$ we see that $S_N \otimes T \in I^2$. Hence $\{S_N \otimes T\}$ is a Cauchy sequence in I^2. By the completeness of I^2, we have that $S_N \otimes T \longrightarrow X$ where $X \in I^2$ and the convergence is taken in the I^2-norm. Hence, by hypothesis (2), $S_N \otimes T \longrightarrow X$ (WOT). But $S_N \longrightarrow D((a_n))$ (WOT) implies that $S_N \otimes T \longrightarrow D((a_n)) \otimes T$ (WOT), whence $D((a_n)) \otimes T = X \in I^2$. Q.E.D.

<u>Note</u>. The assumption that $\|\cdot\|$ is unitarily invariant implies hypothesis (1). This is because $P_n \cong P_m$ whence

$\|P_n\| = \|P_m\|$ for all $n,m \in Z^+$.

The next corollary puts Theorem 5.4 and Lemma 5.5 together. It shows that in some sense, complete normed ideals whose topology is stronger than the uniform operator topology or the weak operator topology resemble the C_p classes in regard to commutators and the present theme.

Corollary 5.6. Suppose I^2 is a complete normed ideal whose norm is unitarily invariant, and convergence in the I^2-norm implies convergence in the weak operator topology. Then $P \in [I,I]$ if and only if $C_1 \subset [I,I]$.

In Section 3, we pointed out that the ideal $I((n^{-1/2}))$ $= I((\alpha_n^{1/2}))$ is the smallest ideal I that we know satisfies $P \in [I,I]$. Theorem 5.3 together with what we know about $\bigcup_{n=1}^{\infty} (\bigotimes_1^n I((\alpha_n^{1/2})))$ yields the following corollary.

Corollary 5.7. If $I = \bigcup_{n=1}^{\infty} (\bigotimes_1^n I((\alpha_n^{1/2})))$ then $C_1 \subset [I,I]$.

The following questions immediately arise.

Questions 5.8. Is the ideal of Corollary 5.7 the smallest ideal I for which $C_1 \subset [I,I]$?

Is $C_1 \subset [I((\alpha_n^{1/2})), I((\alpha_n^{1/2}))]$?

Does there exist an ideal $I \supsetneq C_2$ for which either $C_1 \not\subset [I,I]$ or $P \not\in [I,I]$?

Remark. The answer to the first question is no. There is an ideal I close to $I((\alpha_n^{1/2}))$ for which $C_1 \subset [I,I]$, namely $I = I((\alpha_n^{1/2})) \otimes C_2$. This provides a good setting in which to apply Theorem 5.4.

It is easy to prove that $C_p \otimes C_p = C_p$ for every $0 < p < \infty$. Therefore, even if $I \otimes C_2 \not\subset I$, we must at least have $(I \otimes C_2) \otimes C_2 \subset I \otimes C_2$. Hence the following corollary follows from Theorem 5.4 and the fact that $P \in [I((\alpha_n^{1/2})), I((\alpha_n^{1/2}))] \subset [I((\alpha_n^{1/2}))\otimes C_2, I((\alpha_n^{1/2}))\otimes C_2]$.

<u>Corollary 5.9</u>. $C_1 \subset [I((\alpha_n^{1/2}))\otimes C_2, I((\alpha_n^{1/2}))\otimes C_2]$.

To see that $I((\alpha_n^{1/2})) \otimes C_2$ is strictly smaller than the ideal of Corollary 5.7, it suffices to show that it is strictly smaller than $I((\alpha_n^{1/2})) \otimes I((\alpha_n^{1/2}))$. We will not prove this as the proof is quite lengthy and requires the use of the techniques in Section 3 ('On a Problem of Salinas', pp. 43-50).

This brings us to the last question.

<u>Question 5.10</u>. Is $I = I((\alpha_n^{1/2})) \otimes C_2$ the smallest ideal for which $C_1 \subset [I, I]$?

This concludes Section 5.

6. <u>The Question 'Is $P \in C(K(H))$?'</u>

Open Question 1 asks whether or not $C(K(H)) = K(H)$ and the weaker question which is whether or not $P \in C(K(H))$. If it should turn out that $P \in C(K(H))$, it would then be important to determine which ideals $I \subsetneq K(H)$ have $P \in C(I)$. However, we shall not formally pose the general question since it appears that many mathematicians have considered the question "Is $P \in C(K(H))$?" and yet we know of no significant progress on the problem. As with all the open questions in our list in Section 1, it appears that new techniques need to be developed to handle it. We feel that a positive solution would probably come from one or more new computational techniques.

The main result of this section produces operators $A,B \in K(H)$ that are quite simple, yet special, for which $AB - BA$ is, in some computational sense, close to P. It is hoped that this construct may provide a first step to solving $P = AB - BA$ for some $A,B \in K(H)$; or, in the event that $P \notin C(K(H))$, our $AB - BA$ may provide a candidate for the 'closest' one can get to P using single commutators of compact operators.

The first result is the only negative result we know in the subject of commutators of compact operators. It states that P is not a self-commutator of a compact operator.

Proposition 6.1. For every $A \in K(H)$, $A^*A - AA^* \neq P$.

Proof. We shall prove a little more than the proposition states. We claim that every trace class self-commutator of a compact operator must have trace 0.

Suppose, to the contrary, that $A^*A - AA^* = T \in C_1$ for some operator $A \in K(H)$ such that $Tr(T) \neq 0$. Let $A = M + iN$ be the usual decomposition of A into its real and imaginary parts. Then $A^* = M - iN$ and $A^*A-AA^* = 2i(MN-NM)$, where $M,N \in K(H)$ and are self-adjoint. By the spectral theorem for compact, self-adjoint operators, there exists a basis for which M is a diagonal matrix. With respect to this basis, let $M = D((d_n))$ be this diagonal representation of M, and let $N = (n_{ij})$ be the matrix of N. As we mentioned earlier, a straightforward computation shows that $MN - NM = ((d_i - d_j)n_{ij})$ with respect to this basis, and when $i = j$ the entries are 0. From this it follows that $MN - NM = (1/2i)(A^*A - AA^*) = (1/2i)T$, and so $MN - MN \in C_1$ and in this basis has diagonal entries all 0. Hence $0 = Tr(MN-MN) = Tr((1/2i)T) \neq 0$, which is a contradiction. Q.E.D.

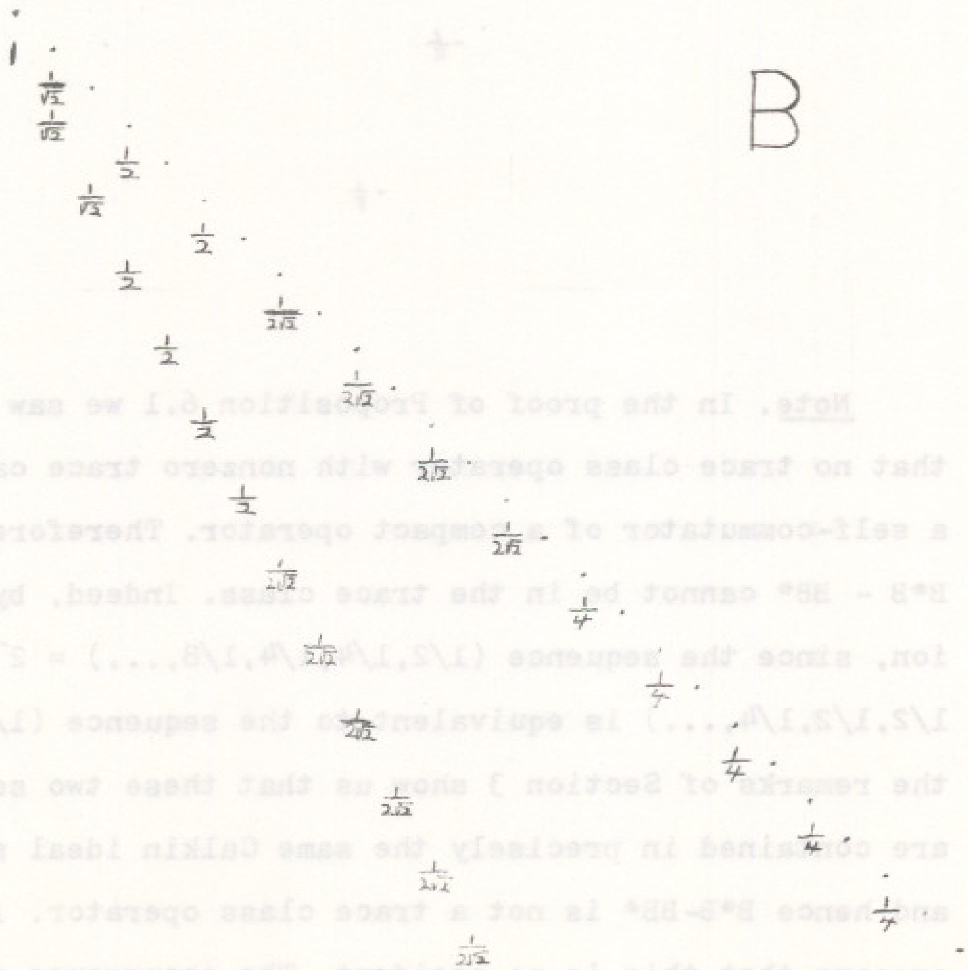
Proposition 6.1 is a little surprising in that both P and each self-commutator are self-adjoint, which might lead one to believe that one could solve $P = A^*A - AA^*$ with $A \in K(H)$.

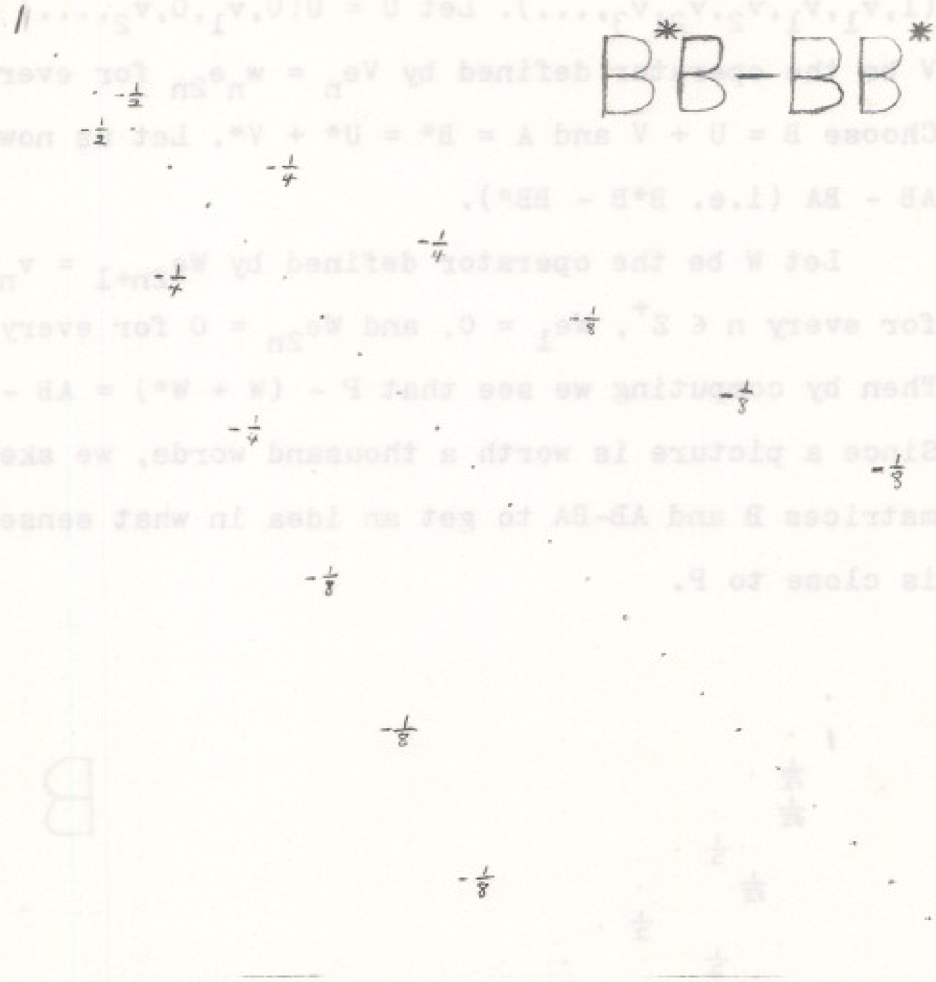
<u>The Main Construction</u>. We shall use an operator $B \in K(H)$ which is the sum of a weighted shift operator of multiplicity 1 and another operator whose nonzero matrix entries occur along the ray which has slope -2 and starts from the $(2,1)$ position. We shall then let $A = B^*$. The operator $AB - BA$ will be a matrix whose nonzero entries occur along two rays starting at the $(1,1)$ position, in which one has a slope of $-1/2$ and the other has a slope of -2, and whose diagonal entries are the same as those of P.

Let $v = (v_n)$ be the decreasing sequence whose entries are $2^{-n/2}$ repeated n times. Let $w = (w_n) =$

$(1, v_1, v_1, v_2, v_2, v_3, \ldots)$. Let $U = U(0, v_1, 0, v_2, \ldots)$ and let V be the operator defined by $Ve_n = w_n e_{2n}$ for every $n \in Z^+$. Choose $B = U + V$ and $A = B^* = U^* + V^*$. Let us now describe $AB - BA$ (i.e. $B^*B - BB^*$).

Let W be the operator defined by $We_{2n+1} = v_n^2 e_{4n}$ for every $n \in Z^+$, $We_1 = 0$, and $We_{2n} = 0$ for every $n \in Z^+$. Then by computing we see that $P - (W + W^*) = AB - BA$. Since a picture is worth a thousand words, we sketch the matrices B and $AB-BA$ to get an idea in what sense $P - (W+W^*)$ is close to P.





Note. In the proof of Proposition 6.1 we saw that no trace class operator with nonzero trace can be a self-commutator of a compact operator. Therefore B*B - BB* cannot be in the trace class. Indeed, by inspection, since the sequence $(1/2,1/4,1/4,1/8,\ldots) = 2^{-1}(1, 1/2,1/2,1/4,\ldots)$ is equivalent to the sequence $(1/n)$, the remarks of Section 3 show us that these two sequences are contained in precisely the same Calkin ideal sets, and hence B*B-BB* is not a trace class operator. It appears that this is no accident. The occurrence of a key sequence related to $(1/n)$ has happened before. Question 3.9 should be considered here, especially if one replaces $[I,I]$ by $C(I)$ in the question.

This concludes Section 6.

7. Cases when $[I,I]_n = [I,I]$ for some n

One of the problems one encounters when considering questions about the structure of $[I,I]$ is that some elements of $[I,I]$ may be represented only as the sum of a large number of commutators of I. We show that in the cases in which we are most interested, this is not true.

Lemma 7.1. If 0 is an infinite-dimensional operator, then

$$[I,I] \oplus 0 \subset [I,I]_3 .$$

Proof. If $\{A_n\}$ is a sequence of operators, let $U(A_1, A_2, \ldots)$ denote the weighted shift operator acting on the Hilbert space $H \oplus H \oplus \cdots$ with operator entries $\{A_n\}$ on its lower diagonal and 0 elsewhere.

Let $T \in [I,I]$. We shall show $T \oplus 0 \in [I,I]_3$. $T = \Sigma_{n=1}^N (X_n Y_n - Y_n X_n)$ for some $\{X_n\}_{n=1}^N$ and $\{Y_n\}_{n=1}^N$ in I. Clearly $T \oplus 0 \cong T \oplus 0 \oplus 0 \oplus \cdots$. Hence it suffices to show that $T \oplus 0 \oplus 0 \oplus \cdots \in [I,I]_3$. Not surprisingly, the trick here is to add and subtract. Write

$$(*) \quad -T \oplus 0 \oplus 0 \oplus \cdots =$$

$$[-T \oplus (X_1Y_1 - Y_1X_1) \oplus (X_2Y_2 - Y_2X_2) \oplus \cdots \oplus (X_NY_N - Y_NX_N) \oplus 0 \oplus \cdots]$$

$$- \quad [0 \oplus (X_1Y_1 - Y_1X_1) \oplus (X_2Y_2 - Y_2X_2) \oplus \cdots \oplus (X_NY_N - Y_NX_N) \oplus \oplus \cdots].$$

The second operator in brackets on the right-hand side of (*) is easily seen to be the commutator of $0 \oplus X_1 \oplus \cdots \oplus X_N \oplus 0 \oplus \cdots$ and $0 \oplus Y_1 \oplus \cdots \oplus Y_N \oplus 0 \oplus \cdots$, which are both in I since the sequences of eigenvalues of their absolute values are obviously in Calk(I). To see that the first operator in brackets on the right-hand side of (*) is a linear combination of 2 commutators of I, write

$$-T \oplus (X_1Y_1 - Y_1X_1) \oplus \cdots \oplus (X_NY_N - Y_NX_N) \oplus 0 \oplus \cdots =$$

$$[-\mathrm{Re}T \oplus \mathrm{Re}(X_1Y_1 - Y_1X_1) \oplus \cdots \oplus \mathrm{Re}(X_NY_N - Y_NX_N) \oplus 0 \oplus \cdots]$$

$$+ \, i[-\mathrm{Im}T \oplus \mathrm{Im}(X_1Y_1 - Y_1X_1) \oplus \cdots \oplus \mathrm{Im}(X_NY_N - Y_NX_N) \oplus 0 \oplus \cdots] ,$$

where for each operator X, ReX and ImX denote the real and imaginary parts of X, respectively. Also $T = \Sigma_{n=1}^N (X_nY_n - Y_nX_n)$ implies that $\mathrm{Re}T = \Sigma_{n=1}^N \mathrm{Re}(X_nY_n - Y_nX_n)$ and $\mathrm{Im}T = \Sigma_{n=1}^N \mathrm{Im}(X_nY_n - Y_nX_n)$, where $\mathrm{Re}(X_nY_n - Y_nX_n)$ and $\mathrm{Im}(X_nY_n - Y_nX_n) \in I^2$ for $1 \leq n \leq N$. Therefore the right-hand side of the previous displayed equation is a linear combination of 2 operators of the form $-S \oplus S_1 \oplus \cdots \oplus S_N \oplus 0 \oplus \cdots$ where $S = \Sigma_{n=1}^N S_n$, $S, S_1, \ldots, S_N$ are self-adjoint and $S_n \in I^2$ for every $1 \leq n \leq N$. Hence it is enough to show that $-S \oplus S_1 \oplus \cdots \oplus S_N \oplus 0 \oplus \cdots$ is a commutator of I.

Denote by $U = U((\Sigma_{n=1}^N S_n)^{1/2}, (\Sigma_{n=2}^N S_n)^{1/2}, \ldots, S_N^{1/2},$

0,...)) the unilateral shift with operator entries $(\Sigma_{n=k}^{N} S_n)^{1/2}$ for $1 \leq k \leq N$ occupying the first N entries on the subdiagonal and 0 elsewhere. The square root of $\Sigma_{n=k}^{N} S_n$ exists (although it is not unique) by the spectral theorem since it is self-adjoint. Also $|U| = D((\Sigma_{n=1}^{N} S_n)^{1/2}, (\Sigma_{n=2}^{N} S_n)^{1/2}, \ldots, S_N^{1/2}, 0, 0, \ldots)$ where we use the diagonal operator notation with operator entries in the obvious way. Since $\Sigma_{n=k}^{N} S_n \in I^2$ for every $1 \leq k \leq N$, it follows that $|U| \in I$, and so $U \in I$. Finally, by computation, $UU^* - U^*U = -S \oplus S_1 \oplus \cdots \oplus S_N \oplus 0 \oplus \cdots$. Q.E.D.

Lemma 7.1 implies that the number of commutators required to build a given operator need not be large, as we see in the following theorem, which is the main theorem of this section.

Theorem 7.2. Suppose I and J are ideals in L(H).

1. If $J \subset [I,I]$ then $J \subset [I,I]_8$.

2. If $J \subset C_1$ and $J^o \subset [I,I]$ then $J^o \subset [I,I]_{14}$.

Proof. To prove the first part, let $T \in J$. It is well-known that $T \cong T'$ where T' acts on $H \oplus H$ and has a 2x2 matrix representation with operator entries. Let $M(T_{11}, T_{12}, T_{21}, T_{22})$ denote the 2x2 matrix with operator entries T_{ij} in the (i,j) position. Then $T' = M(T_{11}, T_{12}, T_{21}, T_{22})$, where $T_{ij} \in J \subset I^2$ for $i,j = 1,2$.

For every operator $A \in J \subset I^2$, if $A = V|A|$ is the polar decomposition of A, then $V|A|^{1/2}$, $|A|^{1/2} \in I$ and

$$M(0,A,0,0) =$$

$M(V|A|^{1/2},0,0,0)\,M(0,|A|^{1/2},0,0)$

$\qquad - M(0,|A|^{1/2},0,0)\,M(V|A|^{1/2},0,0,0)$,

whence $M(0,A,0,0) \in C(I)$. But then because $T_{12},T_{21} \in I^2$ and the fact that $M(0,0,X,0) \cong M(0,X,0,0)$ for every $X \in L(H)$, we obtain $M(0,T_{12},0,0)$ and $M(0,0,T_{21},0) \in C(I)$. Furthermore, for every operator $A \in J \subset [I,I]$, the operator $M(A,0,0,0) \in [I,I] \oplus 0 \subset [I,I]_3$, by Lemma 7.1. Therefore, since $M(X,0,0,0) \cong M(0,0,0,X)$ for every $X \in L(H)$ and $T_{11},T_{22} \in J$, we obtain $M(T_{11},0,0,0), M(0,0,0,T_{22}) \in [I,I]_3$. Therefore $T' = M(T_{11},0,0,0) + M(0,T_{12},0,0) + M(0,0,T_{21},0) + M(0,0,0,T_{22}) \in [I,I]_8$.

To prove the second part of the theorem, let $T \in J^o$ and write $T = T_1 + iT_2$ where T_1 and T_2 are the real and imaginary parts of T, respectively. We saw in Section 4 (in the proof of Lemma 4.4) that $T \in J^o$ implies that $T_1,T_2 \in J^o$. It therefore suffices to show that every self-adjoint operator $S \in J^o$, is contained in $[I,I]_7$.

By the spectral theorem for compact, self-adjoint operators and the unitary invariance of the trace, we have that $S \cong D((a_n))$ where $(a_n) \in \text{Calk}(J) \subset \ell^1$, (a_n) is a real sequence, and $\Sigma a_n = 0$. Let (a_n^+) and $(a_n^-) \in \text{Calk}(J)$ denote the subsequences of (a_n) of non-negative and negative real numbers, respectively. Then $\Sigma a_n^+ = -\Sigma a_n^- \geq 0$. Letting $a = \Sigma a_n^+ = -\Sigma a_n^-$ we obtain (in the case both are infinite sums)

$$D((a_n)) \cong D((a_n^+)) \oplus D((a_n^-))$$
$$= [D((a_n^+))\oplus(-aP)] + [(-aP)\oplus(aP)] + [(aP)\oplus D((a_n^-))] .$$

But $D((a_n^+))\oplus(-aP) \cong D(-a,a_1^+,a_2^+,\ldots)\oplus 0 \in J^o\oplus 0 \subset [I,I]\oplus 0 \subset$

$[I,I]_3$ by Lemma 7.1, and the same holds for $(aP)\oplus D((a_n^-))$. Also $(-aP)\oplus(aP) \cong D(-a,a)\oplus 0 \in C(F(H)) \subset C(I)$, as we have seen before in Section 4 (in the proof of Lemma 4.4). Hence $D((a_n)) \in [I,I]_7$. Q.E.D.

The main point of this theorem is that it proves the following corollary.

Corollary 7.3. Let I be an ideal of L(H). Then $I^2 = [I,I]$ if and only if $I^2 = [I,I]_8 = [I,I]$. Furthermore, if $I \subset C_2$, then $(I^2)^o = [I,I]$ if and only if $(I^2)^o = [I,I]_{14} = [I,I]$.

Proof. Apply Theorem 7.2 setting $J = I^2$. Q.E.D.

Let us point out some of the significant consequences of these results. Corollary 7.3 shows that $C_1^o = [C_2,C_2]$ if and only if $C_1^o = [C_2,C_2]_{14} = [C_2,C_2]$. What is more, $C_1^o = [C_2,C_2]$ if and only if for every sequence (d_n) for which $d_n \downarrow 0$, $\Sigma d_n = d < \infty$, we have that the operator $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_2,C_2]_3$. This suggests that our main construction of Section 4 may be the best possible one with which to deal with $D(-d,d_1,d_2,\ldots)$.

Before we obtained these results, it had seemed likely that there were operators which could only be expressed as a linear combination of commutators of C_2 if we employed a large number of such commutators. In other words, one might have searched for a technique for solving the commutator equation $T = \Sigma_{n=1}^N a_n(X_nY_n - Y_nX_n)$

which depends on taking N large. However, this section proves that there is essentially nothing to be gained by doing so.

This concludes Section 7.

8. Related Commutator Problems

We shall devote this section to some results and questions designed to develop a strategy for improving the results of Section 4.

In Section 4, close examination of the operators $D(-d,d_1,d_2,\ldots)$ and $D(d,d_1,d_2,\ldots) \oplus 0$ where $d_n \downarrow 0$ and $\Sigma d_n = d < \infty$ was a paramount theme. The best results of that section were due to our being able to represent some of these operators as a single commutator or as a linear combination of 3 commutators of Hilbert-Schmidt operators. Our failure to solve the '$C_1^o = [C_2,C_2]$' problem was due to our inability to express all such operators as a finite linear combination of commutators of Hilbert-Schmidt operators. This difficulty is due to a key theme in the main construction of Section 4 which first appears in equation (4.6) together with Corollary 4.8. It permeates the main parts of the construction, especially equation (4.22) and the succeding computations regarding the Hilbert-Schmidt norms of the special solution operators A_1, A_2 and A_3 introduced on page 79. Namely that $D(-d,d_1,d_2,\ldots) = X^*X - XX^*$ where $\|X\|_{C_2}^2 = \Sigma\, nd_n$, and that

$X \in C_2$ if and only if $\Sigma\, nd_n < \infty$. This theme is both good and bad in that it gets us off the ground but limits the power of our end results.

How could the techniques of Section 4 be improved? We expressed $D(-d,d_1,d_2,\ldots) \oplus 0$ as a linear combination of 2 operators, the essential one of which was unitarily equivalent to the countable direct sum of our canonical diagonal operators $D(-x,x_1,x_2,\ldots)$ where $x_n \downarrow 0$ and $\Sigma x_n = x < \infty$ (see Summary 4.25-5 and 4.25-6). We then wrote these diagonal operators as commutators using equation (4.6) and the estimates of Corollary 4.8. There are two obvious strategies to pursue in order to improve these techniques. The first strategy is to try to make a better choice of the set of real numbers that we added and subtracted and then rearranged in order to obtain the diagonal operators that appear in the countable direct sum in equation (4.16). This approach avoids looking for better ways than equation (4.6) to write $D(-x,x_1,x_2,\ldots)$ as a commutator of Hilbert-Schmidt operators, which is a difficult problem for us. However, our work with John Conway proves that such an improvement is impossible. The second strategy is to actually look for better ways to write $D(-x,x_1,x_2,\ldots)$ or $D(-x,x_1,x_2,\ldots) \oplus 0$ as a commutator (or finite linear combination of commutators) of Hilbert-Schmidt operators.

In particular, we should search for solutions X,Y to the operator equation $D(-d,d_1,d_2,\ldots) = XY-YX$ or $D(-d,d_1,d_2,\ldots) \oplus 0 = XY-YX$ for which the Hilbert-Schmidt norms of X and Y are smaller, in the sense that $X,Y \in C_2$ for more decreasing, positive sequences (d_n) than just those for which $\Sigma\, nd_n < \infty$. It is natural to search for a sufficient condition that $X,Y \in C_2$ of the form '$\Sigma\, w_n d_n < \infty$', where the sequence (w_n) is of slower growth than the sequence (n). This is because if w_n is of strictly slower growth than n, then a larger class of decreasing, positive sequences (d_n) satisfy $\Sigma\, w_n d_n < \infty$ than do satisfy $\Sigma\, nd_n < \infty$.

In the preceding remarks we concentrated on the single commutator problem for $D(-d,d_1,d_2,\ldots)$. This is because the problem seems easier with fewer variables. Of course, our main results depend heavily on the greater number of variables inherent in the 'several' commutator problem. Also the single commutator problem is of interest in itself in that it bears on the structure problems for $C(C_2)$.

What are the ways in which an operator can be explicitly written as a commutator of <u>compact</u> operators? We know of only two techniques appearing in the literature that lend themselves to modification in regard to compactness.

The first method is the oldest and was used to prove the original characterization of the commutator class $C(L(H))$ when H is finite-dimensional. We used it earlier (Theorem 4.2) to do just that. The method is as follows. Start with an operator $T \in C_1^o$. Then use the convexity of its numerical range to show that there exists a basis with respect to which the diagonal entries of the matrix representation of T are all 0. Then solve the operator equation

$$D(y)X - XD(y) = ((y_i - y_j)x_{ij}) = (t_{ij}) = T$$

for operators X and D(y) where $X = (x_{ij})$, $y = (y_n)$ and $D(y) = D(y_1, y_2, \ldots)$ with respect to this basis. The problem with this method is that, as far as we know, there is no workable method to determine whether or not the matrix $X = (x_{ij})$ is contained in a particular ideal (we have some results on this problem in Chapter 3, Section 2). The only ideal for which there is a somewhat workable test for containment is the Hilbert-Schmidt class. Another problem is the following conflicting strategy. We need to choose $y \in$ Calk(I), so y must be 'small'; but yet $(x_{ij}) = (t_{ij}/(y_i - y_j)) \in I$, so y must be 'large' in order that x_{ij} may be 'small'. Certainly this method would be quite delicate if it could be made to work at all.

The second method involves our variation on Halmos's identity: $P = U^*U - UU^*$ (where U is the unilateral shift) which is equation (4.6) and Corollary 4.8. It is especially suited to writing diagonal operators as commutators. Our greatest hope for improving the techniques therefore lies in looking for a variation of this method.

The first result actually gives new and better solution operators X,Y for the equation $D(-d, d_1, d_2, \ldots) = XY - YX$. It is a more difficult construction than equation (4.6) and we obtain $\|X\|_{C_2}^{\ 2} = \|Y\|_{C_2}^{\ 2} = \Sigma\ [(n+1)/2]\, d_n$ which is strictly less than $\Sigma\ nd_n$. Unfortunately, although the first sum is quantitatively smaller than the second sum, it is not qualitatively smaller. That is, for each decreasing, positive sequence (d_n), $\Sigma\ [(n+1)/2]\, d_n < \infty$ if and only if $\Sigma\ nd_n < \infty$. Even if we used these better solution operators in the main construction of Section 4, namely in equation (4.20), then all our estimates would be halved. This would be a quantitative improvement but not a qualitative one. In any case, it is an important result because it is the first commutator construction using compact operators we know that involves more than a single weighted shift and that makes use of more than one group of cancellations in the computation.

<u>Proposition 8.1</u>. If $d_n \downarrow 0$ and $\Sigma d_n = d < \infty$, then there exist operators X and Y for which

$$D(-d, d_1, d_2, \ldots) = XY - YX$$

and

$$\|X\|_{C_2}^{\ 2} = \|Y\|_{C_2}^{\ 2} = \Sigma\ [(n+1)/2]\, d_n .$$

<u>Proof</u>. Clearly

$D(-d, d_1, d_2, \ldots) \cong D(-d, d_1, d_3, \ldots) \oplus D(d_2, d_4, \ldots)$.

The right-hand side of this relation acts on $H \oplus H$. Recall the notation $M(T_{11}, T_{12}, T_{21}, T_{22})$ from the proof of Theorem 7.2, which denotes the 2x2 matrix with operator entries T_{ij} (i,j=1,2) in the (i,j) position. Let

$U = U((\Sigma_{n=1}^{\infty} d_{2n-1})^{1/2}, (\Sigma_{n=2}^{\infty} d_{2n-1})^{1/2}, \ldots)$ and

$V = U((\Sigma_{n=2}^{\infty} d_{2n})^{1/2}, (\Sigma_{n=3}^{\infty} d_{2n})^{1/2}, \ldots)$. Furthermore, let

$$A = M(U, -(\Sigma_{n=1}^{\infty} d_{2n})^{1/2} P, 0, -V^*)$$

and

$$B = M(U^*, 0, (\Sigma_{n=1}^{\infty} d_{2n})^{1/2} P, V) .$$

Then by computing we obtain

$$D(-d, d_1, d_3, \ldots) \oplus D(d_2, d_4, \ldots) = AB - BA$$

and

$$\|A\|_{C_2}^2 = \|B\|_{C_2}^2 = \Sigma_{n=1}^{\infty} d_{2n} + \Sigma_{n=1}^{\infty}(\Sigma_{k=n}^{\infty} d_{2k-1}) + \Sigma_{n=2}^{\infty}(\Sigma_{k=n}^{\infty} d_{2k})$$

$$= \Sigma_{n=1}^{\infty} nd_{2n-1} + \Sigma_{n=1}^{\infty} nd_{2n}$$

$$= \Sigma_{n=1}^{\infty}(nd_{2n-1} + nd_{2n})$$

$$= \Sigma \, [(n+1)/2] \, d_n .$$

Hence, via this unitary transformation and the unitary invariance of the Hilbert-Schmidt norm, we see that $D(-d, d_1, d_2, \ldots)$ is a commutator of two operators whose Hilbert-Schmidt norms are both equal to $\Sigma \, [(n+1)/2] \, d_n$. Q.E.D.

<u>Note</u>. This is the first single commutator construction where $Y \neq X^*$.

<u>Related Commutator Problems for Finite Matrices</u>

Let us consider the following commutator problem, which is the finite-dimensional version corresponding to the problem of solving $D(-d, d_1, d_2, \ldots) = XY-YX$ for operators $X, Y \in C_2$.

<u>Problem 8.2</u>. Let $(d_n)_{n=1}^N$ denote a finite positive sequence of length N where $d_n\downarrow$ and $\Sigma d_n = d$. For each of the following equations, find solution matrices X,Y for which $\|X\|_{C_2} = \|Y\|_{C_2}$ is minimal.

a) $D(-d,d_1,\ldots,d_N) = XY - YX$, where X and Y are $(N+1)x(N+1)$ matrices.

b) $D(-d,d_1,\ldots,d_N) \oplus 0 = XY - YX$, where 0,X,Y are finite matrices of sizes rxr,sxs, and sxs, respectively, with $s = (N+1) + r$.

c) $D(-d,d_1,\ldots,d_N) \oplus 0 = XY - YX$, where 0,X,Y are infinite matrices.

Why is Problem 8.2 important? For those who believe that there is justice in the world, since $\|D(-d,d_1,\ldots,d_N)\|_{C_1} = 2d$, one would expect solutions X,Y to exist to one of a,b or c which are uniformly bounded in their Hilbert-Schmidt norms independently of the size of N and the choice of sequence $(d_n)_{n=1}^N$, provided d is held fixed. In other words, the trace norm of $D(-d,d_1, \ldots,d_N)$ should perhaps be the only quantity that bears on the Hilbert-Schmidt norms of the minimal solution operators X,Y. If this were true in a or b, then it would obviously be true for c. Also, if this were true for a,b or c, then we would have a relatively easy proof that $C_1^o = [C_2,C_2]$. We give the details of the proof in the following remark.

<u>Remark 8.3</u>. If there were to exist $M > 0$ such that for every $(d_n)_{n=1}^N$ for which $d_n \geq 0$ and $\Sigma d_n = d$, there exist solution operators $X, Y \in C_2$ of the equation $D(-d, d_1, \ldots, d_N) \oplus 0 = XY - YX$ which satisfy $\|X\|_{C_2} = \|Y\|_{C_2} \leq Md^{1/2}$, then $C_1^o = [C_2, C_2]$. The proof is as follows.

As usual, in order to show that $C_1^o = [C_2, C_2]$, it suffices to show that $D(-d, d_1, d_2, \ldots) \oplus 0 \in [C_2, C_2]$ for every (d_n) for which $d_n \downarrow 0$ and $\Sigma d_n = d$ and where 0 is infinite-dimensional. Let $(\epsilon_k) \in \ell_+^1$ where $\epsilon_k > 0$ for every k. Since $(d_n) \in \ell^1$, we have that for each k, there exists n_k for which $\Sigma_{n=n_k}^{\infty} d_n < \epsilon_k$. Without loss of generality we may assume that (n_k) is a strictly increasing sequence in k. Let $\alpha_k = \Sigma_{n=n_k}^{\infty} d_n$ and let $\alpha = (\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots)$. Then

$$D(-d, d_1, d_2, \ldots) \oplus 0 \cong [D(-d, d_1, d_2, \ldots) \oplus D(\alpha) \oplus 0] - [0 \oplus D(\alpha) \oplus 0] .$$

Furthermore,

$$D(\alpha) \cong \Sigma_n \oplus \alpha_n D(1,-1) = [\Sigma\oplus\, \alpha_n^{1/2} U^*(1,0)][\Sigma\oplus\, \alpha_n^{1/2} U(1,0)] - [\Sigma\oplus\, \alpha_n^{1/2} U(1,0)][\Sigma\oplus\, \alpha_n^{1/2} U^*(1,0)]$$

and $\|\Sigma\oplus\, \alpha_n^{1/2} U(1,0)\|_{C_2}^2 = \|\Sigma\oplus\, \alpha_n^{1/2} U^*(1,0)\|_{C_2}^2 = \Sigma\alpha_n \leq \Sigma\epsilon_n < \infty$. Therefore $0 \oplus D(\alpha) \oplus 0 \in C(C_2)$.

Also

$$D(-d, d_1, d_2, \ldots) \oplus D(\alpha) \oplus 0 \cong [D(-d, d_1, \ldots, d_{n_1 - 1}, \alpha_1) \oplus 0] \oplus [\Sigma\oplus\, (D(-\alpha_k, d_{n_k}, \ldots, d_{n_{k+1}-1}, \alpha_{k+1}) \oplus 0)] .$$

It is easy to verify that the matrices $D(-d,d_1,\ldots,d_{n_1-1},\alpha_1) \oplus 0$ and $D(-\alpha_k,d_{n_k},\ldots,d_{n_{k+1}-1},\alpha_{k+1}) \oplus 0$ satisfy our hypothesis. Hence there exist operators X_k, Y_k for $k = 0,1,\ldots$, where $\|X_k\|_{C_2} = \|Y_k\|_{C_2} \le M\alpha_k^{1/2}$ for $k = 1,2,\ldots$, such that

$$D(-d,d_1,\ldots,d_{n_1-1},\alpha_1) \oplus 0 = X_0Y_0 - Y_0X_0 \text{ and}$$

$$D(-\alpha_k,d_{n_k},\ldots,d_{n_{k+1}-1},\alpha_{k+1}) \oplus 0 = X_kY_k - Y_kX_k .$$

Letting $X = \Sigma\oplus X_k$ and $Y = \Sigma\oplus Y_k$, we obtain $\|X\|_{C_2}^2 = \Sigma\|X_k\|_{C_2}^2 \le \|X_0\|_{C_2}^2 + \Sigma M\alpha_k \le \|X_0\|_{C_2}^2 + M\Sigma\epsilon_k < \infty$, $\|Y\|_{C_2} = \|X\|_{C_2} < \infty$, and

$$[D(-d,d_1,\ldots,d_{n_1-1},\alpha_1) \oplus 0] \oplus [\Sigma\oplus (D(-\alpha_k,d_{n_k},\ldots,d_{n_{k+1}-1},\alpha_{k+1}) \oplus 0)]$$
$$= XY - YX .$$

Hence, via the underlying unitary transformations, $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_2,C_2]$. The conjecture is proved.

Note that this proof resembles the main construction of Section 4. The main construction is a more delicate application of the same principle.

In this result we see a link between the infinite-dimensional problem and its finite-dimensional counterparts. It is not surprising to find such a link since finite-dimensional operators are often used to solve problems about compact, infinite-dimensional operators. A close examination of the techniques of the main construction of Section 4 shows that we could have gotten away with finite matrix approximations, but the arguments would have needed to be more intricate and the results would not be any more worthwhile.

<u>Concrete Commutator Problems for Finite Matrices and The Computer</u>

For what finite sequences $(d_n)_{n=1}^N$ as above are the known solution operators the worst? That is, when is $\Sigma\, nd_n$ or $\Sigma\, [(n+1)/2]\, d_n$ largest (where we assume that the sequence is arranged in decreasing order so as to minimize the sum. See Lemma 4.28.). It is easy to show that if $w_n\uparrow$ and $d_n\downarrow$, then $\Sigma_{n=1}^N w_n d_n$ is maximal for the sequence $d_1 = d_2 = \cdots = d_N = d/N$. Furthermore, if w_n is strictly increasing, then this sequence is the unique sequence which maximizes this sum. Therefore, for the sake of concreteness and in order to be able to apply computer techniques to our problem, we set $d = 1$ and ask the following specialized questions concerning $D(-1,1/N,\ldots,1/N)$, in which $1/N$ appears precisely N times. It is clear that we lose no generality in assuming $d = 1$.

<u>Problem 8.4</u>. For each of the following equations, find solution matrices X,Y for which $\|X\|_{C_2} = \|Y\|_{C_2}$ is minimal.

a) $D(-1,1/N,\ldots,1/N) = XY - YX$, where X and Y are $(N+1)\mathrm{x}(N+1)$ matrices.

b) $D(-1,1/N,\ldots,1/N) \oplus 0 = XY - YX$, where $0,X,Y$ are finite matrices of sizes rxr, sxs, and sxs, respectively, such that $s = (N+1) + r$.

c) $D(-1,1/N,\ldots,1/N) \oplus 0 = XY - YX$, where $0,X,Y$ are infinite matrices.

That $\|X\|_{C_2} = \|Y\|_{C_2}$ cannot be too small is the content of the next proposition.

Proposition 8.5. If X and Y satisfy any of the equations in Problem 8.2, then $\|X\|_{C_2} = \|Y\|_{C_2} \geq d^{1/2}$.

Proof. If $X,Y \in C_2$ and satisfy any one of the equations in Problem 8.2, then

$$2d = \|XY-YX\|_{C_1} \leq \|XY\|_{C_1} + \|YX\|_{C_1} \leq \|X\|_{C_2}\|Y\|_{C_2} + \|Y\|_{C_2}\|X\|_{C_2} = 2\|X\|_{C_2}^{\,2} .$$

Therefore $d^{1/2} \leq \|X\|_{C_2}$. Q.E.D.

This result motivates the next question.

Question 8.6. Can solution operators X,Y be found to the equations in Problems 8.2 or 8.4 which satisfy $\|X\|_{C_2} = \|Y\|_{C_2} = 1$?

Of course, in any case for which the answer is yes, Proposition 8.5 would solve the corresponding problem in 8.2 or 8.4.

What is the role of the computer in our work?

In joint work with Layne Watson, we programmed a computer, using a crude search technique involving the use of gradients, to minimize the function

$$\|D - (XY-YX)\|_{C_2}^{2} + 10^{-3}(\|X\|_{C_2}^{2} + \|Y\|_{C_2}^{2})$$

in which D was taken to be $D(-1,1/N,\ldots,1/N)$ or $D(-1,1/N,\ldots,1/N) \oplus 0$ for the cases $N = 1,2,3,4,5,6$, where 0 was taken to be finite-dimensional of various sizes, and where X,Y are the variables.

This problem is a non-trivial computer problem since it is non-linear. Indeed, it is a "quadratic programming problem" with $(N+1)^2$ variables or more. It is difficult to determine the absolute minimum in this way. The main difficulty seems to lie in the existence of many relative minima. In other words, this method is not too reliable. In any case, we carried out quite a few programs using different "initial points" and obtained data which suggests that the following conjectures seem likely to be true.

<u>Conjectures</u>.

I. The answer to Problems 8.2-a and 8.4-a is $\|X\|_{C_2}^{2} = \|Y\|_{C_2}^{2} = \Sigma_{n=1}^{N} [(n+1)/2]d_n$, where X and Y may be taken to be A and B in the proof of Proposition 8.1 by replacing (d_n) by $(d_n)_{n=1}^{N}$ (if N=2 then the (2,2) matrix position is the number 0). Of course, X and Y are not unique.

II. No improvement is realized by considering $D(-d,d_1,\ldots,d_N) \oplus 0$ instead of $D(-d,d_1,\ldots,d_N)$. In other words, the same minimum holds for each of the Problems 8.2-b, 8.2-c, 8.4-b, and 8.4-c, provided we normalize to d = 1.

What are the ramifications of I and II?

If I were true, then for N = 1 or 2, we have $\Sigma_{n=1}^{N}[(n+1)/2]\,d_n = \Sigma_{n=1}^{N} d_n = d$ and so the answer to Question 8.6 is yes for N = 1,2, but the answer is no if N > 2 (provided $d_3 > 0$). (Equation 4.6 and Corollary 4.8 yield $\|X\|_{C_2}{}^2 = \|Y\|_{C_2}{}^2 = \Sigma_{n=1}^{N} nd_n$, which is greater than 1 when $d_2 > 0$.). Furthermore, we believe that this provides evidence that $C(C_2) \neq C_1^o$, although no completely determining link has actually been proven to exist between the infinite matrices of our theory and these finite ones.

I and II together appear to suggest that our earlier suspicion that very dense matrix forms (matrices with many nonzero entries) would yield smaller solution operators X,Y is wrong. The computer had produced some dense forms, but their Hilbert-Schmidt norms were always roughly the sum $\Sigma_{n=1}^{N}[(n+1)/2]\,d_n$.

The reader should keep in mind that we have no proofs of these conjectures. One of the difficulties is that there are too many variables to contend with. What the computer definitely accomplished was its demonstration that the finite-dimensional case of equation (4.6) with $\|X\|_{C_2}{}^2 =$

$\|Y\|_{C_2}^{\ 2} = \Sigma_{n=1}^N \ nd_n$ was not minimal. Previously, this had been unknown and of interest to us.

If we allow $D(-d,d_1,d_2,\ldots) \oplus 0$ to be written as the sum of two commutators, then the main construction of Section 4 could be made to conform to this problem. In fact, a slight variation on the adding and subtracting theme yields finite matrices A and X for which

$$D(-d,d_1,\ldots,d_N) \oplus 0 = -(A^*A-AA^*) + (X^*X-XX^*)$$

and

$$\Sigma_{n=1}^N [\log_2 n]\, d_n \le \|A\|_{C_2}^{\ 2} + \|X\|_{C_2}^{\ 2} \le 2\Sigma_{n=1}^N [\log_2 n]\, d_n \ .$$

This also holds if (d_n) is an infinite sequence.

With respect to Question 8.4, if in b, the operator 0 is of sufficiently large dimension, then

$$D(-1,1/N,\ldots,1/N) \oplus 0 = -(A^*A-AA^*) + (X^*X-XX^*)$$

with

$$\|A\|_{C_2}^{\ 2} + \|X\|_{C_2}^{\ 2} \le 2(\log_2 e)(\Sigma_{n=1}^N(\log n))/N$$

$$= 2(\log_2 e)(\log N!)/N \ .$$

Proposition 8.1, in its finite matrix form, tells us that $D(-1,1/N,\ldots,1/N) = XY-YX$ with

$$\|X\|_{C_2}^{\ 2} = (\Sigma_{n=1}^N[(n+1)/2])/N = O(N) \ .$$

But $(\log N!)/N = O(\log N)$. Hence, the techniques of Section 4 also have a significant bearing on these problems, which was to be expected.

We close this section by pointing out a well-known problem which has close ties with the conjecture of

Remark 8.3. Namely, 'Does there exist $M > 0$ such that for any finite matrix T (of any size) for which $\|T\| = 1$ and $Tr(T) = 0$, there exist matrices X,Y for which $T = XY - YX$ and $\|X\|, \|Y\| \leq M$, where this norm is the uniform operator norm of $L(H)$?' (for reference see [26, reference 5]).

Note that the only difference between this problem for $T = D(-d, d_1, \ldots, d_N)$ and the hypothesis of the statement in Remark 8.3 is that the relevant norms are different. Also, in this case, the above problem is solved in the above reference.

This concludes Section 8.

9. An Interpretation of the Results

In this section, we close Chapter 1 with an interpretation of the implications of our results in regard to the structure theory of the commutator classes $C(I)$ and $[I,I]$.

The results and comments of Section 8 indicate, with some degree of likelihood, that the operators $D(-1, d_1, d_2, \ldots)$ and $D(-1, d_1, d_2, \ldots) \oplus 0$ with $d_n \downarrow 0$ and $\Sigma d_n = 1$ cannot always be represented as a single commutator of C_2, and also that the operators $D(-1, d_1, \ldots, d_N)$ and $D(-1, d_1, \ldots, d_N) \oplus 0$ cannot be represented as $XY-YX$ if X,Y are chosen to have Hilbert-Schmidt norms that are uniformly bounded, independently of the size of N and the non-negative sequence $(d_n)_{n=1}^{N}$. Indeed, we believe there is a considerable likelihood that

Proposition 8.1 and its construction is the best possible result that one can obtain in the affirmative direction to the '$C(C_2) = C_1^o$' question, and hence that $C(C_2) \neq C_1^o$. Similarly we feel that there is some degree of likelihood that Theorem 4.36 and its underlying construction is the best possible qualitative result in the affirmative direction to the '$[C_2,C_2] = C_1^o$' question. Hence we believe that $[C_2,C_2] \neq C_1^o$.

Examining the constructions of equation (4.6) and Proposition 8.1 reveals that a controlled, efficient process of cancellation occurs. Peculiarly, equation (4.6) depends on the fact that our infinite matrices have a 'beginning edge' (i.e. the first row or first column) and the improvement of Proposition 8.1 depends on the fact that they have more than one 'beginning edge', indeed, they have two of them. One hope for improving the constructions is to construct, via some unitary equivalence, some sort of matrix representation for operators which has more than two beginning edges and which reproduces the same kind of cancellation phenomenon more efficiently. The existence of matrices which do not have two beginning edges is not just speculation. The well-known bilateral shift of multiplicity one has no beginning edges. M.S. Ramanujan [30] considers matrices which are quadruply indexed acting on basis vectors that are doubly indexed . Perhaps there is a natural unitary transformation between L(H) and matrices of this type together with some notion of 'beginning edge' which yields better commutator equations. On the one hand, we expected this to show up in the computer results, which it did not. On the other hand, matrices with several edges must necessarily have a large number of entries, and this might not show up due to computer limitations. This implies that there is hope that severally indexed matrices may provide the improvements we are looking for. Also, increasing the size of the solution matrices, contrary to the evidence mentioned in Section 8, may provide better solutions.

Another reason we believe $[C_2,C_2] \neq C_1^o$ is the results of Section 7 which state that $C_1^o = [C_2,C_2]$ if and only if $D(-d,d_1,d_2,\ldots) \oplus 0 \in [C_2,C_2]_3$. That is, the techniques of employing large numbers of commutators for representations are of no consequence, and so any improvements to be found must be able to be formulated in a construction of at most 3 commutators. This makes our results appear a little closer to being the best possible ones.

It is natural to draw analogous conclusions about P, especially in light of the similarity of P with $D(-d,d_1,d_2,\ldots) \oplus 0$. We believe $P \notin C(K(H))$, and thus $C(K(H)) \neq K(H)$. The one precise result we have in this direction is that P is not a self-commutator of a compact operator (Proposition 6.1).

However, we wish to impress upon the reader that we are merely describing the pattern that is emerging from the known facts. Any quantitative or qualitative improvement on any one of our results could, as likely as not, give rise to complete solutions to one or all of the open questions.

Let us note a theme of the known results and an analogous theme in our results. Pearcy and Topping [26], and Salinas [31] try to obtain ideals $I \supset C_2$ as close to C_2 as possible such that $[I,I] = I^2$. It appears likely that $I((n^{-1/2}))$ is the smallest ideal $I \supset C_2$ for which $P \in [I,I]$, and $\bigcup_{n=1}^{\infty} (\Sigma_1^n I((n^{-1/2})))$ is the smallest ideal I such that $[I,I] = I^2$. Analogously, we have tried to find ideals $I \subset C_2$, as close to C_2 as we could get, for which $[I,I] = (I^2)^\circ$. It appears somewhat likely that the ideal I_o of Section 4 (p. 99) is the largest ideal $I \subset C_2$ for which $(I_o^2)^\circ \subset [C_2, C_2]$. As yet, we have no well-defined candidate for the largest ideal $I \subset C_2$ such that $(I^2)^\circ = [I,I]$.

This concludes Section 9 and Chapter 1.

CHAPTER 2

IDEALS AND THE FUGLEDE COMMUTATIVITY THEOREM

In this chapter we introduce a generalization of the Fuglede Commutativity Theorem which involves ideals of operators. This generalization holds for some ideals but fails to hold for others. We shall give cases where it holds, one case where it fails, and cases where we do not know. We shall also establish a link between this property and the perturbation theory of normal operators.

Let us, for the sake of completeness, state and prove the Fuglede-Putnam generalization in a general B*-algebra setting using an ingenious proof due to Marvin Rosenblum (for the reference see [18, p. 356]). We will then state the Fuglede-Putnam generalization and the Fuglede Commutativity Theorem in a corollary. But first we need some definitions.

Definitions 1. If X is a B*-algebra and $a \in X$, then $\underline{a}$ is said to be self-adjoint if $a = a^*$, and $\underline{a}$ is said to be normal if $a^*a = aa^*$.

Theorem 2. If X is a B*-algebra with identity 1, and if $a_1, a_2 \in X$ are normal, then for $x \in X$, $a_1 x = x a_2$ implies $a_1^* x = x a_2^*$.

Proof. (Rosenblum). If $a_1 x = x a_2$ then induction shows that $a_1{}^n x = x a_2{}^n$ for every $n \in Z^+$. It is well-known that for every $a \in X$, $e^a = \Sigma a^n/n!$ converges in the norm of the B*-algebra, and also that e^{-a} is the inverse of e^a. Therefore, $e^{i\bar{z}a_1}x = xe^{i\bar{z}a_2}$ for every complex number z, and so $x = e^{i\bar{z}a_1}xe^{-i\bar{z}a_2}$ for every complex number z.

Define $F(z) = e^{iza_1^*}xe^{-iza_2^*}$, which maps the set of complex numbers into X. It is clear that F(z) is analytic in the entire complex plane. By the last remark in the preceding paragraph,

$$F(z) = e^{iza_1^*}xe^{-iza_2^*} = e^{iza_1^*}e^{i\bar{z}a_1}xe^{-i\bar{z}a_2}e^{-iza_2^*}$$

$$= e^{i(za_1^* + \bar{z}a_1)}xe^{-i(za_2^* + \bar{z}a_2)} .$$

Note that $za_1^* + \bar{z}a_1$ and $-(za_2^* + \bar{z}a_2)$ are self-adjoint.

We assert that if $h \in X$ is self-adjoint, then e^{ih} is unitary (i.e. $(e^{ih})^* = (e^{ih})^{-1}$) and $\|e^{ih}\| = 1$. If $u \in X$ is unitary, that is, if $u^* = u^{-1}$, then

$$\|u\|^2 = \|u^*u\| = \|u^{-1}u\| = \|1\| = 1 ,$$

and so $\|u\| = 1$. Therefore all we need to show is that e^{ih} is unitary. But by definition,

$$(e^{ih})^* = (\Sigma (ih)^n/n!)^* = \Sigma ((ih)^*)^n/n! = \Sigma (-ih)^n/n! = e^{-ih}.$$

By multiplying the corresponding infinite series and using the fact that they converge absolutely, one can easily show that $e^{ih}e^{-ih} = 1 = e^{-ih}e^{ih}$. Hence $(e^{ih})^* = e^{-ih} = (e^{ih})^{-1}$.

Therefore we obtain

$$\|F(z)\| = \|e^{i(za_1^* + \bar{z}a_1)}xe^{-i(za_2^* + \bar{z}a_2)}\|$$

$$\leq \|e^{i(za_1^* + \bar{z}a_1)}\| \|x\| \|e^{i(-(za_2^* + \bar{z}a_2))}\|$$

$= \|x\|$.

Hence F(z) is bounded in the X-norm and analytic in the entire plane. It is well-known that Liouville's Theorem applies in this setting to yield $F(z) = F(0) = x$ for every complex number z. In other words, we have that $e^{iza_1^*} x e^{-iza_2^*} = x$, thus $e^{iza_1^*} x = x e^{iza_2^*}$, for every complex number z. Differentiating both sides with respect to z, we obtain $(ia_1^* e^{iza_1^*}) x = x (ia_2^* e^{iza_2^*})$ for all z. Setting z = 0 and dividing by i, we obtain $a_1^* x = x a_2^*$.

Q.E.D.

<u>Corollary 3</u>. If $A, B \in L(H)$ and A is normal, then $AB = BA$ implies $A^*B = BA^*$ (the Fuglede Commutativity Theorem). If $A_1, A_2, B \in L(H)$ and A_1 and A_2 are normal, then $A_1B = BA_2$ implies $A_1^*B = BA_2^*$ (the Fuglede-Putnam generalization).

<u>Proof</u>. Fuglede's Commutativity Theorem follows from Theorem 2 by setting $X = L(H)$, $a_1 = a_2 = A$ and $x = B$.

The Fuglede-Putnam generalization follows from Theorem 2 by setting $X = L(H)$, $a_1 = A_1$, $a_2 = A_2$ and $x = B$.

<u>Note</u>. Theorem 2 appears to be more general than the Fuglede-Putnam generalization. This is not the case. Indeed, the Gelfand-Naimark Theorem in the non-commutative case tells us that if X is a B*-algebra with an identity, then X is *-isometrically isomorphic to a uniformly closed

subalgebra of L(H), in which the Fuglede-Putnam generalization applies. Hence, Theorem 2 is equivalent to the Fuglede-Putnam generalization.

Note also that these results and proofs hold in the cases when H is non-separable.

We now introduce some possible generalizations of the Fuglede Commutativity Theorem and the Fuglede-Putnam generalization. Our idea is as follows. One way to state the Fuglede Commutativity Theorem is: "If $A,B \in L(H)$ and A is normal, then $AB - BA \in \{0\}$ implies $A^*B - BA^* \in \{0\}$". It is then natural to ask whether or not the theorem remains true if we replace $\{0\}$ by some other ideal I. That is, does it hold true that 'If $A,B \in L(H)$ and A is normal, then $AB - BA = 0 (\mathrm{mod}\, I)$ implies $A^*B - BA^* = 0 (\mathrm{mod}\, I)$'? It is also natural to ask whether or not one could replace the normality of A in the hypothesis by the normality of A modulo (I^2) (i.e. $A^*A - AA^* = 0(\mathrm{mod}\,(I^2))$). Thus we make the following definitions.

<u>Definition 4</u>. The operator A is said to be <u>normal (I)</u> provided $A^*A - AA^* \in I$.

<u>Definitions 5</u>. Let I be an ideal in L(H).

(1) I has the <u>Generalized Fuglede Property</u> (GFP) if whenever $A,B \in L(H)$ and A is normal, then $AB - BA \in I$ implies $A^*B - BA^* \in I$.

(2) I has the <u>GFP (I)</u> if whenever $A,B \in L(H)$ and A is normal (I^2), then $AB - BA \in I$ implies $A^*B - BA^* \in I$.

(3) I has the <u>Generalized Fuglede-Putnam Property</u> (GFPP) if whenever $A_1, A_2, B \in L(H)$ and A_1 and A_2 are normal, then $A_1B - BA_2 \in I$ implies $A_1^*B - BA_2^* \in I$.

(4) I has the <u>GFPP</u>(I) if whenever $A_1, A_2, B \in L(H)$ and A_1 and A_2 are normal(I^2), $A_1B - BA_2 \in I$ implies $A_1^*B - BA_2^* \in I$.

These four properties are clearly related, as we shall see in the next proposition. The GFP is the simplest and most essential one, especially in view of this proposition.

<u>Proposition 6</u>. If I is an ideal in L(H), then

(2) <==> (4) ==> (3) <==> (1).

<u>Proof</u>. That (4) ==> (2) ==> (1) and (3) ==> (1) are obvious. Therefore all we need show are that (2) ==> (4) and (1) ==> (3).

To see that (1) ==> (3), let $A_1, A_2, B \in L(H)$ for which $A_1B - BA_2 \in I$ and assume (1) holds. Let $A = M(A_1, 0, 0, A_2)$ and $B = M(0, B, 0, 0)$, so $A, B \in L(H \oplus H)$. (Recall that $M(T_{11}, T_{12}, T_{21}, T_{22})$ denotes the 2x2 matrix with operator entries T_{ij} in the (i,j) position.). It is easy to verify that A is normal, and that $AB - BA = M(0, A_1B - BA_2, 0, 0) \in I$ (considered as an ideal in L(H) under our usual identification) and $A^*B - BA^* = M(0, A_1^*B - BA_2^*, 0, 0)$. But if (1) holds, then I has the GFP. It follows that I, considered in $L(H \oplus H)$, has the GFP. Therefore since $AB - BA \in I$, we obtain $A^*B - BA^* = M(0, A_1^*B - BA_2^*, 0, 0) \in I$. Therefore, $A_1^*B - BA_2^* \in I$, considered as an ideal in L(H). That is, (3) holds.

To see that (2) $\Longrightarrow$ (4), apply the same proof as in the case (1) $\Longrightarrow$ (3). Q.E.D.

Therefore we shall only consider the properties GFP and GFP(I).

There is an interesting analogue to the GFP regarding AB instead of AB-BA, and this analogous property holds for every ideal and any Hilbert space of arbitrary dimension. Certainly questions about AB should be considered before the corresponding questions about AB-BA. Fortunately, the answers in this case are elementary and complete. They are given in the next theorem.

Also, this theorem indicates why we choose the hypothesis "A is normal (I^2)" instead of "A is normal (I)" in Definitions 5-2 and 5-4 above.

Theorem 7. Let I be an ideal in L(H).

(a) If $A,B \in L(H)$ and A is normal, then $AB \in I$ implies $A^*B \in I$. Furthermore, if I is a normed ideal such that $\|T\|_I = \| |T| \|_I$ for every $T \in I$, then $AB \in I$ implies $A^*B \in I$ and also $\|AB\|_I = \|A^*B\|_I$.

(b) If $A,B \in L(H)$ and A is normal(I^2), then $AB \in I$ implies $A^*B \in I$.

Proof. Note that

$$|AB| = [(AB)^*(AB)]^{1/2} = [B^*A^*AB]^{1/2} = [B^*AA^*B]^{1/2}$$
$$= [(A^*B)^*(A^*B)]^{1/2} = |A^*B|,$$

since A is normal. That is, if A is normal and $A,B \in L(H)$, then $|AB| = |A^*B|$. Therefore

$AB \in I \Longleftrightarrow |AB| \in I \Longleftrightarrow |A^*B| \in I \Longleftrightarrow A^*B \in I$.

Also $\|AB\|_I = \| |AB| \|_I = \| |A^*B| \|_I = \|A^*B\|_I$. This proves (a).

If $A, B \in L(H)$ and A is normal(I^2), then $A^*A - AA^* = K$ where $K \in I^2$. Hence $|AB|^2 - |A^*B|^2 = B^*(A^*A - AA^*)B = B^*KB \in I^2$. Therefore $|AB|^2 \in I^2$ if and only if $|A^*B|^2 \in I^2$, and so $|AB| \in I$ if and only if $|A^*B| \in I$. Q.E.D.

Note. Theorem 7-b becomes blatantly false if we replace the hypothesis "A is normal(I^2)" by "A is normal(I)". Indeed, let $I = C_1$, $T \in C_2 \setminus C_1$ and let $A = M(0,T^*,0,0)$ and $B = M(1,0,0,0)$ where 1 denotes the identity operator. A simple computation yields $A^*A = M(0,0,0,TT^*) \in C_1$ and $AA^* = M(T^*T,0,0,0) \in C_1$, and so $A^*A - AA^* \in C_1 = I$. That is, A is normal(C_1). Also $AB = 0 \in C_1$ but yet $A^*B = M(0,0,T,0) \notin C_1$, since $T \in C_2 \setminus C_1$.

The next theorem states all that we know about which ideals fully or partially have the GFP, except for a recent advance mentioned in the remark following the theorem. With minor modifications, the theorem holds true for any Hilbert space of arbitrary dimension.

For one of the cases in the theorem, we need the following definition.

Definition. An operator T is called a kernel operator if it can be put into the form

$$(Tf)(x) = \int K(x,y) f(y)\, d\mu(y),$$

where $f \in L^2(\mu)$ and, for μ-almost every x, $K(x,\cdot) \in L^2(\mu)$.

Theorem 8.

(a) $\{0\}$ has the GFP($\{0\}$) (Fuglede's Commutativity Theorem).

(b) K(H) has the GFP(K(H)).

(c) If A is diagonalizable and $AB - BA \in C_2$, then $A^*B - BA^* \in C_2$ and $\|AB - BA\|_{C_2} = \|A^*B - BA^*\|_{C_2}$. In particular, if A is a compact, normal operator and B is a bounded operator for which $AB - BA \in C_2$, then $A^*B - BA^* \in C_2$ and $\|AB - BA\|_{C_2} = \|A^*B - BA^*\|_{C_2}$.

(d) If A is a multiplication operator on $L^2(\mu)$ and B is a kernel operator on $L^2(\mu)$ for which $AB - BA \in C_2$, then $A^*B - BA^* \in C_2$ and $\|AB - BA\|_{C_2} = \|A^*B - BA^*\|_{C_2}$.

(e) If $B \in C_2$ and A is normal, then $AB - BA$ and $A^*B - BA^* \in C_2$ such that $\|AB - BA\|_{C_2} = \|A^*B - BA^*\|_{C_2}$.

(f) The ideal F(H) of finite rank operators does not have the GFP.

Proof. (a). This is precisely the Fuglede Commutativity Theorem.

(b). Apply Theorem 2 setting $X = L(H)/K(H)$.

(c). In some basis, $A = D(z_1, z_2, \ldots)$ where (z_n) is a bounded, complex-valued sequence. As we mentioned in Section 8 (p. 125), by computing, we obtain $AB - BA = ((z_i - z_j) b_{ij})$, where $B = (b_{ij})$ is the matrix representation of B with respect to this basis. Hence

$$\|AB - BA\|_{C_2}^2 = \Sigma_{i,j=1}^{\infty} |(z_i - z_j) b_{ij}|^2 = \Sigma_{i,j=1}^{\infty} |(\bar{z}_i - \bar{z}_j) b_{ij}|^2$$
$$= \|A^*B - BA^*\|_{C_2}^2 .$$

Therefore if $\|AB - BA\|_{C_2} < \infty$, then $\|A^*B - BA^*\|_{C_2} < \infty$

and these norms are equal.

If A were compact and normal, then the spectral theorem would yield that A is diagonalizable.

(d). If A is a multiplication operator on $L^2(\mu)$, then for some $h(x) \in L^\infty(\mu)$, $Af = hf$ for every $f \in L^2(\mu)$. If B is a kernel operator on $L^2(\mu)$, then for some $K(x,y) \in L^2(\mu(y))$ μ-almost everywhere we have that $(Bf)(x) = \int K(x,y)\,f(y)\,d\mu(y)$ for every $f \in L^2(\mu)$. Therefore $(ABf)(x) = \int K(x,y)\,h(x)\,f(y)\,d\mu(y)$ and $(BAf)(x) = \int K(x,y)\,h(y)\,f(y)\,d\mu(y)$. Whence $((AB-BA)\,f)\,(x) = \int K(x,y)\,(h(x)-h(y))\,f(y)\,d\mu(y)$. Similarly $((A^*B-BA^*)\,f)\,(x) = \int K(x,y)(\overline{h(x)} - \overline{h(y)})\,f(y)\,d\mu(y)$. It is well-known that since AB-BA and A*B-BA* are themselves kernel operators with kernels $K(x,y)(h(x)-h(y))$ and $K(x,y)(\overline{h(x)}-\overline{h(y)})$, respectively, we must have

$$\|AB - BA\|_{C_2}^2 = \iint |K(x,y)(h(x) - h(y))|^2 \, d\mu(x)\,d\mu(y)$$

$$= \iint |K(x,y)(\overline{h(x)} - \overline{h(y)})|^2 \, d\mu(x)\,d\mu(y)$$

$$= \|A^*B - BA^*\|_{C_2}^2 .$$

The rest of the proof of (d) is clear.

(e). If A is normal, it is well-known that there exists a sequence (A_n) of diagonalizable operators for which $\|A - A_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, where the norm is the uniform operator norm. But then by part (c) above, we have $\|A_nB - BA_n\|_{C_2} = \|A_n^*B - BA_n^*\|_{C_2}$. This gives us the following inequalities.

$$\left| \|AB - BA\|_{C_2} - \|A^*B - BA^*\|_{C_2} \right| \leq$$

$$\left| \|AB-BA\|_{C_2} - \|A_nB-BA_n\|_{C_2} \right| + \left| \|A_nB-BA_n\|_{C_2} - \|A_n^*B-BA_n^*\|_{C_2} \right|$$

$$+ \left| \|A_n^*B-BA_n^*\|_{C_2} - \|A^*B-BA^*\|_{C_2} \right|$$

$$\leq \|(AB-BA) - (A_nB-BA_n)\|_{C_2} + 0 + \|(A_n^*B-BA_n^*) - (A^*B-BA^*)\|_{C_2}$$

$$= \|(A-A_n)B-B(A-A_n)\|_{C_2} + \|(A_n-A)^*B - B(A_n-A)^*\|_{C_2}$$

$$\leq 4\|A-A_n\| \|B\|_{C_2} \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Therefore $\|AB - BA\|_{C_2} = \|A^*B - BA^*\|_{C_2}$.

(f). To show that F(H) does not have the GFP is the most difficult part of the theorem. It requires an intricate construction. Our strategy is as follows.

We shall produce trace class diagonal operators D_1 and D_2 and a matrix $B_1 \in C_2$ such that $D_1B_1 - B_1D_2$ has rank 1 and $D_1^*B_1 - B_1D_2^*$ has infinite rank. Assuming that we can do this, if we let $A = M(D_1,0,0,D_2)$ and $B = M(0,B_1,0,0)$, we obtain

$$AB - BA = M(0, D_1B_1 - B_1D_2, 0, 0) \quad \text{and}$$

$$A^*B - BA^* = M(0, D_1^*B_1 - B_1D_2^*, 0, 0) \quad ,$$

and clearly A is normal. It is obvious that for every T, the rank of T is the same as the rank of M(0,T,0,0). Therefore the rank of AB - BA is the same as the rank of $D_1B_1 - B_1D_2$, which is 1, but the rank of $A^*B - BA^*$ is the same as the rank of $D_1^*B_1 - B_1D_2^*$, which is infinite. Therefore the proof will be complete when we construct D_1, D_2 and B_1.

Let $D_1 = D(1/2,1/4,1/8,\dots)$. Clearly $D_1 \in C_1$. Formally let $D_2 = D(z_1,z_2,\dots)$ and $B_1 = (b_{ij})$ where $b_{ij} = 4^{-(i+j)}(2^{-i} - z_j)^{-1}$. We shall choose the sequence (z_n) inductively to satisfy several properties. First of all, we want to be sure that $|2^{-i} - z_j|$ is large enough to insure that $B_1 \in C_2$, that is, $b_{ij} \in \ell^2(i,j)$. Secondly, we need to be sure that $|z_n|$ is small enough to insure that $D_2 \in C_1$.

In the first place, if we can choose (z_n) to be in the closed, left half-plane, then $|2^{-i} - z_j| \geq 2^{-i}$, and so $|b_{ij}| \leq 4^{-(i+j)}2^i = 2^{-i}4^{-j} \in \ell^2(i,j)$.

In the second place, we can insure that $D_2 \in C_1$ if we can choose (z_n) such that $|z_n| \leq 2^{-n}$.

In regard to the rank requirements on $D_1B_1 - B_1D_2$ and $D_1^*B_1 - B_1D_2^*$, note that

$$(D_1B_1 - B_1D_2)(i,j) = (2^{-i} - z_j)4^{-(i+j)}(2^{-i} - z_j)^{-1}$$
$$= 4^{-(i+j)},$$

and so the range of $D_1B_1 - B_1D_2$ is the 1-dimensional subspace spanned by the vector $(1,1/4,1/16,\dots)$. Therefore, since

$$(D_1^*B_1 - B_1D_2^*)(i,j) = 4^{-(i+j)}\left((2^{-i} - \bar{z}_j)/(2^{-i} - z_j) \right),$$

it is clear that $D_1^*B_1 - B_1D_2^*$ has an infinite dimensional range provided that we choose the sequence (z_n) with one additional property. Namely, for each positive integer N, the N vectors given by $\left(4^{-(i+j)}((2^{-i}-\bar{z}_j)/(2^{-i}-z_j)) \right)_{i=1}^{\infty}$ for $1 \leq j \leq N$ form a linearly independent set. Clearly, for this to hold, it is sufficient that the N vectors given by $\left(4^{-(i+j)}(2^{-i}-\bar{z}_j)/(2^{-i}-z_j) \right)_{i=1}^{N}$ for $1 \leq j \leq N$ be linearly independent in $\mathbb{C}^n$; equivalently, that the determinant of the NxN matrix $(4^{-(i+j)}((2^{-i}-\bar{z}_j)/(2^{-i}-z_j)))_{i,j=1}^{N}$ be nonzero. Hence, it is sufficient to prove that there exists a sequence (z_n) in the closed, left half-plane for which $|z_n| \leq 2^{-n}$ for every n, and for which $\det ((4^{-(i+j)}((2^{-i}-\bar{z}_j)/(2^{-i}-z_j)))_{i,j=1}^{N}) \neq 0$ for every N. We prove this by induction.

The first possible case where this could hold is the case $N = 2$. For this case, let $z_1 = 0$ and $z_2 = i/2$. Then z_1 and z_2 are in the closed, left half-plane; they satisfy $|z_n| \leq 2^{-n}$; and

$$\det ((4^{-(i+j)}((2^{-i}-\bar{z}_j)/(2^{-i}-z_j)))_{i,j=1}^{2}) =$$

$$\det (M(1/16, i/64, 1/64, (1/256)((2+i)/(2-i))) =$$

$$4^{-4}((1-i)/(2-i)) \neq 0 .$$

For the general case, assume $(z_n)_{n=1}^{N}$ has been chosen to satisfy the induction hypothesis. Let z_{N+1} denote the free complex variable which ranges over the intersection of the open, left half-plane and the open disc $[\,|z| < 2^{-(N+1)}]$. Then

$$f(z_{N+1},\bar{z}_{N+1}) = \det ((4^{-(i+j)}((2^{-i}-\bar{z}_j)/(2^{-i}-z_j)))_{i,j=1}^{N+1})$$

$$= \Sigma_{i=1}^{N+1} (-1)^{N+1-i} D_i 4^{-(i+N+1)}((2^{-i}-\bar{z}_{N+1})/(2^{-i}-z_{N+1})) ,$$

where D_i is the subdeterminant of the $(i,N+1)$ entry which is $4^{-(i+N+1)}((2^{-i}-\bar{z}_{N+1})/(2^{-i}-z_{N+1}))$. Clearly, by inspection and the induction hypothesis,

$$D_{N+1} = \det ((4^{-(i+j)}((2^{-i}-\bar{z}_j)/(2^{-i}-z_j)))_{i,j=1}^{N}) \neq 0.$$

If we let $a_i = (-1)^{N+1-i} D_i 4^{-(i+N+1)}$ and $z = z_{N+1}$, then it suffices to show that there exists a z contained in the intersection of the open, left half-plane and the open disc $[|z| < 2^{-(N+1)}]$ such that

$$f(z,\bar{z}) = \Sigma_{i=1}^{N+1} a_i ((2^{-i}-\bar{z})/(2^{-i}-z)) \neq 0 .$$

To see that such a complex number z exists, suppose to the contrary that $f(z,\bar{z}) = 0$ in this region. Taking the $\bar{z}$ derivative of both sides of this equation, we obtain

$$0 = f_{\bar{z}}(z,\bar{z}) = \Sigma_{i=1}^{N+1} -a_i/(2^{-i}-z) ,$$

for every z in this region. However, $a_{N+1} = D_{N+1} 4^{-2(N+1)} \neq 0$, and so $f_{\bar{z}}(z,\bar{z}) = \Sigma_{i=1}^{N+1} -a_i/(2^{-i}-z)$ becomes unbounded in the open disc $[|z| < 2^{-(N+1)}]$ near the point $z = 2^{-(N+1)}$. Therefore $f_{\bar{z}}(z,\bar{z})$ is not identically 0 in this open disc. But $f_{\bar{z}}(z,\bar{z})$ is clearly analytic in the open disc. This implies that since $f_{\bar{z}}(z,\bar{z}) = 0$ throughout the intersection of the open, left half-plane and the open disc, we must have that $f_{\bar{z}}(z,\bar{z})$ is identically 0 in this open disc, which is a contradiction. This completes the proof of (f). Q.E.D.

Note. Another proof of (e) follows from (d) because every Hilbert-Schmidt operator is a kernel operator in any $L^2(\mu)$-representation of the Hilbert space.

Concerning (f), F(H) was, until recently (see the next remark), the only ideal we knew which fails to have the GFP. It is perhaps the most significant part of the theorem. When we originally discovered the proof of (f), we had first asked and answered the following question. Do there exist $A,B \in L(H)$ where A is normal and the rank of $AB - BA$ is 1, but the rank of A^*B-BA^* is 2? We found that if $\dim H \leq 3$, then the rank of $AB-BA$ is the same as the rank of A^*B-BA^*. However, in the case when $\dim H = 4$, the construction in the proof of (f) answers this question in the affirmative (i.e. set $D_1 = D(1/2,i/4)$, $D_2 = D(0,i/2)$, $B_1 = M(1/8,1/32(1-i),$ $1/16,1/128(1-2i))$, $A = D_1 \oplus D_2$, and $B = M(0,B_1,0,0)$).

Also note that the proof might have been slightly easier had we not insisted that $A \in C_1$ and $B \in C_2$, but we preferred to give the most general counter-example that our construction could reasonably provide.

Remark. We have also recently shown, independently of this thesis, that C_p $(0 < p < 1)$ does not have the GFP.

Theorem 8 leaves the following questions open, which are especially important to this theory.

Question 9. Does C_2 have the GFP?

Question 10. Is there any ideal, other than the uniformly closed ideals (in the case when H is separable, other than L(H) and K(H)), which possesses the GFP? Are there any ideals, other than F(H) and C_p $(0 < p < 1)$, which fail to have the GFP?

Question 11. If A is a diagonal operator and $AB-BA \in C_1$, must $A^*B-BA^* \in C_1$?

Question 11 is the statement of Theorem 8-c replacing C_2 by C_1. Question 10 indicates how incomplete this theory is, at present. One of the difficulties one encounters here is in deciding when an operator is or is not in a particular ideal. Question 9 is the one of most importance. It has an interesting connection to an important unsolved problem in perturbation theory due to I.D. Berg [3] and P.R. Halmos. We shall describe the problem and the connection in the following paragraphs.

'Does C_2 have the GFP?' and a problem of Halmos and Berg.

In 1909 Hermann Weyl [37] proved that each bounded, self-adjoint operator acting on a separable Hilbert space can be written as the sum of a diagonalizable operator and a compact operator. In 1935 J. von Neumann [36] proved that the compact operator can actually be chosen to be Hilbert-Schmidt and also observed that boundedness is unnecessary (in fact, the theory can be used to show that for each $p > 1$, the compact operator can chosen to be in C_p). P.R. Halmos [19] inquired whether or not the Weyl result and the von Neumann result can be extended to normal operators. I.D. Berg [3] later proved that the Weyl result can be so extended, but he left unsolved whether or not the von Neumann result can be so extended. That is, Berg proved that every normal operator is the sum of a diagonalizable operator and a compact operator, but he left unsolved whether or not the compact operator can be chosen to be Hilbert-Schmidt. Berg did show, however, that the compact operator can be chosen to have arbitrarily small uniform norm, and in some special cases (e.g. when its spectrum is contained in a rectifiable curve), the compact operator can be chosen to be Hilbert-Schmidt with an arbitrarily small Hilbert-Schmidt norm. It is interesting to note that it has been known for some time that not every self-adjoint operator is the sum of a diagonalizable operator and a trace class operator. J.H. Anderson gave a nice proof of this in [1], and we give a variation of this as follows.

Let U be the unilateral shift of multiplicity 1 and let $S = U + U^*$. Clearly S is self-adjoint. We assert that S is not the sum of a diagonalizable operator and a trace class operator. To see this, suppose to the contrary that $S = D + T$ where D is diagonalizable and $T \in C_1$. A computation obtains

$$P = (U+U^*)U - U(U+U^*) = SU - US = (DU-UD) + (TU-UT) .$$

In a basis which diagonalizes D, the diagonal entries of $DU-UD$ are all 0 (see p. 125). But $T \in C_1$ implies that $\mathrm{Tr}(TU-UT) = 0$ (Chapter 1, Proposition 4.1). Therefore, in the basis which diagonalizes D, the sum of the diagonal entries of $(DU-UD) + (TU-UT)$ is 0. But $\mathrm{Tr}(P) = 1$ and is independent of basis. Hence we have a contradiction.

The connection between the GFP with respect to C_2, and the unsolved problem of whether or not every normal operator is the sum of a diagonalizable operator and a Hilbert-Schmidt operator, is described as follows. We assert that if it is true that 'every normal operator $A = D + T$ for some diagonalizable operator and some $T \in C_2$', then C_2 must have the GFP. Furthermore, if it is true that this decomposition can be made with a T of arbitrarily small Hilbert-Schmidt norm (as Berg [3] shows in special cases), then if A is normal and $AB-BA \in C_2$, it follows that $A^*B-BA^* \in C_2$ and $\|AB-BA\|_{C_2} = \|A^*B-BA^*\|_{C_2}$. The proof goes as follows.

If A is normal, then by assumption, $A = D + T$ for some diagonalizable operator D and some $T \in C_2$. Hence if $AB-BA \in C_2$, then $DB-BD + TB-BT \in C_2$. Since $T \in C_2$, we have $DB-BD \in C_2$. By Theorem 8-c, $D^*B-BD^* \in C_2$, and so $A^*B-BA^* = D^*B-BD^* + T^*B-BT^* \in C_2$. This proves the first assertion. If furthermore $A = D_n + T_n$ where, for each n, D_n is diagonalizable and $T_n \in C_2$ such that $\|T_n\|_{C_2} < 1/n$, and if $AB-BA \in C_2$, then $D_nB-BD_n \in C_2$, $D_n^*B-BD_n^* \in C_2$, and $\|D_nB-BD_n\|_{C_2} = \|D_n^*B-BD_n^*\|_{C_2}$. Therefore

$$\left| \|AB-BA\|_{C_2} - \|A^*B-BA^*\|_{C_2} \right| \leq$$

$$\left| \|AB-BA\|_{C_2} - \|D_nB-BD_n\|_{C_2} \right| + \left| \|D_nB-BD_n\|_{C_2} - \|D_n^*B-BD_n^*\|_{C_2} \right|$$

$$+ \left| \|D_n^*B-BD_n^*\|_{C_2} - \|A^*B-BA^*\|_{C_2} \right|$$

$$\leq \|(A-D_n)B - B(A-D_n)\|_{C_2} + 0 + \|(A-D_n)^*B - B(A-D_n)^*\|_{C_2}$$

$$\leq 4\,\|T_n\|_{C_2}\|B\| \leq 4\,\|B\|/n \quad , \text{ for every } n.$$

Hence $\|AB-BA\|_{C_2} = \|A^*B-BA^*\|_{C_2}$.

Berg states in [3] that he believes that not every normal operator is the sum of a diagonalizable operator and a Hilbert-Schmidt operator, and he suggests a candidate for such a normal operator. It is multiplication by z operating on $L^2(\mu)$ where μ is Lebesgue measure on the unit square in the plane. If Berg is right in that not every normal operator can be so decomposed, then a possible strategy or test for showing this is to prove that C_2 does not have the GFP. If he is right in that the multiplication operator above cannot be so decomposed, then if we call this operator A, perhaps one could find a $B \in L(H)$ such that $AB-BA \in C_2$, but yet $A^*B-BA^* \notin C_2$. Then by a preceding remark, A could not be the sum of a diagonalizable and a Hilbert-Schmidt operator. The advantage of this test is that for each A and B, it is almost purely computational, and the C_2-norm is the easiest of norms to compute. The difficult part is in locating a proper B which is bounded.

One might suspect that instead of using AB-BA as a test, one could use AB instead. That is, if A is normal and $A = D + T$ as above, then $AB \in C_2 \Rightarrow A^*B \in C_2$. This can be proved from the fact that $\|DB\|_{C_2} = \|D^*B\|_{C_2}$, the proof of which is similar to that of Theorem 8-c. Hence, if A is normal, $AB \in C_2$, and $A^*B \notin C_2$, then we would have $A \neq D + T$. However Theorem 7 will not allow the situation that $AB \in C_2$ but yet $A^*B \notin C_2$. It is quite possible that there is a similar theorem for AB-BA. In other words, C_2 may have the GFP but still it may not be true that every normal operator is decomposable as above. In the next paragraphs, we show a curious fact by an intricate computation which might be construed as evidence that C_2 possesses the GFP and has the C_2-norm preserving property.

<u>Heuristics</u>. There are some quantities in the theory of infinite matrices which, when formally computed (in the sense that the order of summation of doubly infinite series are reversed without regard to validity), behave exactly as one would naively expect. A prime example of this is the following.

If $A = (a_{ij})$ and $B = (b_{ij})$ in a particular basis, then $(AB-BA)(i,i) = \Sigma_k(a_{ik}b_{ki} - b_{ik}a_{ki})$.

But then the formal trace of AB-BA is given by

$$\Sigma_i\Sigma_k\ (a_{ik}b_{ki} - b_{ik}a_{ki}) = \Sigma_i\ \Sigma_k\ a_{ik}b_{ki} - \Sigma_i\ \Sigma_k\ a_{ki}b_{ik}$$

$$= \Sigma_i\ \Sigma_k\ a_{ik}b_{ki} - \Sigma_k\ \Sigma_i\ a_{ki}b_{ik}$$

$$= \Sigma_i\ \Sigma_k\ a_{ik}b_{ki} - \Sigma_{i'}\Sigma_{k'}a_{i'k'}b_{k'i'}$$

$$= 0\ ,$$

where the first equality is treated as a formal difference, the second equality follows formally by reversing the order of summation in the second sum, the third equality follows rigorously by substituting i' = k and k' = i in the second sum, and the last equality is obvious. In other words, formally the trace of AB-BA is always 0. Of course this is, in fact, blatantly false. A good example to see what is going on is to analyze this argument in regard to the equation $P = U^*U - UU^*$ where U is the unilateral shift of multiplicity 1.

A more involved argument of the same kind can be used to formally show that if A is normal and $AB \in C_2$, then $A^*B \in C_2$ and $\|AB\|_{C_2} = \|A^*B\|_{C_2}$. However we already know that this is true by Theorem 7.

We shall now formally show that if $AB-BA \in C_2$, then $A^*B-BA^* \in C_2$ and $\|AB-BA\|_{C_2} = \|A^*B-BA^*\|_{C_2}$.

Let $A = (a_{ij})$ and $B = (b_{ij})$ in some basis. Then $(AB-BA)(i,j) = \Sigma_k\, (a_{ik}b_{kj} - b_{ik}a_{kj})$. Then

$$\|AB-BA\|_{C_2}{}^2 = \Sigma_{i,j}\left|\Sigma_k\, (a_{ik}b_{kj} - b_{ik}a_{kj})\right|^2$$

$$= \Sigma_{i,j}(\Sigma_k(a_{ik}b_{kj}-b_{ik}a_{kj}))(\overline{\Sigma_m(a_{im}b_{mj}-b_{im}a_{mj})})$$

$$= \Sigma_{i,j}\, \Sigma_{k,m}\, (a_{ik}b_{kj}-b_{ik}a_{kj})(\bar{a}_{im}\bar{b}_{mj}-\bar{b}_{im}\bar{a}_{mj})$$

$$= \Sigma_{i,j}\, \Sigma_{k,m}\, (a_{ik}\bar{a}_{im}b_{kj}\bar{b}_{mj} + b_{ik}\bar{b}_{im}a_{kj}\bar{a}_{mj} - a_{ik}\bar{a}_{mj}b_{kj}\bar{b}_{im} - b_{ik}\bar{b}_{mj}a_{kj}\bar{a}_{im})\ ,$$

where the first equality follows by definition, the second follows easily by substituting $k = m$ in $\overline{\Sigma_k(a_{ik}b_{kj}-b_{ik}a_{kj})}$, the third equality holds since $\overline{\Sigma z_n} = \Sigma \bar{z}_n$ and also, for each fixed (i,j), $(a_{ik}b_{kj}-b_{ik}a_{kj}) \in \ell^1(k)$, and so $(\Sigma_k(a_{ik}b_{kj}-b_{ik}a_{kj}))(\Sigma_m(\bar{a}_{im}\bar{b}_{mj}-\bar{b}_{im}\bar{a}_{mj})) =$

$$\Sigma_{k,m}(a_{ik}b_{kj}-b_{ik}a_{kj})(\bar{a}_{im}\bar{b}_{mj}-\bar{b}_{im}\bar{a}_{mj}) ,$$

and the fourth equality is obvious. Note that, thus far, our argument is rigorous. If we use the same argument together with the fact that $A^* = (\bar{a}_{ji})$, we obtain

$$\|A^*B-BA^*\|_{C_2}^{\,2} = \Sigma_{i,j}\,\Sigma_{k,m}\,(\bar{a}_{ki}a_{mi}b_{kj}\bar{b}_{mj} + b_{ik}\bar{b}_{im}\bar{a}_{jk}a_{jm} - \bar{a}_{ki}a_{jm}b_{kj}\bar{b}_{im} - b_{ik}\bar{b}_{mj}\bar{a}_{jk}a_{mi}) .$$

That A is normal gives us the series identity

$$(A^*A)(i,j) = \Sigma_k\,\bar{a}_{ki}a_{kj} = (AA^*)(i,j) = \Sigma_k\,a_{ik}\bar{a}_{jk} .$$

From this point on we use some formal, but sometimes invalid identities. From an above equality,

$$\|AB-BA\|_{C_2}^{\,2} = (\Sigma_{i,j}\Sigma_{k,m}a_{ik}\bar{a}_{im}b_{kj}\bar{b}_{mj}) + (\Sigma_{i,j}\Sigma_{k,m}b_{ik}\bar{b}_{im}a_{kj}\bar{a}_{mj}) - (\Sigma_{i,j}\Sigma_{k,m}a_{ik}\bar{a}_{mj}b_{kj}\bar{b}_{im}) - (\Sigma_{i,j}\Sigma_{k,m}b_{ik}\bar{b}_{mj}a_{kj}\bar{a}_{im}) .$$

If in the first two double sums, we use the normality of A and interchange the orders of summation, we obtain

$$\Sigma_{i,j}\,\Sigma_{k,m}\,a_{ik}\bar{a}_{im}b_{kj}\bar{b}_{mj} = \Sigma_j\,\Sigma_{k,m}\,\Sigma_i\,a_{ik}\bar{a}_{im}b_{kj}\bar{b}_{mj}$$

$$= \Sigma_j\,\Sigma_{k,m}\,(\Sigma_i\,a_{ik}\bar{a}_{im})b_{kj}\bar{b}_{mj} = \Sigma_j\,\Sigma_{k,m}\,\overline{(\Sigma_i\,\bar{a}_{ik}a_{im})}b_{kj}\bar{b}_{mj}$$

$$= \Sigma_j\,\Sigma_{k,m}\,\overline{(\Sigma_i\,a_{ki}\bar{a}_{mi})}b_{kj}\bar{b}_{mj} = \Sigma_j\,\Sigma_{k,m}\,\Sigma_i\,\bar{a}_{ki}a_{mi}b_{kj}\bar{b}_{mj}$$

$$= \Sigma_{i,j}\,\Sigma_{k,m}\,\bar{a}_{ki}a_{mi}b_{kj}\bar{b}_{mj} ,$$

and

$$\Sigma_{i,j}\Sigma_{k,m} b_{ik}\bar{b}_{im}a_{kj}\bar{a}_{mj} = \Sigma_i \Sigma_{k,m}\Sigma_j b_{ik}\bar{b}_{im}a_{kj}\bar{a}_{mj}$$

$$= \Sigma_i \Sigma_{k,m} b_{ik}\bar{b}_{im}(\Sigma_j a_{kj}\bar{a}_{mj}) = \Sigma_i \Sigma_{k,m} b_{ik}\bar{b}_{im}(\Sigma_j \bar{a}_{jk}a_{jm})$$

$$= \Sigma_{i,j}\Sigma_{k,m} b_{ik}\bar{b}_{im}\bar{a}_{jk}a_{jm} .$$

Note that these last double sums are the first two double sums in our expression for $\|A^*B-BA^*\|_{C_2}{}^2$. We assert further that the sum of the last two double sums in the expression for $\|AB-BA\|_{C_2}{}^2$ is the same (formally) as that for $\|A^*B-BA^*\|_{C_2}{}^2$. Namely,

$$\Sigma_{i,j}\Sigma_{k,m} (a_{ik}\bar{a}_{mj}b_{kj}\bar{b}_{im} + b_{ik}\bar{b}_{mj}a_{kj}\bar{a}_{im})$$

$$= \Sigma_{i,j}\Sigma_{k,m} (\bar{a}_{ki}a_{jm}b_{kj}\bar{b}_{im} + b_{ik}\bar{b}_{mj}\bar{a}_{jk}a_{mi}) .$$

That this holds follows from the following equations: if we make the substitutions $i \longrightarrow m \longrightarrow j \longrightarrow k \longrightarrow i$, we obtain

$$\Sigma_{i,j}\Sigma_{k,m} a_{ik}\bar{a}_{mj}b_{kj}\bar{b}_{im} = \Sigma_{m,k}\Sigma_{i,j} b_{ik}\bar{b}_{mj}\bar{a}_{jk}a_{mi} ;$$

if we make the substitutions $i \longrightarrow k \longrightarrow j \longrightarrow m \longrightarrow i$, we obtain

$$\Sigma_{i,j}\Sigma_{k,m} b_{ik}\bar{b}_{mj}a_{kj}\bar{a}_{im} = \Sigma_{k,m}\Sigma_{j,i} \bar{a}_{ki}a_{jm}b_{kj}\bar{b}_{im} .$$

Hence the expression for $\|AB-BA\|_{C_2}{}^2$ is the same 'formally' as the expression for $\|A^*B-BA^*\|_{C_2}{}^2$. This demonstrates what we wish to show.

In summary, we have seen that such a heuristic argument shows something which is actually false (i.e. $Tr(AB-BA) = 0$ for every $A,B \in L(H)$); it shows something that is true (i.e. if A is normal and $AB \in C_2$, then $A^*B \in C_2$ and $\|AB\|_{C_2} = \|A^*B\|_{C_2}$); and it leaves Question 9 unsolved, as far as we know. We tend to believe that C_2 does not have the GFP. However if C_2 does have the GFP, then our formal argument might play a role in a rigorous proof.

Another Strategy to the problem of Halmos and Berg. We wish to suggest another possible strategy for showing that not every normal operator is the sum of a diagonalizable operator and a Hilbert-Schmidt operator.

1. Choose an operator $S \in C_2$ such that the diagonal of S in any basis is not contained in ℓ^1. The diagonal operator $D((1/n))$ is an example of such an operator (see Chapter 3, Section 2). (Note that Deckard and Pearcy have an opposite result, namely, that for each operator in C_1^o, there exists a basis in which its diagonal entries are all 0.).

2. Find a normal operator N and an operator $X \in C_2$ such that $NX-XN = S$.

3. Conclusion: $N \neq D + T$ where D is diagonalizable and $T \in C_2$. This holds because if $N = D + T$ as above, then $S = NX-XN = DX-XD + TX-XT$ and the diagonal of the right-hand side, in that basis which diagonalizes D and hence in which the diagonal entries of $DX-XD$ are all 0, is contained in ℓ^1. This gives a contradiction to 1 above.

Note that by [3, Corollary 4] N must necessarily have some pure continuous spectrum of positive planar area.

This concludes Chapter 2.

CHAPTER 3

IDEALS, INDECOMPOSABLE OPERATORS, AND DIAGONALS

1. Indecomposable Operators and a Problem of Brown, Douglas, and Fillmore.

Let H be a separable, infinite-dimensional, complex Hilbert space. Let (N) denote the class of normal operators in L(H). For each bounded operator A, let $\sigma_e(A)$ denote the essential spectrum of A.

L.G. Brown, R.G. Douglas, and P.A. Fillmore [8, Theorem 11.2] characterized (N) + K(H) by proving that an operator A is decomposable into the sum of a normal operator and a compact operator (i.e. $A \in (N) + K(H)$) if and only if $A^*A - AA^* \in K(H)$ and index $(A-\lambda I) = 0$ for every $\lambda \notin \sigma_e(A)$.

One then asks under what circumstances the ideal of compact operators can be replaced by the ideal C_1 in this result. Indeed, if $A \in (N) + C_1$, then A not only satisfies the two conditions: (1) $A^*A - AA^* \in C_1$ and (2) index $(A-\lambda I) = 0$ for every $\lambda \notin \sigma_e(A)$, but in addition, the trace of $A^*A - AA^*$ is clearly 0. In fact, more must be true. If $A \in (N) + C_1$, then the Helton and Howe trace invariant [20] vanishes for A. This follows since if $A = N + C$ where N is normal and $C \in C_1$, then the trace invariant for N + C is the same as that for N, and the trace invariant of a normal operator vanishes. One then asks:

Question 1.1. Is it true that '$A \in (N) + C_1$ if and only if (1) $A^*A - AA^* \in C_1$, (2) index $(A-\lambda I) = 0$ for every $\lambda \notin \sigma_e(A)$, and (3) A has vanishing Helton and Howe trace invariant'?

Brown, Douglas and Fillmore conjectured that this is not the case, and in this connection they posed the following question in [8, pp. 123-124] and at the 1973 Wabash International Conference on Banach Spaces.

Question 1.2. Prove that not every Hilbert-Schmidt operator is decomposable into the sum of a normal operator and a trace class operator (i.e. $C_2 \not\subset (N) + C_1$).

An affirmative solution to Question 1.2 answers Question 1.1 in the negative. This follows from the facts that any Hilbert-Schmidt operator A for which $A \notin (N) + C_1$ nevertheless satisfies conditions (1), (2) and (3) in Question 1.1. That conditions (1) and (2) hold for every Hilbert-Schmidt operator is well-known. That condition (3) holds for every Hilbert-Schmidt operator follows from the fact every compact operator with a trace class self-commutator has a vanishing Helton and Howe trace invariant.

The authors also asked the following related question.

Question 1.3. If A is a Hilbert-Schmidt operator, does there exist a normal operator N such that $A \oplus N \in (N) + C_1$?

In what follows, we answer all three questions and generalize the results. In particular, we answer Question

1.1 in the negative by proving that $C_2 \not\subset (N) + C_1$, thereby solving Question 1.2. For this we produce a large class of operators A in C_2 that are not contained in $(N) + C_1$. Furthermore we show that each such A in C_2 leads to a solution, in the negative, of Question 1.3. In fact, we show that for each such A in C_2 and every $Q \in L(H)$ (Q need not be normal) we obtain $A \oplus Q \not\in (N) + C_1$. Our techniques apply to more general ideals and not merely to C_2 and C_1. We shall state and prove our results in this more general setting.

Let us recall some facts. Any ideal contained in L(H), or $L(H \oplus H)$, or $L(H \oplus H \oplus H)$ is thought of as simultaneously lying in L(H), $L(H \oplus H)$, and $L(H \oplus H \oplus H)$.

As usual, $M(T_{11}, T_{12}, T_{21}, T_{22})$ denotes the operator in $L(H \oplus H)$ which is represented by the 2x2 matrix with operator entries T_{ij} in the (i,j) position, for i,j = 1,2. Also as usual, if I is an ideal of L(H), and so, under our identification, I is an ideal in $L(H \oplus H)$, then I, considered to be in $L(H \oplus H)$ is precisely

$$\left\{ M(T_{11}, T_{12}, T_{21}, T_{22}) : T_{ij} \in I \text{ for } i,j = 1,2 \right\}.$$

We now state and prove our results.

<u>Theorem 1.4</u>. (The Main Theorem). If I is an ideal in L(H) and $A \not\in I$, then

$$M(0,0,A,0) = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \not\in (N) + I.$$

Proof. Suppose to the contrary that $A \notin I$ and yet $M(0,0,A,0) \in (N) + I$. Then $M(0,0,A,0) = N - M(X,Y,S,T)$ for some normal operator $N \in L(H \oplus H)$ and operators $X,Y,S,T \in I$ where I is considered to be in $L(H)$. Then $N = M(X,Y,A+S,T)$ and $N^* = M(X^*,(A+S)^*,Y^*,T^*)$. Since N is normal, $N^*N = NN^*$. Substituting and computing using the last two equations, we obtain the equation

$$M(X^*X + (A+S)^*(A+S),\cdot,\cdot,\cdot) = M(XX^*+YY^*,\cdot,\cdot,\cdot).$$

Therefore, $X^*X + (A+S)^*(A+S) = XX^* + YY^*$ or, equivalently, $|A+S|^2 = |X^*|^2 + |Y^*|^2 - |X|^2$. Since $X,Y \in I$, and since every ideal is closed under the operation of taking adjoints, we obtain $|X^*|^2, |Y^*|^2, |X|^2 \in I^2$, and therefore $|A+S|^2 \in I^2$. Hence $|A+S| \in I$, $A+S \in I$, and finally $A \in I$, which contradicts our assumption that $A \notin I$. Q.E.D.

Corollary 1.5. If I and J are ideals in $L(H)$ for which $J \not\subset I$, then $J \not\subset (N) + I$. In particular $C_2 \not\subset (N) + C_1$.

Proof. For every $A \in J \setminus I$, $M(0,0,A,0) \in J$ and by Theorem 1.4, $M(0,0,A,0) \notin (N) + I$. Therefore $J \not\subset (N) + I$. Q.E.D.

Theorem 1.6. If I is an ideal in $L(H)$ and $A \notin I$, then $M(0,0,A,0) \oplus Q \notin (N) + I$ for every $Q \in L(H)$.

Proof. Use the proof of Theorem 1.4, using 3x3 matrices with operator entries to represent operators in $L(H \oplus H \oplus H)$ in place of the 2x2 matrices. As in the proof of Theorem 1.4, apply the equation $N^*N = NN^*$ and after computing, consider only the (1,1) position. Q.E.D.

Corollary 1.7. If $A \in C_2 \setminus C_1$, then $M(0,0,A,0) \in C_2$, but for every $Q \in L(H)$, $M(0,0,A,0) \oplus Q \notin (N) + C_1$.

Proof. This follows trivially from Theorem 1.6. Q.E.D.

Let us now reformulate Question 1.1 for an arbitrary ideal I and ask another related question.

Question 1.8a. Is it true that '$A \in (N) + I$ if and only if $A^*A-AA^* \in I$ and index $(A-\lambda I) = 0$ for every $\lambda \notin \sigma_e(A)$' ?

Question 1.8b. If $I \subset C_1$, is it true that '$A \in (N) + I$ if and only if $A^*A-AA^* \in I$, index $(A - \lambda I) = 0$ for every $\lambda \notin \sigma_e(A)$, and the Helton and Howe trace invariant for A vanishes' ?

We stated earlier that $I = K(H)$ solves Question 1.8a in the affirmative. It is the only ideal we know that solves Question 1.8a in the affirmative. However, the next corollary yields many ideals which solve Question 1.8a in the negative.

Corollary 1.9. If I is an ideal for which $I \neq I^{1/2}$, then it is false that '$A \in (N) + I$ if and only if $A^*A-AA^* \in I$ and index $(A-\lambda I) = 0$ for every $\lambda \notin \sigma_e(A)$'.

Proof. Choose $A \in I^{1/2} \setminus I$ and let $A_1 = M(0,0,A,0)$.

Then $A_1 \in I^{1/2}$ and so $A_1^*A_1 - A_1A_1^* \in I$ and index $(A_1-\lambda I) = 0$ for every $\lambda \notin \sigma_e(A_1)$. But by Theorem 1.4, $A_1 \notin (N) + I$.
Q.E.D.

A Point of View. The point of view which led to these solutions motivates the following question, which may be important in decomposition theory. Is every Hilbert-Schmidt weighted shift operator of finite multiplicity decomposable into the sum of a normal operator and a trace class operator? Note that our Hilbert-Schmidt operator M(0,0,A,0) seems 'far' from a weighted shift operator of finite multiplicity in that its nonzero entries lie 'far' from the diagonal. There appears to be an important theme arising here. Loosely speaking, some operators whose nonzero entries are near or on the diagonal are not unitarily equivalent to operators whose diagonal entries are far from the diagonal, and some are. Which ones are and which ones are not appears, at times, to be the central issue. This theme has arisen in regard to commutators. Often in Chapter 1, diagonal operators were not commutators, or they presented difficult commutator problems, whereas operators of the form M(0,0,A,0) were easily written as the right kind of commutators. It is becoming well-known that entries on the diagonal are harder to handle than entries off the diagonal, in some contexts. J.H. Anderson makes some of the same observations in [1, Remark 4.4].

The next theorem answers the previous question in the negative, but leads to two other questions. We give two proofs of this theorem. The first one is a technique which can be used to obtain a more general result. The second one cannot, but it is shorter and depends on Theorem 1.4.

Theorem 1.10. If U is the unilateral weighted shift operator with weights (w_n) where $w_{2n-1} = n^{-3/4}$ and $w_{2n} = 0$ for every n, then $U \in C_2$ and $U \notin (N) + C_1$.

Proof I. Clearly $U \in C_2$. Suppose to the contrary that $U = N + C$ where N is a normal operator and $C \in C_1$. Then $N = U - C$ and

$$0 = N^*N - NN^* = (U^*U - UU^*) - (U^*C - CU^*) - (C^*U - UC^*) + (C^*C - CC^*),$$

and hence $U^*U - UU^* = (U^*C - CU^*) + (C^*U - UC^*) - (C^*C - CC^*)$. It is well-known that the product of an operator in C_2 and an operator in C_1 is an operator in $C_{2/3}$. Therefore the right-hand side of the previous equation is in $C_{2/3}$, and so $U^*U - UU^* \in C_{2/3}$. However, by computing we see that $U^*U - UU^*$ is the diagonal operator with entries $|w_1|^2$, $|w_2|^2 - |w_1|^2$, $|w_3|^2 - |w_2|^2, \ldots$. Hence, if we choose $w_{2n-1} = n^{-3/4}$ and $w_{2n} = 0$ for every n, then by computing we see that $U^*U - UU^*$ is the diagonal operator with diagonal entries $(1, -1, 2^{-3/2}, -2^{-3/2}, 3^{-3/2}, -3^{-3/2}, \ldots)$, which is not contained in $\ell^{2/3}$. Therefore $U^*U - UU^* \notin C_{2/3}$, which is a contradiction.

Proof II. It is easy to show that since $w_{2n} = 0$ for every n, U is unitarily equivalent to $M(0,0,D,0)$ where D is the diagonal matrix whose entries are the numbers w_{2n-1}.

Hence $D \notin C_1$. Therefore, by Theorem 1.4, $U \notin (N) + C_1$.

Q.E.D.

The following two questions are concerned with characterizing $(N) + C_1$.

<u>Question 1.11</u>. Which Hilbert-Schmidt weighted shift operators are contained in $(N) + C_1$?

<u>Question 1.12</u>. Does there exist a Hilbert-Schmidt weighted shift operator which is not contained in C_1 but which is contained in $(N) + C_1$?

The answer to Question 1.12 is yes. I.D. Berg solved this, and we give a simple version of his proof.

Let U_n be the unilateral shift on the n-dimensional Hilbert space $H = \{e_k\}_{k=1}^{n}$. Let K_n be the nxn matrix for which $K_n e_k = 0$ if $1 \leq k \leq n-1$, and $K_n e_n = -e_1$. It is clear that $V_n = U_n - K_n$ is an nxn matrix in which every row and every column has precisely one nonzero entry, and that entry is 1. It follows that V_n is unitary. Also the C_p-norm $(1 \leq p < \infty)$ of $K_n = U_n - V_n$ is 1 and that of $U_n = V_n + K_n$ is $(n-1)^{1/p}$. Letting $N = \Sigma\oplus\, n^{-2} V_n$, $S = \Sigma\oplus\, n^{-2} U_n$, and $K = \Sigma\oplus\, n^{-2} K_n$, we see that N is normal and $S \in C_2 \setminus C_1$ and is a weighted shift. Also $S = N + K$ and $\|K\|_{C_1} = \Sigma\, n^{-2} < \infty$.

Question 1.11 remains unsolved. To this end Theorem 1.10 and the affirmative solution to Question 1.12 may be a beginning.

This concludes Section 1.

2. Diagonals of Operators in Ideals

The problem of deciding whether or not a particular operator lies in a specific ideal is often difficult. We have already encountered this problem in Chapter 2, Question 11, where given a matrix representation for a particular operator A, we would like to determine whether or not A is contained in C_1 or some other specific ideal. In Chapter 2, the theory was carried only as far as this problem could be solved. We wish to find new criteria which make the problem more tractable.

Our approach is matricial, being motivated by the well-known precise and complete solution to the problem in the case of the Hilbert-Schmidt class. We shall determine, for several important ideals, necessary and sufficient conditions on the entries of the matrices representing an operator which will place that operator inside the ideal. We begin with some definitions that are motivated by a known fact [14] that a positive operator $T \in C_1$ if and only if, in each basis, the sequence of its diagonal entries is contained in ℓ^1.

Definition 2.1.

(a) If T is the operator with matrix representation (t_{ij}) with respect to a fixed basis, let Diag(T) denote the diagonal operator $D((t_{ii}))$, which has the same diagonal as T.

(b) An ideal I is said to have property D if $T \in I$ <==> in each basis, $\text{Diag}(T) \in I$.

(c) An ideal I is said to have <u>property D^+</u> if for every operator $T \geq 0$, $T \in I \iff$ in each basis, $Diag(T) \in I$.

Note that 'Diag' depends on the basis.

The following two results are concerned with property D and linking property D with property D^+, and the first one provides sufficient conditions on the matrices' entries to insure membership in an ideal.

<u>Lemma 2.2</u>. If I is any ideal, then $Diag(T) \in I$ for each basis $\Longrightarrow T \in I$.

<u>Proof</u>. We shall prove the contrapositive. Suppose $T \notin I$. Let $T = T_1 + iT_2$ where T_1 is the real part and T_2 is the imaginary part of T. For every self-adjoint operator S and any basis $\{e_n\}$, $(Se_n, e_n) = (e_n, Se_n) = \overline{(Se_n, e_n)}$ for every n, and so the diagonal entries of any matrix representation of S are real numbers. Therefore if $\{e_n\}$ is any basis, (T_1e_n, e_n) and (T_2e_n, e_n) are real for every n, and thus $|(Te_n, e_n)| \geq \max(|(T_1e_n, e_n)|, |(T_2e_n, e_n)|)$ for every n. Since $T \notin I$, either T_1 or $T_2 \notin I$. If $T_1 \notin I$ and $\{e_n\}$ is a basis which diagonalizes T_1, then $|(Te_n, e_n)| \geq |(T_1e_n, e_n)|$ for all n. But then $D((T_1e_n, e_n)) = Diag(T_1) \notin I$ and therefore $Diag(T) = D((Te_n, e_n)) \notin I$. On the other hand, if $T_2 \notin I$, then similarly, in some basis, $Diag(T) \notin I$. Q.E.D.

<u>Proposition 2.3</u>. If I is an ideal in L(H), then I has property D if and only if I has property D^+.

Proof. It is clear that if I has property D, then I has property D^+, and so it suffices to show that if I has property D^+, then I has property D.

Suppose I has property D^+. By Lemma 2.2, in order to show that I has property D, it suffices to show that $T \in I$ implies that $\mathrm{Diag}(T) \in I$ for each basis. Hence, suppose $T \in I$ and let $\{e_n\}$ be an arbitrary basis. We need to show that $D((Te_n, e_n)) \in I$. If $T = U|T|$ is the polar decomposition of T, where $|T|$ is the positive part, and U is a partial isometry, then for each $f \in H$,

$$|(Tf,f)| = |(U|T|f,f)| = |(|T|^{1/2}f, |T|^{1/2}U^*f)|$$

$$\leq \||T|^{1/2}f\| \cdot \||T|^{1/2}U^*f\|$$

$$= (|T|f,f)^{1/2}(U|T|U^*f,f)^{1/2} .$$

Since $T \in I$, we have $|T| \in I$ and $U|T|U^* \in I$. Also $|T| \geq 0$, and since $(U|T|U^*f,f) = \||T|^{1/2}U^*f\|^2 \geq 0$ for every $f \in H$, we have $U|T|U^* \geq 0$. If I has property D^+, then

$\mathrm{Diag}(|T|) = D((|T|e_n, e_n)) \in I$ and also

$\mathrm{Diag}(U|T|U^*) = D((U|T|U^*e_n, e_n)) \in I$. Therefore

$$D((|T|e_n,e_n)(U|T|U^*e_n,e_n)) = D((|T|e_n,e_n))D((U|T|U^*e_n,e_n)) \in I^2,$$

and so $(D(|(Te_n,e_n)|))^2 = D(|(Te_n,e_n)|^2) \in I^2$. Hence $D(|(Te_n,e_n)|) \in I$, and this implies $D((Te_n,e_n)) \in I$. Q.E.D.

Corollary 2.4. An ideal I has property D if and only if, for every positive operator $T \in I$, $\mathrm{Diag}(T) \in I$ in each basis.

Proof. Use Lemma 2.2 and Proposition 2.3. Q.E.D.

Let us examine property D in the context of some specific ideals.

Question 2.5. Which ideals have property D?

A complete list of the ideals that we have determined do or do not have property D appears in the main theorem of this section. However we need to develop some preliminaries first. The next lemma contains basic inequalities and appears in [22].

Lemma 2.6. Let $T \in L(H)$, $T \geq 0$, $f \in H$, and $p > 0$. Then

(1) if $0 < p \leq 1$, $(T^p f, f) \leq (Tf, f)^p \| f \|^{2(1-p)}$;

(2) if $1 \leq p < \infty$, $(T^p f, f) \geq (Tf, f)^p \| f \|^{2(1-p)}$.

Furthermore if $p \neq 1$, then equality in (1) or (2) implies that f is an eigenvector of T.

Proof. If $T \in L(H)$ and $T \geq 0$, then by the spectral theorem for positive operators, there exists a measure space (X, μ) and a function $h \in L^\infty(X, \mu)$ such that $h \geq 0$ μ- almost everywhere and $T \cong M_h$, where M_h denotes the multiplication operator of multiplication by h acting on $L^2(X, \mu)$. This unitary equivalence implies an underlying unitary transformation from H onto $L^2(X, \mu)$, and so if $f \in H$, we may denote the image of f under this unitary transformation again by f. The quantities referred to in (1), (2) and the last statement in the lemma are invariant if we replace T by M_h, and the inner product on H by the usual integral type inner product on $L^2(X,\mu)$, etc.. It therefore suffices to prove Lemma 2.6 for operators M_h acting on $L^2(X,\mu)$, where $h \geq 0$ μ-almost everywhere and $h \in L^\infty(X,\mu)$.

First we shall prove (2). Let $1 \leq p < \infty$. Then

$$(M_h f,f) = \int h|f|^2 d\mu = \int h|f|^{2/p}|f|^{2/q} d\mu$$

$$\leq (\int h^p |f|^2 d\mu)^{1/p} (\int |f|^2 d\mu)^{1/q}$$

$$= (M_h^p f,f)^{1/p} \|f\|^{2/q} ,$$

where q is the conjugate of p (i.e. $p^{-1} + q^{-1} = 1$ or $q = p/(p-1)$) and the inequality follows from Holder's inequality. Therefore

$$(M_h^p f,f) \geq (M_h f,f)^p \|f\|^{-2p/q} = (M_h f,f)^p \|f\|^{2(1-p)} ,$$

which proves (2).

Now (1) follows easily from (2). Let $0 < p \leq 1$. If $r = p^{-1}$, then $1 \leq r < \infty$, $rp = 1$, and (2) applies to r and M_{h^p}, producing

$$(M_h f,f) = (M_{h^p}^r f,f) \geq (M_{h^p} f,f)^r \|f\|^{2(1-r)}$$

$$= (M_h^p f,f)^r \|f\|^{2(1-r)} ,$$

and so

$$(M_h^p f,f) \leq (M_h f,f)^{1/r} \|f\|^{2(r-1)/r} = (M_h f,f)^p \|f\|^{2(1-p)} ,$$

which proves (1).

If equality holds in (1) or (2) when $p \neq 1$, then the corresponding Holder's inequality becomes an equality. But we know that Holder's inequality for $p \neq 1$ becomes an equality if and only if one of the two functions in the equation is μ-almost everywhere equal to a scalar multiple of the other. Hence $Tf = cf$ μ-almost everywhere for some complex number c. Q.E.D.

With this lemma we can prove an interesting result about C_p which is concerned with the ℓ^p-norm of the sequence of ℓ^2-norms of the columns of a matrix. This result is stronger than property D in certain ways. In particular, it introduces a criterion which depends upon only one basis instead of every basis as in property D.

<u>Theorem 2.7</u>. Let $\{e_n\}$ be an arbitrary basis for H.

(1) If $0 < p \leq 2$, then $(\|Te_n\|) \in \ell^p \Longrightarrow T \in C_p$.

(2) If $2 \leq p < \infty$, then $T \in C_p \Longrightarrow (\|Te_n\|) \in \ell^p$.

<u>Proof</u>. Recall that $T \in C_p \Longleftrightarrow |T| \in C_p \Longleftrightarrow |T|^p \in C_1$.

To prove (1), suppose that for some basis $\{e_n\}$, $(\|Te_n\|) \in \ell^p$ where $0 < p \leq 2$. Then $\Sigma \|Te_n\|^p < \infty$ and $0 < p/2 \leq 1$. Hence, by Lemma 2.6-1,

$$\|Te_n\|^p = (|T|^2 e_n, e_n)^{p/2} \geq (|T|^p e_n, e_n).$$

Therefore $\||T|^p\|_{C_1} = Tr(|T|^p) = \Sigma\, (|T|^p e_n, e_n) \leq \Sigma \|Te_n\|^p < \infty$, and so $|T|^p \in C_1$, $|T| \in C_p$, and finally $T \in C_p$.

To prove (2), suppose $2 \leq p < \infty$, $T \in C_p$, and $\{e_n\}$ is an arbitrary basis. Then $|T| \in C_p$, $|T|^p \in C_1$, and

$1 \leq p/2 < \infty$. Hence, by Lemma 2.6-2,

$$\|Te_n\|^p = (|T|^2 e_n, e_n)^{p/2} \leq (|T|^p e_n, e_n) \ ,$$

and so $\Sigma \|Te_n\|^p \leq \Sigma (|T|^p e_n, e_n) < \infty$. Whence $(\|Te_n\|) \in \ell^p$.
Q.E.D.

We shall use this theorem to prove the main result that will settle Question 2.5 for C_p. It can be stated in a stronger form than we shall need, but for the sake of completeness, we give the stronger form in the following.

<u>Corollary 2.8</u>. Let (n_k) and (m_k) be two strictly increasing sequences of positive integers, let $\{e_n\}$ be an arbitrary basis for H, and let $1 \leq p < \infty$. If $T \in C_p$, then $((Te_{n_k}, e_{m_k}))_{k=1}^{\infty} \in \ell^p$. In particular, if $T \in C_p$, then $\mathrm{Diag}(T) \in C_p$.

<u>Proof</u>. If $1 \leq p < \infty$ and $T \in C_p$, then $|T| \in C_p$ and $|T|^{1/2} \in C_{2p}$, where $2 \leq 2p < \infty$. Then Theorem 2.7-2 applies to $|T|^{1/2}$ and $|T|^{1/2}U^*$, where $T = U|T|$ is the polar decomposition of T in which $|T|$ is the positive part and U is a partial isometry, and where $U^*U|T| = |T|$. This gives $(\| |T|^{1/2} e_n \|)$ and $(\| |T|^{1/2} U^* e_n \|) \in \ell^{2p}$. Therefore

$$|(Te_{n_k}, e_{m_k})|^p = |(|T|^{1/2} e_{n_k}, |T|^{1/2} U^* e_{m_k})|^p$$

$$\leq \| |T|^{1/2} e_{n_k} \|^p \, \| |T|^{1/2} U^* e_{m_k} \|^p$$

$$\leq \max (\| |T|^{1/2} e_{n_k} \|^{2p}, \| |T|^{1/2} U^* e_{m_k} \|^{2p}) \ .$$

But both $(\| |T|^{1/2} e_{n_k} \|)$ and $(\| |T|^{1/2} U^* e_{m_k} \|)$ are subsequences of sequences in ℓ^{2p} and hence, are themselves sequences in ℓ^{2p}. Therefore, by the above inequality,

$$\Sigma \left|(Te_{n_k}, e_{m_k})\right|^p \leq \Sigma \left\| |T|^{1/2} e_{n_k} \right\|^{2p} + \Sigma \left\| |T|^{1/2} U^* e_{m_k} \right\|^{2p} < \infty.$$

To prove $\mathrm{Diag}(T) \in C_p$, use the first part of the corollary letting $n_k = m_k = k$. Q.E.D.

We now have the preliminaries to state and prove the main theorem.

<u>Theorem 2.9</u>. (The Main Theorem).

(1) $K(H)$ has property D.

(2) If $1 \leq p < \infty$, then C_p has property D.

(3) If an ideal I has property D, then $C_1 \subset I$. That is, C_1 is the minimal ideal possessing property D. In particular, $F(H)$ and C_p for $0 < p < 1$ fail to have property D.

(4) The union and intersection of ideals that have property D themselves have property D. In particular, if $q \geq 1$, then $\bigcap_{p>q} C_p$ has property D, and if $q > 1$, then $\bigcup_{p<q} C_p$ has property D. Also $\bigcup_{p>0} C_p$ has property D.

<u>Proof</u>. In light of Corollary 2.4, for each ideal I above, it suffices to determine whether or not $\mathrm{Diag}(T) \in I$ for every positive operator $T \in I$ and every basis.

To prove (1), note that if $\{e_n\}$ is any basis for H and $T \in K(H)$, then $\|Te_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore $\left|(Te_n, e_n)\right| \leq \|Te_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. In other words, $((Te_n, e_n)) \in c_o$, and so $\mathrm{Diag}(T) \in K(H)$.

To prove (2), apply Corollary 2.8.

To prove (3), it clearly suffices to show that if $C_1 \not\subseteq I$, then I fails to have property D. If $C_1 \not\subseteq I$, then $\ell_+^1 = \text{Calk}(C_1) \not\subseteq \text{Calk}(I)$. Therefore there is some positive sequence $x = (x_n)$ for which $x \in \text{Calk}(C_1) \setminus \text{Calk}(I)$, and so $D(x) \in C_1 \setminus I$. Multiplying the sequence x by a positive scalar does not alter its membership in Calkin ideal sets. Therefore, without loss of generality, we may assume that $\Sigma x_n = 1$.

The rank one orthogonal projection operator $P \in F(H)$, and so P is contained in every ideal. It is easy to verify that if we set $A = (a_{ij})$ where $a_{ij} = x_i^{1/2} x_j^{1/2}$ and $\Sigma x_n = 1$, then $P \cong A$. This implies that there is a basis in which $\text{Diag}(P) = D(x) \in C_1 \setminus I$, and so $P \in I$, $P \geq 0$, $\text{Diag}(P) \notin I$, and therefore I fails to have property D.

The rest is clear. Q.E.D.

Remark. In the introduction to Chapter 1 we mentioned that we would make attempts to generalize the concept of the trace. Strictly speaking, the trace is a numerical quantity. However, in Chapter 1 and in this chapter, an important theme is implied which is that the character of the diagonals of matrices often play a crucial role in the study of a particular problem. It is this character, or different aspects of it, that we have tried to understand and exploit. In other words, the 'trace' of a matrix is more than just a numerical quantity.

<u>Aside</u>. In the proof of Theorem 2.9-3 we took notice of a matrix that was unitarily equivalent to P. Since P has been central to our work, we shall state some useful facts about P which are not difficult to prove, but whose proofs we shall omit.

(1) Every matrix $A = (a_{ij})$ for which $P \cong A$ is given by the equation $a_{ij} = x_i y_j$ where (x_n) and $(y_n) \in \ell^2$, $x_i y_j = \bar{x}_j \bar{y}_i$ for all i,j, and $\Sigma\, x_i y_i = 1$.

(2) For every rank one operator T (not necessarily a projection operator), the following identity holds: $T^2 = (\mathrm{Tr}(T))T$. Such an operator T is similar to P if and only if $\mathrm{Tr}(T) = 1$.

This concludes Section 2 and Chapter 3.

CHAPTER 4

INFINITE SERIES REARRANGEMENTS OF A NEW TYPE

A new type of rearrangement problem for infinite series has arisen in connection with the results of Chapter 1, Section 4 which has appeal to other areas of mathematics and is reminiscent of Riemann's theorem on rearrangements of conditionally convergent series. We present here the problems, the known results and their applications to commutator structures. (W.A.J. Luxembourg has recently pointed out to us the existence of related work in this area (see [10] and [11]) which deals with related questions in a similar setting.)

In Chapter 1, Section 4, equation 4.6 together with Corollary 4.8 states that (if d = 1) $D(-1,d_1,d_2,\ldots)=XY-YX$ with $\|X\|_{C_2}{}^2 = \|Y\|_{C_2}{}^2 = \Sigma\, nd_n$, and by Summary 4.24 of that section, we obtain $D(-1,d_1,d_2,\ldots) \oplus 0 = (XY-YX) + (ST-TS)$ where $\|X\|_{C_2} = \|Y\|_{C_2}$, $\|S\|_{C_2} = \|T\|_{C_2}$, and $\|X\|_{C_2}{}^2 + \|S\|_{C_2}{}^2 = \Sigma\, b_n d_n$, where (b_n) (see p. 89) is related to the Fibonacci numbers. Both the infinite series $\Sigma\, nd_n$ and $\Sigma\, b_n d_n$ are of the form $\Sigma\, a_n x_n$ where $a_n \uparrow \infty$ and $x_n \downarrow 0$. In Chapter 1, Section 4, Problem II and Lemma 4.28 concern themselves with the first aspects of this theory. We now expand our view in the following way.

Let us examine different solutions to the equation $D(-1,d_1,d_2,\ldots) = XY-YX$. Our original construction (Chapter 1, Section 4, equation 4.6) did not actually need that (d_n) be decreasing, but Lemma 4.28 of that section indicated that the best chance that $\|X\|_{C_2}{}^2 = \|Y\|_{C_2}{}^2 = \Sigma\, nd_n$ has to be finite is when (d_n) is arranged in decreasing order (hence we might as well have assumed it). However, if we rearrange (d_n) and call the rearranged sequence (d_n'), then equation 4.6 can be used to yield $D(-1,d_1',d_2',\ldots) = XY-YX$ with $\|X\|_{C_2}{}^2 = \|Y\|_{C_2}{}^2 = \Sigma\, nd_n'$. But this diagonal matrix has exactly the same entries as $D(-1,d_1,d_2,\ldots)$, only rearranged, and hence is unitarily equivalent to it. Hence we can solve $D(-1,d_1,d_2,\ldots) = XY-YX$ with $\|X\|_{C_2}{}^2 = \|Y\|_{C_2}{}^2 = \Sigma\, nd_n'$, which is greater than or equal to $\Sigma\, nd_n$. It is then natural to ask what values can arise for $\Sigma\, nd_n'$. That is, what size Hilbert-Schmidt norms can be realized by solution operators X,Y in the framework of this construction. The same question can be asked of $\Sigma\, b_n d_n$ and $\Sigma[(n+1)/2]\,d_n$ (see Chapter 1, Proposition 8.1). This is one of the basic motivations for this chapter.

It appears that examining the different solutions to the commutator equations via their Hilbert-Schmidt norms in this way is not crucial to the study of the structure of commutator classes. Although, Lemma 4.28 of Chapter 1 was crucial in that it identified the direction that we needed to take in order to obtain our results. In any case, this theory leads into some interesting and perhaps important questions on infinite series.

We shall first introduce a definition which in part occurred in Problem II of Chapter 1, Section 4 in a less general setting.

<u>Definition 1</u>. Let $a = (a_n)$ and $x = (x_n)$ be two non-negative sequences of real numbers. Let <u>$S(a,x)$</u> be the set of non-negative real numbers defined by the following equation.

$$S(a,x) = \left\{ \Sigma\, a_n x_{\pi(n)} : \pi \text{ is any permutation of } Z^+ \right\}.$$

The general problem is as follows.

<u>Problem 2</u>. For each fixed pair of non-negative sequences a and x, what is the structure of $S(a,x)$? Also, what sets in the positive real line are realizable as of the form $S(a,x)$ for some such a and x?

Various obvious facts about $S(a,x)$ may now be stated. Let a and x be two arbitrary non-negative sequences.

(1) $S(a,x) \subset [0,\infty]$. The values 0 and ∞ may be attained.

(2) If a and x are strictly positive sequences (or contain at most a finite number of 0 entries), then $S(a,x) \subset (0,\infty]$.

(3) There are 2^c subsets of $[0,\infty]$, but only c subsets of the form $S(a,x)$, because there are only c pairs (a,x). Therefore only c of the 2^c subsets of $[0,\infty]$ are realizable as of the form $S(a,x)$. (Observed by Paul Erdos)

The first of the following two examples is a non-trivial case where the structure of $S(a,x)$ is known. We omit the details.

<u>Examples</u>.

Let $a = (0,2,0,2,\dots)$ and $x = (3^{-n})$. Then $S(a,x)$ is precisely the Cantor set except for those non-negative real numbers whose ternary expansion consists of a tail of 0's or a tail of 2's.

If a or x is only finitely nonzero, then $S(a,x)$ is a countable set.

Problem 2 is too broad for us to say much more about with our present knowledge. For this reason, in what follows we specialize to the cases when a is a non-decreasing sequence and x is a non-increasing sequence.

If $a = 0$ or $x = 0$, the problem is trivial and $S(a,x)=\{0\}$. If $a \neq 0$ and $x_n \not\to 0$, the problem is trivial and $S(a,x)=\{\infty\}$. Furthermore if a is bounded by M, then $S(a,x) \subset [0, M\Sigma x_n]$. In any case, hereafter we shall assume $x_n \downarrow 0$ and $a_n \uparrow \infty$, unless otherwise specified.

Note that Lemma 4.28 and its proof remain valid even if $a_n \not\to \infty$ or $x_n \not\to 0$.

<u>Proposition 3</u>. If $a_n\uparrow$ and $x_n\downarrow$, and if $m = a\cdot x = \Sigma a_n x_n$, then $S(a,x) \subset [m,\infty]$, where m is attained.

If $a_n \uparrow \infty$ and $x_n > 0$ for infinitely many values of n (x_n need not be decreasing), then $\infty \in S(a,x)$.

<u>Proof</u>. The first assertion follows from Lemma 4.28 (without the inessential hypotheses that $a_n \to \infty$ or $x_n \to 0$).

To prove the second assertion, if $a_n \uparrow \infty$ and $x_n > 0$ for infinitely many n, then for each k we can choose $n_k \geq 2k-1$ such that $a_{n_k} x_{2k-1} \geq 1$. Let π be any permutation of Z^+ for which $\pi(n_k) = 2k-1$ (e.g. map $Z^+ \setminus \{n_k\}$ arbitrarily one-to-one onto $\{2k\}$). Then $\Sigma a_n x_{\pi(n)} \geq \Sigma_k a_{n_k} x_{2k-1} = \infty$. Q.E.D.

In view of Proposition 3, we ask the main question of this section.

__Question 4.__ For which a,x with $a_n \uparrow \infty$ and $x_n \downarrow 0$ is $S(a,x) = [a \cdot x, \infty]$?

On first sight, one might think that it is never the case that $S(a,x) = [a \cdot x, \infty]$ or that it is quite rare. However, our investigations began with the surprising fact that, with regard to $\Sigma\, nd_n$, if $d_n > 0$ for every n and $d_n \downarrow$, then $S((n),(d_n)) = [\Sigma\, nd_n, \infty]$. After this it appeared that there was a possibility that $S(a,x) = [a \cdot x, \infty]$ for every such a,x. However, this was ruled out by an example due to Robert Young. Namely, let $a_n = 2^{2^n}$ and let $x_n = 2^{-2^{n+1}}$. We omit the proof since a more general result due to Paul Erdos is forthcoming with a proof that proceeds along the same lines. Nevertheless, if a is any of the sequences (n), $([(n+1)/2])$, the Fibonacci sequence, $(\log n)$, and if $d_n \underset{\neq}{\downarrow} 0$, then $S(a,(d_n)) = [\Sigma\, a_n d_n, \infty]$.

Hugh Montgomery discovered a general sufficient condition on the sequence a that implies that '$S(a,x) = [a \cdot x, \infty]$ for every x for which $x_n \underset{\neq}{\downarrow} 0$', namely that

a_{n+1}/a_n remain bounded. Paul Erdos proved that this condition is also necessary. Note the similarity between this condition and the well-known ratio test for power series. What we really want to know is what determines whether or not $S(a,x) = [a\cdot x,\infty]$. However, the above condition is the only complete characterization we have of any aspect of the theory. We develop the results in both of these directions simultaneously, giving our own proofs in what follows.

Theorem 5. (The Main Theorem). Let $a = (a_n)$ where $a_n > 0$ for every n, and $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Consider the following conditions.

(1) a_{n+1}/a_n is bounded.

(2) For the non-negative sequence $x = (x_n)$, there exist subsequences (a_{n_k}) and (x_{m_k}) of a and x, respectively, such that

a) $a_{n_k} x_{m_k} \longrightarrow 0$ as $k \longrightarrow \infty$, and

b) $\Sigma_k a_{n_k} x_{m_k} = \infty$.

(3) $S(a,x) = [a\cdot x,\infty]$.

Then (1) implies that (2) holds for every strictly positive sequence $x = (x_n)$ that tends to 0. Furthermore, if $a_n \uparrow \infty$ and $x_n \downarrow 0$ where $a_n, x_n > 0$ for all n, then (2) $\Longrightarrow$ (3).

Proof. To prove that (1) implies that (2) holds for every strictly positive sequence $x = (x_n)$ that tends to 0, suppose $a_{n+1}/a_n \leq M$ for all n. We assert that for every positive integer k, there exist arbitrarily large positive integers n_k and m_k for which $(k+1)^{-1} \leq a_{n_k} x_{m_k} \leq M k^{-1}$.

If this assertion were true, then clearly we could choose two strictly increasing subsequences of positive integers (n_k) and (m_k) such that $a_{n_k} x_{m_k} \longrightarrow 0$ as $k \longrightarrow \infty$ and $\Sigma_k a_{n_k} x_{m_k} \geq \Sigma_k (k+1)^{-1} = \infty$, and (2) would be proved. Therefore it suffices to prove the assertion.

For each fixed positive integer k, $(k+1)^{-1} \leq a_n x_m \leq Mk^{-1}$ if and only if $x_m \in [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$. All we need show is that there exist arbitrarily large n,m for which $x_m \in [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$.

Suppose to the contrary that there exists a positive integer N for which $x_m \notin [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$ for every $n,m \geq N$. (In other words, for every $m \geq N$, $x_m \notin \bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$. It is interesting to note that this would imply that $\bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$ cannot contain any interval of the form $(0,\epsilon)$ for some $\epsilon > 0$, since $x_m \longrightarrow 0$ as $m \longrightarrow \infty$. However, this is not the case. Indeed, the forthcoming proof can be used to show that for every N, there exists $\epsilon > 0$ such that $(0,\epsilon) \subset \bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$.).

For each $m \geq N$, let n_m denote the least positive integer n such that $M(a_{n+1}k)^{-1} < x_m$, which exists since $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and hence $M(a_{n+1}k)^{-1} \longrightarrow 0$ as $n \longrightarrow \infty$. Except for a finite number of positive integers m for which $x_m > M(a_1 k)^{-1}$, we have $M(a_{n_m+1}k)^{-1} < x_m \leq M(a_{n_m}k)^{-1}$. Also, since $M(a_{n_m+1}k)^{-1} < x_m$ and $x_m \longrightarrow 0$ as $m \longrightarrow \infty$, we have $m \longrightarrow \infty$ implies $a_{n_m+1} \longrightarrow \infty$ and hence $n_m \longrightarrow \infty$. Therefore $n_m \geq N$ for all but a finite number of integers m, and for these m, $x_m \notin [(a_{n_m}(k+1))^{-1}, M(a_{n_m}k)^{-1}]$. Hence, for infinitely many m, we have $x_m \le M(a_{n_m}k)^{-1}$ and $x_m \notin [(a_{n_m}(k+1))^{-1}, M(a_{n_m}k)^{-1}]$. Therefore, for infinitely many m, we have $M(a_{n_m+1}k)^{-1} < x_m < (a_{n_m}(k+1))^{-1}$. This implies that $M(a_{n_m+1}k)^{-1} < (a_{n_m}(k+1))^{-1}$ for infinitely many m, or equivalently, $a_{n_m+1}/a_{n_m} > M(k+1)/k > M$ for infinitely many m, which contradicts our assumption that $a_{n+1}/a_n \le M$ for all n. Hence (2) is proved.

To prove (2) $\Longrightarrow$ (3) whenever $a_n \uparrow \infty$ and $x_n \underset{\neq}{\downarrow} 0$, suppose (2) holds for a and x, so that there exist subsequences (a_{n_k}) and (x_{m_k}) such that $a_{n_k}x_{m_k} \longrightarrow 0$ as $k \longrightarrow \infty$, and $\Sigma_k a_{n_k}x_{m_k} = \infty$. We first assert that without loss of generality we may assume that $a \cdot x = \Sigma\, a_n x_n < \infty$. To see this, suppose $a \cdot x = \Sigma\, a_n x_n = \infty$. Then by Lemma 4.28 of Chapter 1, we have that $S(a,x) = \{\infty\}$, and hence (3) holds.

Assuming that $\Sigma\, a_n x_n < \infty$, we next assert that without loss of generality we can assume that $n_k > m_k$ for every k. To see this, let $Z_>$ denote the set $\{k\colon n_k > m_k\}$ and let $Z_\le$ denote the set $\{k\colon n_k \le m_k\}$. Then

$$\infty = \Sigma_k a_{n_k}x_{m_k} = \Sigma_{k\in Z_>} a_{n_k}x_{m_k} + \Sigma_{k\in Z_\le} a_{n_k}x_{m_k}.$$

But $\Sigma_{k\in Z_\le} a_{n_k}x_{m_k} \le \Sigma_{k\in Z_\le} a_{n_k}x_{n_k} \le \Sigma_n a_n x_n < \infty$. Therefore $\Sigma_{k\in Z_>} a_{n_k}x_{m_k} = \infty$. Let $Z_>$ determine subsequences of (n_k) and (m_k), which for simplicity we again call (n_k) and (m_k), respectively, by taking only those entries n_k, m_k (in increasing order) for which $k \in Z_>$. This gives us subsequences (a_{n_k}) and (x_{m_k}) of a and x which satisfy conditions a and b in the 2nd condition of the theorem, and in addition satisfy $n_k > m_k$ for all k.

Next we assert that without loss of generality we may assume n_k and m_k can be chosen so that $n_k \neq m_j$ for all i,j. The proof is as follows.

We shall use induction to partition the set of ordered pairs $\{(n_k,m_k)\}_{k=1}^{\infty}$ into 3 disjoint sets, each having the property that no integer appears more than once as an entry. We induct on k. Let S_1, S_2 and S_3 denote the sets to be determined. Put $(n_1,m_1) \in S_1$, $(n_2,m_2) \in S_2$, and $(n_3,m_3) \in S_3$. Since, by hypothesis, (a_{n_k}) and (x_{m_k}) are subsequences of a and x, the subsequences (n_k) and (m_k) must be strictly increasing. Therefore n_4 can appear as an entry in at most 1 of the 3 sets S_1,S_2,S_3. The same holds true for m_4. Therefore the ordered pair (n_4,m_4) can have an entry in common with an entry of an ordered pair in at most 2 of the sets S_1,S_2,S_3. Hence put (n_4,m_4) into that set whose ordered pair does not have either n_4 or m_4 as an entry. Inducting on k, assume that S_1,S_2 and S_3 are determined for the ordered pairs (n_i,m_i) where $i = 1,2,\ldots,k$, and so no entry of any ordered pair in any of these sets appears as an entry in any other ordered pair in that set. Then, as in the case of (n_4,m_4), the entry (n_{k+1},m_{k+1}) can have an entry in common with an entry of an ordered pair in at most 2 of the sets S_1,S_2,S_3. Hence put

(n_{k+1}, m_{k+1}) into that set whose ordered pairs do not have either n_{k+1} or m_{k+1} as an entry. This completes the induction.

Each set S_1, S_2 and S_3 determines a subset of Z^+ as follows. Let $Z_i = \{k: (n_k, m_k) \in S_i\}$, for $i = 1,2,3$. Clearly $Z_1 \cup Z_2 \cup Z_3 = Z^+$. Hence we obtain

$$\infty = \Sigma_k a_{n_k} x_{m_k} = \Sigma_{k \in Z_1} a_{n_k} x_{m_k} + \Sigma_{k \in Z_2} a_{n_k} x_{m_k} + \Sigma_{k \in Z_3} a_{n_k} x_{m_k}.$$

Therefore one of these sums is infinite, say $\Sigma_{k \in Z_1} a_{n_k} x_{m_k} = \infty$ (we omit the cases for Z_2 and Z_3 as the proofs are identical). Using Z_1, we can choose subsequences of (n_k) and (m_k) which, for simplicity, we again call (n_k) and (m_k), respectively, and which satisfy $a_{n_k} x_{m_k} \longrightarrow 0$, $\Sigma_k a_{n_k} x_{m_k} = \infty$, $n_k > m_k$ and $n_k \neq m_j$ for all k,j. This proves the assertion.

Now consider the series $\Sigma_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$.

Since $n_k > m_k$, we have $0 \leq a_{n_k} - a_{m_k} \leq a_{n_k}$ and $0 \leq x_{m_k} - x_{n_k} \leq x_{m_k}$, and so $0 \leq (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) \leq a_{n_k} x_{m_k} \longrightarrow 0$ as $k \longrightarrow \infty$. Furthermore, since $\Sigma_k a_{n_k} x_{m_k} = \infty$, $a_{m_k} x_{n_k} \geq 0$, $\Sigma_k a_{n_k} x_{n_k} \leq a \cdot x < \infty$, and $\Sigma_k a_{m_k} x_{m_k} \leq a \cdot x < \infty$, we have

$$\Sigma_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) = \Sigma_k (a_{n_k} x_{m_k} + a_{m_k} x_{n_k} - a_{n_k} x_{n_k} - a_{m_k} x_{m_k}) = \infty .$$

We shall now show that for every $\epsilon > 0$, there exists a subsequence (k_n) of positive integers such that $\epsilon = \Sigma_{k \in \{k_n\}} (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$. This follows from the following more general fact.

Suppose $(d(k))$ is a non-negative sequence for which $d(k) \longrightarrow 0$ as $k \longrightarrow \infty$ and $\Sigma\, d(k) = \infty$. We assert that for every $\epsilon > 0$, there exists a subsequence (k_n) of the increasing sequence of all positive integers such that $\epsilon = \Sigma_{k \in k_n} d(k)$. The proof of this fact proceeds along the same lines as the proof of Riemann's theorem on rearrangements of conditionally convergent series. Fix $\epsilon > 0$ and choose $n_1 \geq N_1$ so that $d(k) < \epsilon$ for every $k \geq N_1$, and so that n_1 is the greatest integer greater than N_1 such that $\Sigma_{k=N_1}^{n_1} d(k) < \epsilon$. Hence $\Sigma_{k=N_1}^{n_1} d(k) < \epsilon \leq \Sigma_{k=N_1}^{n_1+1} d(k)$. This can be done since $d(k) \longrightarrow 0$ as $k \longrightarrow \infty$ and $\Sigma\, d(k) = \infty$. Choose $N_2 > n_1$ so that $d(k) < (\epsilon - \Sigma_{k=N_1}^{n_1} d(k))/2$ for every $k \geq N_2$ and then choose n_2 to be the largest integer greater than N_2 such that $\Sigma_{k=N_2}^{n_2} d(k) < \epsilon - \Sigma_{k=N_1}^{n_1} d(k)$. Hence $\Sigma_{k=N_2}^{n_2} d(k) < \epsilon - \Sigma_{k=N_1}^{n_1} d(k) \leq \Sigma_{k=N_2}^{n_2+1} d(k)$. Proceeding inductively in this way, we obtain sequences (N_p) and (n_p) of positive integers for which $n_p \geq N_p > n_{p-1}$ and $0 \leq d(k) \leq (\epsilon - \Sigma_{q=1}^{p-1} \Sigma_{k=N_q}^{n_q} d(k))/2^{p-1}$, for every p and every $k \geq N_p$, and

$$\Sigma_{k=N_p}^{n_p} d(k) < \epsilon - \Sigma_{q=1}^{p-1} \Sigma_{k=N_q}^{n_q} d(k) \leq \Sigma_{k=N_p}^{n_p+1} d(k) \ .$$

This implies that

$$0 < \epsilon - \Sigma_{q=1}^{p} \Sigma_{k=N_q}^{n_q} d(k) \leq d(n_p+1) \leq (\epsilon - \Sigma_{q=1}^{p-1} \Sigma_{k=N_q}^{n_q} d(k))/2^{p-1}$$
$$\leq \epsilon/2^{p-1} \longrightarrow 0 \text{ as } p \longrightarrow \infty.$$

Therefore $\epsilon = \Sigma_{q=1}^{\infty} \Sigma_{k=N_q}^{n_q} d(k)$. Hence, if we choose (k_n) to be the strictly increasing sequence of positive integers k, where k is taken to range over the set $\bigcup_{p=1}^{\infty} \{k: N_p \leq k \leq n_p\}$, we have $\epsilon = \Sigma_{k \in \{k_n\}} d(k)$.

Applying this result to the sequence $(a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$, since it is non-negative, tends to 0, and sums to ∞ , we obtain that for every $\epsilon > 0$, there exist subsequences of (n_k) and (m_k) , which we shall again denote by (n_k) and (m_k) , for which $\epsilon = \Sigma_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$.

Now recall that we wish to show that $S(a,x) = [a \cdot x, \infty]$. We already know $a \cdot x$ and $\infty \in S(a,x)$. Suppose $a \cdot x < r < \infty$. It suffices to show $r \in S(a,x)$. Let $\epsilon = r - a \cdot x$ and choose subsequences (n_k) and (m_k) so that

$$\epsilon = \Sigma_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) .$$

We now choose π, the requisite permutation on Z^+, as follows. Let $\pi(n_k) = m_k$,and $\pi(m_k) = n_k$ for each k, and let π fix all other integers n (i.e. for which $n \neq n_k, m_k$ for every k). The permutation π is well-defined since $n_i \neq m_j$ for every i,j. Let Z_π denote the set $\{n: n=n_k \text{ or } n=m_k \text{ for some } k\}$. Hence $\pi(n) = n$ for all $n \notin Z_\pi$. Then

$$\begin{aligned}\Sigma_n a_n x_{\pi(n)} &= \Sigma_{n \notin Z_\pi} a_n x_n + \Sigma_k (a_{n_k} x_{m_k} + a_{m_k} x_{n_k}) \\ &= \Sigma_{n \notin Z_\pi} a_n x_n + \Sigma_k ((a_{n_k} x_{n_k} + a_{m_k} x_{m_k}) + \\ &\qquad (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}))\end{aligned}$$

$$= \Sigma a_n x_n + \Sigma_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$$

$$= a \cdot x + \epsilon = r \ ,$$

and so $r \in S(a,x)$, which proves (3). Q.E.D.

<u>Theorem 6</u>. Let $a = (a_n)$ where $a_1 > 0$ and $a_n \uparrow \infty$. Then a_{n+1}/a_n is bounded if and only if, for every $x = (x_n)$ for which $x_n \underset{\neq}{\downarrow} 0$, $S(a,x) = [a \cdot x, \infty]$.

<u>Proof</u>. If a_{n+1}/a_n is bounded, then by Theorem 5, if $x_n \underset{\neq}{\downarrow} 0$, then $x = (x_n)$ satisfies condition (2) of the theorem. Also by Theorem 5, since $a_n \uparrow \infty$ and $a_1 > 0$, condition (3) of the theorem is satisfied by x. That is, $S(a,x) = [a \cdot x, \infty]$.

Conversely, if $S(a,x) = [a \cdot x, \infty]$ for every $x = (x_n)$ for which $x_n \underset{\neq}{\downarrow} 0$, we claim that a_{n+1}/a_n must remain bounded. This result and its forthcoming proof are due to Paul Erdos.

Suppose to the contrary that a_{n+1}/a_n is not bounded. Let $h(n)$ denote the least positive integer k for which $k \geq n$ and $a_{k+1}/a_k \geq 4^n$. Clearly $h(n)$ is a non-decreasing function of n. Define $x_n = (a_{h(n)} 3^n)^{-1}$. Then $x_n \underset{\neq}{\downarrow} 0$. Letting $x = (x_n)$, we claim that $S(a,x) \neq [a \cdot x, \infty]$. In fact, we claim that $a \cdot x < 1$ but $1 \notin S(a,x)$. Indeed,

$$a \cdot x = \Sigma a_n x_n = \Sigma a_n (a_{h(n)} 3^n)^{-1} \leq \Sigma 3^{-n} = 1/2 < 1.$$

Furthermore, letting π be any permutation of Z^+, if $\pi^{-1}(k) > h(k)$ for some k, then

$$\Sigma a_n x_{\pi(n)} \geq a_{\pi^{-1}(k)} x_k \geq a_{h(k)+1} x_k = a_{h(k)+1} (a_{h(k)} 3^k)^{-1}$$

$\geq 4^k 3^{-k} > 1$.

On the other hand, if $\pi^{-1}(k) \leq h(k)$ for every k, then

$$\Sigma\, a_n x_{\pi(n)} = \Sigma\, a_{\pi^{-1}(n)} x_n \leq \Sigma\, a_{h(n)} x_n = \Sigma\, 3^{-n} = 1/2 < 1.$$

In any case, $\Sigma\, a_n x_{\pi(n)} \neq 1$, hence $1 \notin S(a,x)$. Q.E.D.

__Note.__ In regard to Theorem 5, each time we constructed a permutation π to solve the equation $\Sigma\, a_n x_{\pi(n)} = r$, it sufficed to use only disjoint 2-cycles. That is, each such π that we constructed was the product of disjoint 2-cycles. This seems odd and leads us to ask if there are any circumstances in which the use of infinite-cycles or n-cycles yields more. In other words, is it always true that S(a,x) is the same as $\{\Sigma\, a_n x_{\pi(n)} : \pi$ is a permutation of Z^+ which is a product of disjoint 2-cycles$\}$?

The following question seems likely to have an affirmative answer. If so, this would give a characterization for those sequences a and x where $a_n \uparrow \infty$, $a_1 > 0$, and $x_n \downarrow_{\neq} 0$, which satisfy $S(a,x) = [a\cdot x, \infty]$. However, it remains unsolved.

__Question 7.__ If a and x are as above, does (3) $\Longrightarrow$ (2) in Theorem 5 ?

Finally, we wish to point out that Theorems 5 and 6 imply analogous theorems in which a and x switch roles. Indeed, the following two corollaries follow easily by setting $a_n' = 1/x_n$ and $x_n' = 1/a_n$ and applying Theorems 5 and 6 to the sequences (a_n') and (x_n').

<u>Corollary 8</u>. Let $x = (x_n)$ where $x_n > 0$ for all n, and $x_n \longrightarrow 0$ as $n \longrightarrow \infty$. Consider the following conditions.

(1) x_n/x_{n+1} is bounded below.

(2) For the non-negative sequence $a = (a_n)$, there exist subsequences (a_{n_k}) and (x_{m_k}) of a and x, respectively, such that

a) $a_{n_k} x_{m_k} \longrightarrow 0$ as $k \longrightarrow \infty$, and

b) $\Sigma_k a_{n_k} x_{m_k} = \infty$.

Then (1) implies that (2) holds for every strictly positive sequence $a = (a_n)$ that tends to ∞.

<u>Corollary 9</u>. Let $x = (x_n)$ be a non-negative sequence. Then x_n/x_{n+1} is bounded below if and only if, for every $a = (a_n)$ for which $a_n \uparrow \infty$ and $a_1 > 0$, $S(a,x) = [a \cdot x, \infty]$.

This concludes Chapter 4.

CHAPTER 5

MISCELLANEOUS TOPICS

1. Sparse Matrices for Two Special Normal Operators

In Chapter 2, we mentioned that the Halmos and Berg question [3] asking whether or not every normal operator is the sum of a diagonalizable operator and a Hilbert-Schmidt operator is related to our question on whether or not the ideal C_2 possesses the generalized Fuglede property. Indeed, any normal operator A for which there exists a bounded operator B so that $AB-BA \in C_2$ but yet $A^*B-BA^* \notin C_2$ must necessarily fail to be the sum of a diagonalizable and a Hilbert-Schmidt operator.

The results in Berg's paper imply that if such a normal operator exists, then it must necessarily have some pure continuous spectrum of positive planar area. He also conjectures that every normal operator which has some pure continuous spectrum of positive planar area fails to be decomposable into the sum of a diagonalizable and a Hilbert-Schmidt operator. For this reason, and since the techniques of Chapter 2 related to the GFP are essentially matricial, it appears that the next step in the strategy (to find $A \in (N)$, $B \in L(H)$ so that $AB-BA \in C_2$ but $A^*B-BA^* \notin C_2$) is to produce a normal operator with some pure continuous spectrum of positive planar area which has a sparse enough matrix to make the necessary calculations possible.

In what follows, we exhibit two normal operators which have some pure continuous spectrum of positive planar area, and for each one, we give a matrix representation which is sparse. For one of them, the matrix we exhibit is as sparse as we could hope for (this can be proven, but we shall not do it here).

To exhibit the matrices, we first need some notation. Let $L^2(T^2)$ denote the Hilbert space of square Lebesgue integrable functions on the torus, where here T^2 denotes the torus. Let $L^2([-\pi,\pi]x[-\pi,\pi])$ denote the square Lebesgue integrable functions on the square $[-\pi,\pi]x[-\pi,\pi]$. Let U denote the bilateral shift of multiplicity 1 acting on $\ell^2(Z)$ where Z denotes the set of all integers. Let T denote the two-way infinite Toeplitz matrix whose entries are constant along the diagonal as well as every subdiagonal and superdiagonal, which is given by $T = (t_{ij})$ where $t_{ij} = (-1)^{i-j}/(i-j)$ if $i \neq j$, and $t_{ij} = 0$ if $i = j$.

In both Hilbert spaces above, it is well-known that the functions $e_{n,m} = e^{i(nx+my)}$ for $n,m \in Z$, form a basis. By calculating the Fourier coefficients and carefully grouping them, we obtain the next proposition. We omit the details.

Proposition 1.1.

(1) For M_{z+w} acting on $L^2(T^2)$, $M_{z+w} \cong I\otimes U + U\otimes I$.

(2) For M_{z+w} acting on $L^2([-\pi,\pi]x[-\pi,\pi])$, $M_{z+w} \cong I\otimes T + i(T\otimes I)$.

Remark. Note the similarity between the forms $I\otimes U + U\otimes I$ and $I\otimes T + i(T\otimes I)$. Note that T is pure imaginary (i.e. $T^* = -T$). Note the degree of sparseness of $I\otimes T + i(T\otimes I)$. Finally, note the degree of sparseness of $I\otimes U + U\otimes I$ which, in some sense, has only two nonzero subdiagonals, and has 0 entries elsewhere.

Note. We mentioned earlier that we proved elsewhere that C_p $(0 < p < 1)$ fails to have the GFP. It is $I\otimes U + U\otimes I$ which is the normal operator we used to prove this.

This concludes Section 1.

2. The "Stretch" Axiom for Ideals

In this section, we introduce a new axiom for the characteristic set of an ideal. Recall axiom 3 of a characteristic set C (see p. 58) which states that $(x_n) \in C$ implies $(x_1,x_1,x_2,x_2,x_3,\ldots) \in C$. In other words, "stretching" the sequence so that each original entry appears twice keeps the stretched sequence inside C. Often, stretching a sequence more than this chases it out of C. For example, one can show that if $C = \{(x_n): x_n\downarrow \text{ and } x_n = O(1/n)\}$, which we already know is a characteristic set (see p. 58), then $(1/n) \in C$, but yet $(1,1/2,1/2,1/3,1/3,1/3,1/4,\ldots) \notin C$.

A precise way to formulate this idea is as follows. Let $m = (m_n)$ be an arbitrary sequence of positive integers. Let $x = (x_n) \in C$.

Definition 2.1.

(a) $\underline{x^{(m)}} = (x_1,\ldots,x_1,x_2,\ldots,x_2,x_3,\ldots)$ where each x_n is repeated m_n times.

(b) (m_n), when used in this way, is called a frequency sequence.

(c) C is closed with respect to the frequency sequence (m_n) provided $x \in C \Longrightarrow x^{(m)} \in C$.

(d) $\underline{F_C}$ denotes the set of all frequency sequences with respect to which C is closed.

Note. Axiom 3 of characteristic sets implies that all constant positive integer sequences are contained in F_C for every characteristic set C. In fact, F_C contains all bounded frequency sequences. Sometimes F_C contains no more than these frequency sequences, as is the case for the characteristic set $C = \left\{(x_n) : x_n\downarrow \text{ and } x_n = O(1/n)\right\}$(We leave the proof to the reader). On the other hand, sometimes F_C is quite full. For example, F_C is the set of all positive integer sequences if C is the characteristic set corresponding to the Calkin ideal set c_o^+, alias corresponding to the ideal K(H).

The following proposition gives some of the properties that F_C inherits from C. We omit the proof.

Proposition 2.2. Let C be a characteristic set. Then F_C satisfies the following closure laws.

(1) $m_n' \le m_n$ for all n and $(m_n) \in F_C \Longrightarrow (m_n') \in F_C$.

(2) $(m_n'), (m_n) \in F_C \Longrightarrow (m_n' + m_n) \in F_C$.

(3) $(m_n) \in F_C \Longrightarrow (m_{n+p}) \in F_C$ for each fixed positive integer p.

Note. For a characteristic set C, F_C contains all the positive integer sequences if and only if the ideal corresponding to C is K(H).

Remark. Independently of this thesis, in joint work with Andreas Blass, we use related ideas to solve a problem of Brown, Pearcy, and Salinas [7] that is central to the theory of operator ideals. Namely, we prove that K(H) = I + J for two ideals of L(H) for which $I, J \subsetneq K(H)$. The proof uses these ideas to establish an entirely new characterization of operator ideals which is quite different from Calkin's ideal sets and from Salinas's characteristic sets. It also requires that we assume the continuum hypothesis (or a weaker version of it) as well as the axiom of choice and transfinite induction. This part of the argument is due to Andreas Blass.

This concludes Section 2.

3. Admissible Functions and a Problem of Brown, Pearcy, and Salinas

Central to the work of Brown, Pearcy, and Salinas [7] is their concept of "admissible function". An admissible function is a non-decreasing real-valued function f(t) defined on $[0,\infty)$ satisfying (1) f(0) = 0 ; (2) $\lim_{t \to 0+} f(t) = 0$;

and (3) $f(t) > 0$ for $t > 0$. They prove [7, Theorem 4.2] that a necessary and sufficient condition on $f(t)$ so that $\{(\lambda_n) \in c_o^+ : \Sigma_n f(\lambda_n) < \infty\}$ is a Calkin ideal set is that there exists $\alpha > 1$ so that $f(\alpha t)/f(t)$ is bounded on $(0,\epsilon)$ for some $\epsilon > 0$.

Then they asked a question [7, p. 377] which, by their results, is equivalent to the following question. If $f(t)$ is an admissible function for which $f(\alpha t)/f(t)$ is bounded on $(0,\epsilon)$ for some $\alpha > 1$ and some $\epsilon > 0$, does there exist a continuous admissible function $g(t)$ so that $f(t)/g(t)$ and $g(t)/f(t)$ are bounded on $(0,\epsilon)$ for some $\epsilon > 0$? Simultaneously with and independently of Salinas [32] we answered this question in the affirmative. The result is as follows.

<u>Proposition 3.1</u>. Let $f(t)$ be an admissible function where, for some $\alpha > 1$, $f(\alpha t)/f(t)$ is bounded on $(0,\epsilon)$ for some $\epsilon > 0$. Define $g(t)$ as follows:

$$g(t) = t^{-1} \int_t^{\alpha t} f(u)\, du = \int_1^{\alpha} f(tx)\, dx ,$$

where the integration is Lebesgue integration. Then $g(t)$ is continuous, admissible, and g/f and f/g are bounded on $(0,\epsilon)$ for some $\epsilon > 0$.

<u>Proof</u>. The above equality between the integrals follows from the substitution $u = tx$, which is a valid technique in real variable theory since $f(t)$ is a monotone function.

Using the first integral, it is clear that $g(t)$ is continuous. Using the second integral, it is clear that

$$(\alpha-1)\, f(t) \leq g(t) \leq (\alpha-1)\, f(\alpha t) \quad .$$

Therefore f/g is bounded, and since $f(\alpha t)/f(t)$ is bounded, we obtain that g/f is bounded (that is, on the appropriate intervals). Therefore it suffices to show that $g(t)$ is admissible.

Clearly $g(0) = 0$ and $\lim_{t \to 0+} g(t) = 0$. Hence all we need show is that $g(t)$ is non-decreasing. The second integral yields this easily. Q.E.D.

<u>Remark</u>. If we define $g_1(t) = \int_1^\alpha g(tx)\, dx$, then the same proof shows that $g_1(t)$ is also a solution to the above question. In addition, g_1 is clearly differentiable. Proceeding in this way we can obtain solutions that are n-differentiable. However, we do not know if one can obtain solutions that are infinitely differentiable on $(0,\epsilon)$ for some $\epsilon > 0$.

This concludes Section 3 and Chapter 5.

REFERENCES

1. J.H. Anderson, "Derivations, commutators, and the essential numerical range". Dissertation, Indiana University, 1971.

2. J.H. Anderson and J.G. Stampfli, "Commutators and compressions". Israel J. Math. 10 (1971), 433-441.

3. I.D. Berg, "An extension of the Weyl-von Neumann Theorem to normal operators". Trans. Amer. Math. Soc. 160 (1971), 365-371.

4. C.A. Berger and B.I. Shaw, "Self-commutators of multicyclic hyponormal operators are always trace class". Bull. Amer. Math. Soc. 79 (1973), 1193-1199.

5. A. Brown, P.R. Halmos, and C. Pearcy, "Commutators of operators on Hilbert space". Canad. J. Math. 17 (1965), 695-708.

6. A. Brown and C. Pearcy, "Structure of commutators of operators". Ann. of Math. (2) 82 (1965), 112-127.

7. A. Brown, C. Pearcy, and N. Salinas, "Ideals of compact operators on Hilbert space". Michigan Math. J. 18 (1971), 373-384.

8. L.G. Brown, R.G. Douglas, and P.A. Fillmore, "Unitary equivalence modulo the compact operators and extensions of C*-algebras". Springer-Verlag, 345 (1973), 58-128.

9. J.W. Calkin, "Two-sided ideals and congruences in the ring of bounded operators on Hilbert space". Ann. of Math. (2) 42 (1941), 839-873.

10. Peter W. Day, "Rearrangement inequalities". Canad. J. Math. 24 (1972), 930-943.

11. Peter W. Day, "Decreasing rearrangements and doubly stochastic operators". Trans. Amer. Math. Soc. 178 (1973), 383-392.

12. J.A. Deddens and J.G. Stampfli, "On a question of Douglas and Fillmore". Bull. Amer. Math. Soc. 79 (1973), 327-330.

13. I.M. Gelfand and N. Ja Vilenkin, Generalized Functions Vol. 4. Applications of Harmonic Analysis, Fizmatgiz, Moscow 1961. English translation, Academic Press, New York, 1964.

14. I.C. Gohberg and M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, Rhode Island, 1969.

15. P.R. Halmos, "Commutators of operators". Amer. J. Math. 74 (1952), 273-240.

16. P.R. Halmos, "Commutators of operators II". Amer. J. Math. 76 (1954), 191-198.

17. P.R. Halmos, "A glimpse into Hilbert space". Lectures on Modern Mathematics, Vol. 1, pp. 1-22, Wiley, New York, 1963.

18. P.R. Halmos, A Hilbert Space Problem Book. D. Van Nostrand Co., Inc., Princeton, New Jersey-Toronto, Ontario-London, 1967.

19. P.R. Halmos, "Ten problems in Hilbert space". Bull. Amer. Math. Soc. 76 (1970), 887-933.

20. J.W. Helton and R.E. Howe, "Integral operators: commutators, traces, index and homology". Springer-Verlag 345 (1973), 141-209.

21. D.C. Kleineke, "On operator commutators". Proc. Amer. Math. Soc. 8 (1957), 535-536.

22. Charles A. McCarthy, "C_p". Israel J. Math. 5 (1967), 249-271.

23. David Morris and Norberto Salinas, "Semiprime ideals and irreducible ideals of the ring of bounded operators on Hilbert space". Indiana Univ. Math. J. 23 (1973/74), 575-589.

24. Carl Pearcy, "Some unsolved problems in operator theory". Preprint circulated in 1972.

25. C. Pearcy, "On commutators of operators in Hilbert space". Proc. Amer. Math. Soc. 16 (1965), 53-59.

26. C. Pearcy and David Topping, "On commutators in ideals of compact operators". Michigan Math. J. 18 (1971), 247-252.

27. C.R. Putnam, "Commutation properties of Hilbert space operators and related topics". Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 36. Springer-Verlag, Inc., New York, 1967.

28. H. Radjavi, "The structure of A*A-AA*". J. Math. Mech. 16 (1966), 19-26.

29. M.S. Ramanujan, "Power series spaces $\Lambda(\alpha)$ and associated $\Lambda(\alpha)$-nuclearity". Math. Ann. 189 (1970), 161-168.

30. M.S. Ramanujan, "On a class of double sequence transformations". Ann. Polon. Math. 5 (1958), 55-65.

31. N. Salinas, "Ideals of commutators of compact operators". Acta. Sci. Math. 36 (1974), 131-144.

32. N. Salinas, "Ideal sets and ideals of compact operators on Hilbert space". Indiana Univ. Math. J. 22 (1973/74), 505-521.

33. N. Salinas, "Symmetric norm ideals and relative conjugate ideals". Trans. Amer. Math. Soc. 188 (1974), 213-240.

34. R. Schatten, "Norm ideals of completely continuous operators". Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., Heft 27. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.

35. F.V. Shirokov, "Proof of a conjecture of Kaplansky". Uspekhi Mat. Nauk (N.S.) 11 (1956), no. 4(70), 167-168.

36. J. von Neumann, "Charakterisierung des Spektrums eines Integraloperators". Actualites Sci. Indust., no. 229, Hermann, Paris, 1935.

37. H. Weyl, "Über beschränkte quadratischen Formen deren Differenz vollstetig ist". Rend. Circ. Mat. Palermo 27 (1909), 373-392.

38. H. Wielandt, "Über die unbeschränktheit der Operatoren der Quantenmechanik". Math. Ann. 121, 21 (1949).

39. A. Wintner, "The unboundedness of quantum-mechanical matrices". Physical Rev. (2) 71 (1947), 738-739.