A limiting random analytic function related to the CUE

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September 2014

Introduction, presentation of the model

Random matrix theory is a very large and rich mathematical subject, which has much developed in the last decades, and which is related to different parts of mathematics and theoretical physics. Two of the most classical examples of random matrices are the following:

- The *Gaussian Unitary Ensemble*, corresponding to a random hermitian matrix for which the entries above the diagonal are independent, complex gaussian random variables.
- The *Circular Unitary Ensemble*, corresponding to a random unitary matrix following the Haar (i.e. uniform) measure on a unitary group.

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In this presentation, we will focus our study on the Circular Unitary Ensemble (CUE). When a random matrix model is considered, it is natural to study the corresponding distribution of the eigenvalues. In particular, a fundamental object is the characteristic polynomial of the random matrix. More precisely:

- We consider, for $n \geq 1$, u_n a Haar-distributed matrix on the unitary group $U(n)$.
- We denote by $(\lambda_k^{(n)})_{1 \leq k \leq n}$ the eigenvalues of u_n, ordered counterclockwise starting from 1: recall that all the eigenvalues have modulus 1.
- For $1 \leq k \leq n$, we denote by $\theta_k^{(n)} \in [0, 2\pi)$ the argument of $\lambda_k^{(n)}$. We extend the notation to all $k \in \mathbb{Z}$, in the unique way such that $\theta_{k+n}^{(n)} = \theta_k^{(n)} + 2\pi$.

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- The characteristic polynomial Z_n of u_n is then defined as follows: for $z \in \mathbb{C}$,

$$Z_n(z) := \det(zI_n - u_n) = \prod_{k=1}^{n} (z - \lambda_k^{(n)}).$$

- One can prove that for $|z| < 1$, $n \to \infty$, $Z_n(z)$ converges in law to a limiting random variable (consequence of a result by Diaconis and Shahshahani on the distribution of eigenvalues).
- Such a convergence does not hold for $|z| \geq 1$. For $|z_n| = 1$, Keating and Snaith have proven that $\log |Z_n(z)| / \sqrt{\log n}$ converges to a gaussian random variable.
- In our article, we consider another way to get convergence of Z_n on, or near the unit circle.

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Statement of the main result

- We make a zoom around a given point of the unit circle, say 1, and we renomalize the characteristic polynomial. More precisely, we consider the ratio:

$$\xi_n(z) := \frac{Z_n(e^{2i\pi z/n})}{Z_n(1)}.$$

- This ratio defines a random holomorphic function $(\xi_n(z))_{z\in\mathbb{C}}$. We show the following fact: there exists a random holomorphic function $(\xi_\infty(z))_{z\in\mathbb{C}}$ such that $(\xi_n(z))_{z\in\mathbb{C}}$ converges in law to $(\xi_\infty(z))_{z\in\mathbb{C}}$ when $n \to \infty$, in the space of continuous functions from $\mathbb{C}$ to $\mathbb{C}$, endowed with the topology of uniform convergence on compact sets.

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Some properties of ξ_∞ are the following:

- All the zeros of ξ_∞ are real, as well as the zeros of ξ_n for all $n \geq 1$.
- The set E of the zeros of ξ_∞ is a *determinantal sine-kernel point process*, i.e. for $m \geq 1$, f nonnegative and measurable from $\mathbb{R}^m$ to $\mathbb{R}$,

$$\mathbb{E}\left[\sum_{x_1 \neq \cdots \neq x_m \in E} f(x_1, \ldots, x_n)\right]$$

$$= \int_{\mathbb{R}^m} f(y_1, \ldots, y_m) \rho_m(y_1, \ldots, y_m) dy_1 \ldots dy_m$$

where

$$\rho_m(y_1, \ldots, y_m) = \det\left(\frac{\sin(\pi(y_p - y_q))}{\pi(y_p - y_q)}\right)_{1 \leq p,q \leq m}.$$

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- The function ρ_m is called the *m-point correlation function* of the point process E.
- ρ_1 is identically equal to 1, so E has the same 1-point correlation as a Poisson point process of intensity 1.
- The 2-point correlation function is smaller than or equal to 1, so the points of E tend to repel each other. When $x_1 - x_2 \to 0$, $\rho_2(x_1, x_2)$ is equivalent to $\pi^2(x_1 - x_2)^2/3$, and in particular tends to zero.
- The function ξ_∞ has order one in the sense of the complex analysis. More precisely, there almost surely exist $C > c > 0$ such that for all $z \in \mathbb{C}$, $x \in \mathbb{R}$,

$$|\xi_\infty(z)| \leq e^{C|z|\log(2+|z|)}, |\xi_\infty(ix)| \geq ce^{c|x|}.$$

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The notion of virtual isometry

A classical result (essentially due to Dyson) about the CUE is the following: if we multiply the eigenangles of u_n by $n/2\pi$, then the corresponding point process

$$E_n = \{y_k^{(n)} := n\theta_k^{(n)}/2\pi, k \in \mathbb{Z}\}$$

weakly converges to a determinantal sine-kernel process E.

- This weak convergence means the following: for all functions f, continuous with compact support from $\mathbb{R}$ to $\mathbb{R}$,

$$\sum_{x \in E_n} f(x) \underset{n \to \infty}{\longrightarrow} \sum_{x \in E} f(x)$$

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- The convergence of E_n toward E is a weak convergence: if we want to get a strong convergence, we need to define all the matrices $(u_n)_{n\geq 1}$ on the same probability space.
- If $(u_n)_{n\geq 1}$ are independent, then a strong convergence cannot occur, by the zero-one law.
- In articles with Bourgade, Maples and Nikeghbali, we study a particular coupling of the dimensions n, in such a way that an almost sure convergence occurs.
- In our construction, the sequence $(u_n)_{n\geq 1}$ is almost surely a so-called *virtual isometry*.

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A virtual isometry is a sequence $(u_n)_{n \geq 1}$ which can be constructed as follows:

- For $n \geq 1$, we consider a point x_n on the unit sphere of $\mathbb{C}^n$.
- If e_n is the last canonical basis vector of $\mathbb{C}^n$, we set $r_{e_n} := I_n$, and if $x_n \neq e_n$, r_{x_n} is the unique element of $U(n)$ such that $r_{x_n}(e_n) = x_n$ and $r_{x_n} - I_n$ has rank one.
- One has $u_1 = x_1$, and for all $n \geq 2$, $u_n = r_{x_n} \mathrm{Diag}(u_{n-1}, 1)$.
- From now, we assume that for $n \geq 1$, x_n is uniform on the unit sphere of $\mathbb{C}^n$, and that $(x_n)_{n \geq 1}$ are independent. It is then possible to prove that u_n follows the Haar measure on $U(n)$ for all $n \geq 1$.

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With the coupling defined just above, one has a strong convergence of E_n toward E. More precisely:

- By using representation theory, Borodin, Olshanski and Vershik have proven the almost sure convergence of E_n towards a determinantal sine-kernel process E.
- In our joint paper with Bourgade and Nikeghbali, we give another proof, more elementary and purely probabilistic, and we obtain an estimate for the corresponding rate of convergence.
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Our precise result is the following:

- There exists a random sequence $(y_k)_{k\in\mathbb{Z}}$ such that almost surely, for all $\varepsilon > 0$, there exists C_ε such that for all $k \in \mathbb{Z}$, $n \geq k^4$,

$$|y_k^{(n)} - y_k| \leq C_\varepsilon (1 + k^2) n^{-\frac{1}{3}+\varepsilon}.$$

- In particular, almost surely, for all $k \in \mathbb{Z}$, $y_k^{(n)}$ tends to y_k when n goes to infinity.
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Application to the characteristic polynomial

In our article with Chhaibi and Nikeghbali, we use the virtual isometries in order to prove the convergence of ξ_n towards ξ_∞ stated above.

- An elementary computation gives the following formula:

$$\xi_n(z) = e^{i\pi z} \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}} \right).$$

- The infinite product just above is not absolutely convergent. The absolute convergence occurs when we regroup the term of index k with the term of index $1 - k$, for $k \geq 0$.

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- With the coupling of the virtual isometries, we have almost surely $y_k^{(n)} \to y_k$ for $n \to \infty$.
- From this convergence, it is natural to expect the following result

$$\xi_n(z) \underset{n\to\infty}{\longrightarrow} e^{i\pi z} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{y_k}\right) =: \xi_\infty(z).$$

- By using some estimates on the distribution of the points of a determinanatal sine-kernel process, deduced from results by Costin, Lebowitz, Meckes and Soshnikov, one shows that the previous infinite product converges if we regroup the terms of indices k and $1-k$.

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- By using the estimate of $y_k^{(n)} - y_k$ stated above, and some bounds on the global distribution of eigenvalues of u_n, we are able to prove that almost surely, ξ_n converges to ξ_∞, uniformly on compact sets of $\mathbb{C}$.
- If we forget about the coupling between the different dimensions, we get the weak convergence stated at the beginning of the talk.
- From the infinite product giving ξ_∞, one deduces that the zeros of this function are exactly $(y_k)_{k \in \mathbb{Z}}$: hence, they are all real and form a determinantal sine-kernel process
- One also uses the infinite product in order to show that ξ_∞ is an holomorphic function of order 1.

- By using the estimate of $y_k^{(n)} - y_k$ stated above, and some bounds on the global distribution of eigenvalues of u_n, we are able to prove that almost surely, ξ_n converges to ξ_∞, uniformly on compact sets of $\mathbb{C}$.
- If we forget about the coupling between the different dimensions, we get the weak convergence stated at the beginning of the talk.
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- For any $z \in \mathbb{C}$, the random variable $\xi_\infty(z)$ has no moment of order 1 or higher. This fact is an obstacle for a direct proof of the convergence in law of ξ_n towards ξ_∞, without using the coupling of the virtual isometries.
- However, the logarithmic derivative of ξ_∞ has moments of any order, at any point on $\mathbb{C}\backslash\mathbb{R}$.
- We have computed the moment of order 1, and the joint moments of order 2. For example, for $z \notin \mathbb{R}$:

$$\mathbb{E}\left[\frac{\xi'_\infty(z)}{\xi_\infty(z)}\right] = 2i\pi \mathbb{1}_{\Im(z)<0}$$

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$$\mathbb{E}\left[\left|\frac{\xi'_\infty(z)}{\xi_\infty(z)}\right|\right] = 4\pi^2 \mathbb{1}_{\Im(z)<0} + \frac{1 - e^{-4\pi|\Im(z)|}}{4(\Im(z))^2}.$$

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Link with the Riemann zeta function

The convergence of the renormalized characteristic polynomial toward ξ_∞ is directly related to the convergence of the point process E_n towards a determinantal sine-kernel process. Now, let ζ be the Riemann zeta function, i.e. the unique meromorphic function on $\mathbb{C}$ such that for all s of real part strictly larger than 1,

$$\zeta(s) = \sum_{n \geq 1} n^{-s}.$$

- The Riemann hypothesis says that all the zeros of ζ whose real part are in $[0,1]$ are in fact on the critical line $\{s \in \mathbb{C}, \Re(s) = 1/2\}$.
- A conjecture by Montgomery, generalized by Rudnik and Sarnak, says, in a sense which can be made precise, that the distribution of zeros of ζ, properly renormalized, tends to the distribution of a determinanatal sine-kernel process, when the imaginary part goes to infinity.

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- Using this conjecture and a classical expression of ζ as a Hadamard product, we have stated the following conjecture. If U is a uniform variable on $[0,1]$, then we have the convergences in law:

$$\left(\frac{\zeta(\frac{1}{2}+iTU-\frac{2i\pi z}{\log T})}{\zeta(\frac{1}{2}+iTU)}\right)_{z\in\mathbb{C}} \underset{T\to\infty}{\longrightarrow} (\xi_\infty(z))_{z\in\mathbb{C}},$$

$$\left(\frac{-2i\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iTU-\frac{2i\pi z}{\log T}\right)\right)_{z\in\mathbb{C}\backslash\mathbb{R}} \underset{T\to\infty}{\longrightarrow} \left(\frac{\xi'_\infty}{\xi_\infty}(z)\right)_{z\in\mathbb{C}\backslash\mathbb{R}}.$$

- For this last convergence in law, we also expect the corresponding convergence for the joint moments. Such a convergence is directly related to conjectures by Goldston, Gonek and Montgomery on the moments of ζ'/ζ near the critical line.

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Thank you for your attention!