The normal matrix model with monomial potential and multi-orthogonality on a star

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Main ideas in the talk

- There is a natural connection between:
  1) The **global asymptotic distribution of eigenvalues** in the normal matrix model with monomial potential.
  2) The **limiting zero distribution** of a certain sequence of polynomials.

  The limiting zero distribution of the sequence of polynomials is part of the solution to a (vector) equilibrium problem.

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- The polynomials are multi-orthogonal with respect to a system of weights defined on a star-like set.

- The problem is investigated in a pre-critical regime (for a certain parameter in the model).

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that induces on D^n (the space of eigenvalues) the probability distribution

$$\frac{1}{Z_n} \exp\left(-n \sum_{i=1}^{n} \mathcal{V}(z_i)\right) \prod_{i<j} |z_i - z_j|^2 \, \mathrm{d}A(z_1) \ldots \mathrm{d}A(z_n),$$

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Wiegmann-Zabrodin ’00

Elbau-Felder ’05

Ameur-Hedenmalm-Makarov ’11

Bleher-Kuijlaars ’12

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NMM is naturally tied with the study of **(Bergmann) orthogonal polynomials** associated with the inner product

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(1) is a **determinantal point process** with correlation kernel

$$K_n(z, w) = \exp\left(-\frac{n}{2} (\mathcal{V}(w) + \mathcal{V}(z))\right) \sum_{k=0}^{n-1} q_{k,n}(z) \, \overline{q_{k,n}(w)},$$

where $(q_{k,n})_{k=0}^{\infty}$ are the orthonormal polynomials associated to (2), i.e.,

$$\langle q_{k,n}, q_{l,n} \rangle_D = \delta_{kl}, \qquad q_{k,n}(z) = \gamma_k \, z^k + \cdots, \quad \gamma_k > 0.$$

Global asymptotic distribution of eigenvalues

$$\mathcal{V}(z) := \frac{1}{t_0}(|z|^2 - V(z) - \overline{V(z)}), \qquad t_0 > 0,$$

$$V(z) := \sum_{k=1}^{d+1} \frac{t_k}{k} z^k, \qquad \{t_k\}_{k=1}^{d+1} \subset \mathbb{C},$$

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Theorem (Elbau-Felder '05)

Under certain assumptions on $\mathcal{V}$ and D, for all $t_0 > 0$ ***small enough****,*

$$\frac{1}{n} K_n(z,z)\, \mathrm{d}A(z) \xrightarrow[n\to\infty]{*} \lambda_\Omega, \qquad \lambda_\Omega : \textit{ normalized area measure on } \Omega,$$

where $\Omega \subset D$ is a Jordan domain with $0 \in \operatorname{int}(\Omega)$.

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where $\Omega \subset D$ is a Jordan domain with $0 \in \operatorname{int}(\Omega)$. Moreover, $\mathrm{area}(\Omega) = \pi t_0$ and

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial\Omega} \overline{z} z^{-k}\,\mathrm{d}z = \begin{cases} t_k, & k \in \{1,\dots,d+1\}, \\ 0, & k \in \mathbb{N} \setminus \{1,\dots,d+1\}. \end{cases}$$

Dynamics of $\Omega = \Omega(t_0)$**:** Laplacian growth, Hele-Shaw flow (work of Wiegmann, Zabrodin, Teodorescu, Lee, Bettelheim, Elbau, Ameur, Makarov and others).

Problem: Relation between the eigenvalue asymptotic distribution in the NMM and the zero asymptotic distribution of the orthogonal polynomials $q_{n,n}$.

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In the case

$$V(z) = \frac{t_{d+1}}{d+1} z^{d+1}, \qquad t_{d+1} > 0,$$

we establish a relation between the eigenvalue asymptotic distribution in the NMM and the zero asymptotic distribution of a sequence of **multi-orthogonal polynomials** $P_{n,n}$ associated with weights supported on a star-like set. The zero asymptotic distribution solves a (vector) equilibrium problem. This generalizes work of **Bleher-Kuijlaars** '12 for $d = 2$.

The approach of Bleher-Kuijlaars to the NMM

Inner product

$$\langle f, g\rangle_D = \iint_D f(z)\,\overline{g(z)}\,e^{-n\mathcal{V}(z)}\,\mathrm{d}A(z),$$
$$\mathcal{V}(z) = \frac{1}{t_0}(|z|^2 - V(z) - \overline{V(z)}).$$

Applying Green's formula on D, for polynomials p and q,

$$t_0\langle p, q'\rangle_D - n\langle zp, q\rangle_D + n\langle p, V'q\rangle_D = \frac{t_0}{2i}\oint_{\partial D} p(z)\,\overline{q(z)}\,e^{-n\mathcal{V}(z)}\,\mathrm{d}z.$$

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Bleher-Kuijlaars neglect the boundary term on the right-hand side and this leads to the study of **sesquilinear forms** $\langle\cdot,\cdot\rangle$ satisfying the structure relation

$$t_0\langle p, q'\rangle - n\langle zp, q\rangle + n\langle p, V'q\rangle = 0.$$

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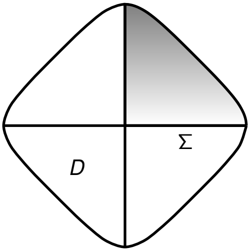
Bleher-Kuijlaars conjecture: For any polynomial

$$V(z) = \sum_{k=1}^{d+1} \frac{t_k}{k} z^k,$$

there is a suitable choice of a sesquilinear form $\langle \cdot, \cdot \rangle$ satisfying (3) such that, for t_0 small enough, the orthogonal polynomials associated with the sesquilinear form and the Bergmann orthogonal polynomials in the NMM will have the same asymptotic behavior.

The monomial case $V(z) = \frac{t_{d+1}}{d+1} z^{d+1}$, $t_{d+1} > 0$

D: simply-connected, Jordan domain with 0 in its origin, invariant under $z \mapsto \exp\left(\frac{2\pi \mathrm{i}}{d+1}\right) z$ and $z \mapsto \overline{z}$.



Let $\Sigma = \{z \in D : z^{d+1} \in \mathbb{R}^+\}$, the $(d+1)$-star.

Green's theorem applied on the sectors of D gives

$$2\mathrm{i} \iint_D Q(z)\, \overline{z}^j e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})} \mathrm{d}A(z) = \int_\Sigma Q(z)\, w_{j,n}(z)\, \mathrm{d}z + \oint_{\partial D} Q(z)\, \widetilde{w}_{j,n}(z)\, \mathrm{d}z$$

See also **Balogh-Bertola-Lee-McLaughlin**'12.

$$2\mathrm{i} \iint_D Q(z)\, \overline{z}^j e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})} \mathrm{d}A(z) = \int_\Sigma Q(z)\, w_{j,n}(z)\, \mathrm{d}z + \oint_{\partial D} Q(z)\, \widetilde{w}_{j,n}(z)\, \mathrm{d}z$$

For $z \in \Sigma$,

$$w_{j,n}(z) = \int_{\Gamma(\ell)} s^j\, e^{-\frac{n}{t_0}(sz - V(s) - V(z))}\, \mathrm{d}s, \qquad \arg z = \frac{2\pi}{d+1}\,\ell,$$

for $z \in \partial D$,

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Definition

Let $d \geq 2$, and $\widehat{x}, t_0, t_{d+1} > 0$. Then for every $k, n \in \mathbb{N}$, we let $P_{k,n}(z) = z^k + \cdots$ be the monic polynomial that satisfies

$$\int_\Sigma P_{k,n}(z)\, w_{j,n}(z)\, \mathrm{d}z = 0, \qquad j = 0, \ldots, k-1,$$

where $\Sigma = \bigcup_{\ell=0}^{d} \omega^\ell\, [0, \widehat{x}]$, $\omega = \exp(2\pi\mathrm{i}/(d+1))$.

The orthogonality conditions

$$\int_{\Sigma} P_{k,n}(z)\, w_{j,n}(z)\, \mathrm{d}z = 0, \qquad j = 0, \ldots, k-1,$$

can be written in the form

$$\langle P_{k,n}(z), z^j \rangle = 0, \qquad j = 0, \ldots, k-1,$$

with the sesquilinear form $\langle \cdot, \cdot \rangle$ defined by

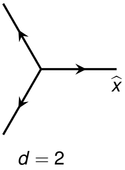
$$\langle p, q \rangle = \sum_{\ell=0}^{d} \int_0^{\omega^\ell \widehat{x}} \mathrm{d}z \int_{\Gamma(\ell)} \mathrm{d}s\, p(z)\, q(s)\, e^{-\frac{n}{t_0}(sz - V(s) - V(z))},$$

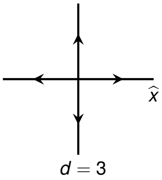
which also satisfies the structure relation

$$t_0 \langle p, q' \rangle - n \langle zp, q \rangle + n \langle p, V' q \rangle = 0.$$

Multi-orthogonality on a star

$$\Sigma := \bigcup_{\ell=0}^{d} \omega^{\ell}\,[0, \widehat{x}], \qquad \widehat{x} > 0, \qquad \omega = \exp\left(\frac{2\pi \mathrm{i}}{d+1}\right).$$





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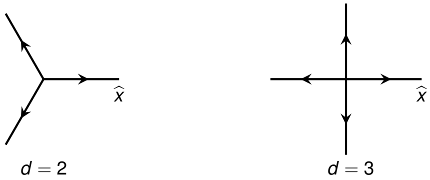

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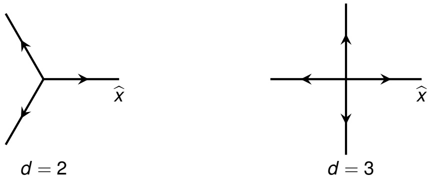

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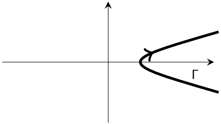
$d = 2 \longrightarrow$ **Airy differential equation.**

$$p(z) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \exp\left(\frac{s^{d+1}}{d+1} - sz\right) \mathrm{d}s,$$

where $\Gamma : e^{-\frac{\pi \mathrm{i}}{d+1}} \infty \longrightarrow e^{\frac{\pi \mathrm{i}}{d+1}} \infty$.



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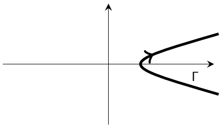

Orthogonality weights $w_{0,n}(z), \ldots, w_{d-1,n}(z)$

$$w_{j,n}(x) := \exp\left(\frac{nV(x)}{t_0}\right) p^{(j)}(c_n x), \qquad x \in [0, \widehat{x}],$$

where

$$V(x) = \frac{t_{d+1}}{d+1} x^{d+1}, \qquad c_n = \left(\frac{n^d}{t_0^d t_{d+1}}\right)^{\frac{1}{d+1}}.$$

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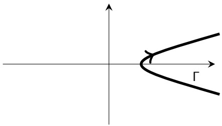

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The definition of $w_{j,n}(z)$ is extended to the whole star Σ so that

$$w_{j,n}(\omega z) = \omega^{d-j} w_{j,n}(z), \qquad z \in \Sigma, \qquad \omega = \exp\left(\frac{2\pi \mathrm{i}}{d+1}\right).$$

Proposition

Fix $t_0, t_{d+1}, \widehat{x} > 0$*. Then the polynomial* $P_{n,n}(z) = z^n + \cdots$ *is multi-orthogonal with respect to the system of weights* $w_{0,n}(z), w_{1,n}(z), \ldots, w_{d-1,n}(z)$*. We have for each* $j = 0, \ldots, d-1$,

$$0 = \int_{\Sigma} P_{n,n}(z)\, z^k\, w_{j,n}(z)\, \mathrm{d}z, \qquad k = 0, \ldots, \frac{n}{d} - 1.$$

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Connection between the polynomials $P_{n,n}$ and the NMM?

Kuijlaars-L.: In a precritical regime for t_0, for a suitable choice of $\widehat{x}$ we will have

$$\frac{1}{n} \sum_{P_{n,n}(z)=0} \delta_z \longrightarrow \mu_1^*,$$

where μ_1^* is a rotationally invariant probability measure with $\mathrm{supp}(\mu_1^*) = \Sigma^* \subset \Sigma$, one has $\Sigma^* = \Sigma^*(t_0) \subset \Omega(t_0)$ and $\Omega(t_0)$ is a **harmonic quadrature domain** for μ_1^*.

Moreover, in the pre-critical regime for t_0, the curve $\partial\Omega(t_0)$ is a hypotrochoid:

$$\partial\Omega(t_0) = \psi([|w| = 1]), \qquad \psi(w) = rw + \frac{t_{d+1} r^d}{w^d},$$

where r is the smallest positive root of $t_0 = r^2 - d\, t_{d+1}^2\, r^{2d}$.

The Schwarz function $S(z)$ associated with $\partial\Omega(t_0)$ is the function

$$S(z) = t_{d+1}\, z^d + \int_{\Sigma^*} \frac{\mathrm{d}\mu_1^*(s)}{z - s}.$$

The measure μ_1^* is the first component of the solution to a vector equilibrium problem for logarithmic potentials.

Vector equilibrium problem

$$\Sigma_1 := \bigcup_{\ell=0}^{d} \omega^{\ell}\,[0, x^*], \qquad x^* > 0, \qquad \omega = \exp\left(\frac{2\pi \mathrm{i}}{d+1}\right),$$

for $k = 2, \ldots, d$,

$$\Sigma_k := \begin{cases} \{z \in \mathbb{C} : z^{d+1} \in \mathbb{R}_-\}, & \text{for } k \text{ even,} \\ \{z \in \mathbb{C} : z^{d+1} \in \mathbb{R}_+\}, & \text{for } k \text{ odd.} \end{cases}$$

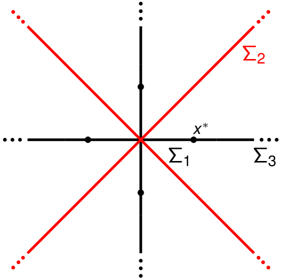

Figure : The stars Σ_k in the case $d = 3$.

$$I(\mu) = \iint \log \frac{1}{|x-y|} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y), \qquad I(\mu,\nu) = \iint \log \frac{1}{|x-y|} \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y).$$

Vector equilibrium problem

Fix $x^*, t_0, t_{d+1} > 0$. Minimize the energy functional

$$E(\mu_1, \mu_2, \ldots, \mu_d)$$
$$= \sum_{k=1}^{d} I(\mu_k) - \sum_{k=1}^{d-1} I(\mu_k, \mu_{k+1}) + \frac{1}{t_0} \int \left(\frac{d}{d+1} \frac{1}{t_{d+1}^{1/d}} |z|^{\frac{d+1}{d}} - \frac{t_{d+1}}{d+1} z^{d+1} \right) \mathrm{d}\mu_1(z)$$

among all positive Borel measures $\mu_1, \ldots, \mu_d$ satisfying:

(1)
$$\|\mu_k\| = \frac{d-k+1}{d}, \qquad k = 1, \ldots, d,$$

(2)
$$\mathrm{supp}(\mu_k) \subset \Sigma_k, \qquad k = 1, \ldots, d.$$

This VEP is weakly admissible, see **Hardy-Kuijlaars '12**, so it admits a unique minimizer $(\mu_1^*, \ldots, \mu_d^*)$.

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Theorem (Kuijlaars-L.)

Let $d \geq 2$ be an arbitrary integer. Fix $t_{d+1} > 0$ and set

$$t_{0,\mathrm{crit}} = t_{d+1}^{-\frac{2}{d-1}} \left(d^{-\frac{2}{d-1}} - d^{-\frac{d+1}{d-1}}\right) > 0.$$

Let $0 < t_0 < t_{0,\mathrm{crit}}$ *and define*

$$x^* = (d+1)\, d^{-\frac{d}{d+1}}\, t_{d+1}^{\frac{1}{d+1}}\, r^{\frac{2d}{d+1}},$$

where r denotes the smallest positive solution of the equation

$$t_0 = r^2 - d\, t_{d+1}^2\, r^{2d}.$$

This VEP is weakly admissible, see **Hardy-Kuijlaars '12**, so it admits a unique minimizer $(\mu_1^*, \dots, \mu_d^*)$.

Theorem (Kuijlaars-L.)

Let $d \geq 2$ be an arbitrary integer. Fix $t_{d+1} > 0$ and set

$$t_{0,\mathrm{crit}} = t_{d+1}^{-\frac{2}{d-1}} \left(d^{-\frac{2}{d-1}} - d^{-\frac{d+1}{d-1}}\right) > 0.$$

Let $0 < t_0 < t_{0,\mathrm{crit}}$ and define

$$x^* = (d+1)\, d^{-\frac{d}{d+1}}\, t_{d+1}^{\frac{1}{d+1}}\, r^{\frac{2d}{d+1}},$$

where r denotes the smallest positive solution of the equation

$$t_0 = r^2 - d\, t_{d+1}^2\, r^{2d}.$$

Under these assumptions on $t_{d+1}, t_0, x^* > 0$,

(1) *μ_1^* has full support, i.e., $\mathrm{supp}(\mu_1^*) = \Sigma_1 = \bigcup_{\ell=0}^{d} \omega^\ell [0, x^*]$.*

(2) *The density of μ_1^* vanishes like a square root at x^*.*

The spectral curve

The Schwarz function

$$\xi = S(z) = t_{d+1}\, z^d + t_0 \int \frac{\mathrm{d}\mu_1^*(t)}{z-t}.$$

associated with the curve $\partial\Omega(t_0)$ satisfies an algebraic equation of the form

$$P(z,\xi) = \xi^{d+1} + z^{d+1} - \sum_{k=1}^{d} c_k z^k\, \xi^k + \beta = 0,$$

where $c_k > 0$ for all $k = 1, \ldots, d$, and $\beta > 0$.

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The Schwarz function admits analytic continuation to a $(d+1)$-sheeted compact Riemann surface $\mathcal{R}$ of genus zero with sheets

$$\mathcal{R}_1 = \overline{\mathbb{C}} \setminus \Sigma_1, \qquad \mathcal{R}_k = \overline{\mathbb{C}} \setminus (\Sigma_{k-1} \cup \Sigma_k), \quad 2 \le k \le d, \qquad \mathcal{R}_{d+1} = \overline{\mathbb{C}} \setminus \Sigma_d.$$