Asymptotics of orthogonal polynomials in normal matrix ensemble

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Joint work with **Roman Riser.**

Many discussions with **Marco Bertola, Robert Buckingham, Maurice Duits, Kenneth McLaughlin,** ...

Main actors:

- Orthogonal polynomials
- Two dimensional Coulomb gas
- Hele-Shaw flow

Orthogonal polynomials on $\mathbb{C}$

Orthogonal polynomials: $p_n(z) = z^n + \dots$

$$\int_{\mathbb{C}} p_j(z)\,\overline{p_k(z)}\,\mathrm{e}^{-NQ(z)}\mathrm{d}A(z) = h_j\delta_{jk}.$$

$Q : \mathbb{C} \to \mathbb{R}$ is the external field; N is a positive parameter.

Examples:

– When $Q(z) = |z|^2$,

$$p_n(z) = z^n.$$

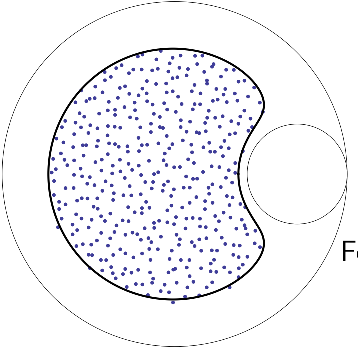
– When $Q(z) = (1-t)(\mathrm{Re}\,z)^2 + (1+t)(\mathrm{Im}\,z)^2$,

$$p_n(z) \propto H_n\left(\sqrt{2n}\frac{z}{F_0}\right); \qquad F_0 = 2\sqrt{\frac{t\,n}{(1-t^2)N}}.$$

2D Coulomb gas (Eigenvalues)

Using the same Q, probablity density function of n point particles, $\{z_1, \cdots, z_n\} \subset \mathbb{C}$, are given by

$$\mathrm{PDF}(\{z_j\text{'s}\}) = \frac{1}{Z_n} \exp\Bigg[-N \underbrace{\Bigg(\sum_{j=1}^{n} Q(z_j) + \frac{2}{N} \sum_{j<k} \log \frac{1}{|z_j - z_k|} \Bigg)}_{\text{2D Coulomb energy}} \Bigg].$$



For $Q(z) = |z|^2 - c \log |z - a|$.

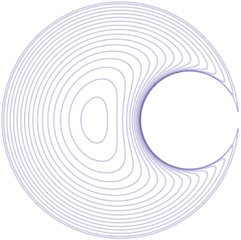
Droplet K (compact set in $\mathbb{C}$)

– Support of the equilibrium measure.

– Throughout this talk, we assume that $\Delta Q = \text{const.}$

– For logarhthmic/rational Hele-Shaw potential, the exterior of K^c is a **quadrature domain**.

– As $T := n/N$ grows, K grows monotonically in T:



We call $T := n/N$ the **total charge** or **(Hele-Shaw) time**. The deformation of K under T follows **Hele-Shaw flow**.

Exterior conformal map of K

For simplicity, we assume that K is **simply connected** so that we can define the unique **riemann mapping**

$$f : K^c \to \overline{\mathbb{D}}^c$$

such that

$$f(z) = \frac{z}{\rho} + \mathcal{O}(1), \quad \rho > 0, \qquad \text{as } |z| \to \infty.$$

Geometry of K is encoded in f.

For example, the **curvature** of the boundary of K is given by

$$\kappa = \mathrm{Re}\left(1 - \frac{f'' f}{(f')^2}\right)|f'|$$

where the prime $'$ stands for the complex derivative.

Similar cases:

– Bergman orthogonal polynomials:

$$\int_D p_n(z)\overline{p_m(z)}dA(z) = h_n \delta_{nm}.$$

$$p_n(z) = \rho^{n+1} f'(z)\, f(z)^n (1 + (\text{corrections})).$$

– Szegö orthogonal polynomials:

$$\oint_\Gamma p_n(z)\overline{p_m(z)}|dz| = h_n \delta_{nm}.$$

$$p_n(z) = \rho^n \sqrt{\rho f'(z)}\, f(z)^n (1 + (\text{corrections})).$$

In both cases, if the relevant geometry has a **smooth boundary**, the correction term is exponentially small in n.

Conjecture

(If the potential Q is such that K has real analytic boundary,) the **strong asymptotics** of $p_n(z)$ as $n \to \infty$ and $N \to \infty$ while $T := n/N$ is finite, is given by

$$p_n(z) = \sqrt{\rho f'(z)}\, e^{ng(z)} \left(1 + \frac{1}{N}\Psi(z) + \mathcal{O}\left(\frac{1}{N^2}\right)\right), \quad z \notin K.$$

The function g (called g-function) is the complex logarithmic potential generated by the measure $\mathbf{1}_K$:

$$g(z) = \frac{1}{\pi T} \int_K \log(z - \zeta) dA(\zeta).$$

The function Ψ is in the next page.

The function Ψ is given by

$$\overline{\Psi(z)} = \frac{i}{2\pi} \oint_{\partial K} \frac{\Phi(\zeta)\, df(\zeta)}{f(\zeta)\left(\overline{f(z)}\, f(\zeta) - 1\right)},$$

where

$$\Phi := \frac{\kappa^2}{12} + \frac{1}{2}\kappa(|f'| - \kappa) + \frac{1}{4}\mathrm{Re}\left(\frac{f''' f^2}{f'^2} - \frac{1}{2}\frac{f''^2 f^2}{f'^4}\right)|f'|^2 + \frac{i}{2}\partial_{\|}|f'|.$$

Remark. The method (that we will explain) can generate the corrections in the arbitrary order in $1/N$.

Known examples of strong asymptotics:

$Q(z) = |z|^2$: K is a disk

$Q(z) = |z|^2 + a \operatorname{Re} z^2$: K is ellipse (Felder-Riser '13)

$Q(z) = |z|^2 + a \operatorname{Re} z^3$: K is a hypotrochoid (Bleher-Kuijlaars '12)

$Q(z) = |z|^2 + a \operatorname{Re} z^p$: (Kuijlaars - Lopez-Garcia)

$Q(z) = |z|^2 - c \log |z - a|$: K is a Joukowsky airfoil (Balogh-Bertola-Lee-McLaughlin '13)

*** The correction term is checked explicitly only for the first two cases.

Restating the conjecture...

Claim. If the following (WKB) expansion

$$\textbf{(A1)} \qquad p_n(z) = \exp\left[n\,g(z) + \Psi_0(z) + \frac{1}{N}\Psi_1(z) + \mathcal{O}\left(\frac{1}{N^2}\right)\right],$$

holds (in some region around the boundary), and if the kernel satisfies certain asymptotic behavior such that the density is given by

$$\textbf{(A2)} \qquad \rho(z) = \frac{1}{\pi} + \mathcal{O}\left(\frac{1}{N^2}\right),$$

(uniformly) inside (a compact subset of) K, then the conjecture is true.

Relation between OP and CG:

Several fundamental facts:

– OP = Average of characteristic polynomial:

$$p_n(z) = \mathbb{E}\Big(\prod_{j=1}^{n}(z - z_j)\Big).$$

– Density of the CG = Sum of the absolute square of OPs:

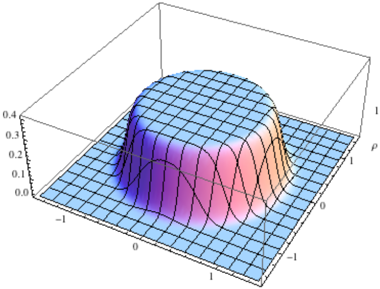
$$\rho(z) = \frac{1}{N}\sum_{j=0}^{n-1} |p_j(z)|^2 \mathrm{e}^{-NQ(z)}.$$

$$\left(\quad K_n(z,w) = \frac{1}{N}\sum_{j=0}^{n-1} p_j(z)\overline{p_j(w)}\mathrm{e}^{-\frac{N}{2}\big(Q(z)+Q(w)\big)}. \quad \right)$$

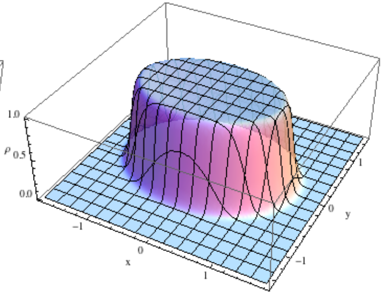
Hele-Shaw potential

The **density** of the Coulomb gas is given by

$$\rho(z) := \int \mathrm{PDF}(z, z_2, \cdots, z_n) \prod_{j=2}^{n} dA(z_j) \to \frac{\Delta Q}{4\pi} \text{ when } z \in K.$$



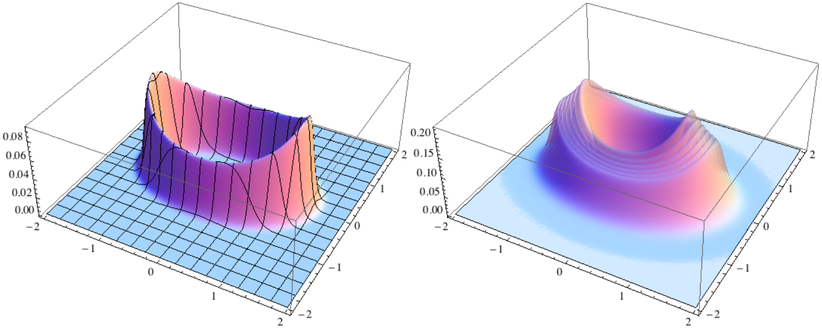
$Q(z) = |z|^2$



$Q(z) = |z|^2 - t\mathrm{Re}(z^2)$

Quantum Hele-Shaw flow

The plot of $|p_n(z)|^2 e^{-NQ(z)}$: (Left: single; Right: several consecutive)



Gaussian peak along the boundary is from

$$e^{-N\left(Q(z)-Tg(z)-T\overline{g(z)}\right)}.$$

D-bar approach

From the **orthogonality** we have

$$\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{p_n(w)}\,\mathrm{e}^{-NQ(w)}}{z-w}\mathrm{d}A(w) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right).$$

Again by the orthogonality, we have

$$\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{p_n(w)}\,\mathrm{e}^{-NQ(w)}}{z-w}\mathrm{d}A(w) = \frac{1}{\pi}\frac{1}{p_n(z)}\int_{\mathbb{C}} \frac{p_n(w)\,\overline{p_n(w)}\,\mathrm{e}^{-NQ(w)}}{z-w}\mathrm{d}A(w).$$

The numerator in RHS has the following property.

Theorem (Ameur-Hedenmalm-Makarov)

$$|p_n(z)|^2 e^{-NQ(z)} dA(z) \to \textit{Harmonic measure on } K^c$$

$1/N$-expansion of Cauchy transform

For a smooth test function f,

$$\int_{\mathbb{C}} f(\zeta)\, e^{-N\left(Q(\zeta)-g(\zeta)-\overline{g(\zeta)}+\ell\right)}\, dA(\zeta)$$
$$= \sqrt{\frac{\pi}{2N}} \oint_{\partial K} \left(f(\zeta) + \frac{1}{N}\left(\frac{\kappa^2}{12} f(\zeta) + \frac{3\kappa}{8}\partial_{\mathbf{n}} f(\zeta) + \frac{1}{8}\partial_{\mathbf{n}}^2 f(\zeta)\right) + \mathcal{O}\left(\frac{1}{N^2}\right)\right) |d\zeta|.$$

(This is obtained by using Schwarz function.)

We take

$$f(\zeta) = \frac{|\widehat{p}_n(\zeta)|^2}{\zeta - z}$$

where $\widehat{p}_n$ is all the subleading parts of p_n:

$$\widehat{p}_n(z) := p_n(z)\, e^{-ng(z)} = e^{\Psi_0}\left(1 + \frac{1}{N}\Psi_1 + \mathcal{O}\left(\frac{1}{N^2}\right)\right).$$

$1/N$-expansion of Cauchy transform (cont.)

One obtains the following.

$$\widehat{C}_n(z) = \frac{1}{\widehat{p}_n(z)}\sqrt{\frac{\pi}{2N}} \oint \left[\frac{|\widehat{p}_n(w)|^2}{z-w} + \frac{1}{N}\left(\frac{\kappa^2}{12} + \frac{3\kappa}{8}\partial_{\mathbf{n}} + \frac{1}{8}\partial_{\mathbf{n}}^2\right)\frac{|\widehat{p}_n(w)|^2}{z-w} + \mathcal{O}\left(\frac{1}{N^2}\right)\right]|dw|.$$

- Note that this is the "electric force" from the measure $|p_n|^2 e^{-NQ} dA$.
- By using the "convergence to harmonic measure" the leading term of $\widehat{C}(z)$ must vanish **inside** K.

Therefore, in the leading order,

$$|\widehat{p}_n(w)|^2 \approx |e^{2\Psi_0}| \propto |f'|.$$

And this leads to

$$e^{\Psi_0(z)} = \sqrt{\rho f'(z)}.$$

(This is not the main point.)

To calculate the next order, we claim that $\widehat{C}_n$ vanishes even at the second order. This is not proven in general, however it follows from certain asymptotics of the kernel (which is also not proven in general).

Kernel → Cauchy transform

Recall

$$\rho_n^{(1)}(z) := \int \mathrm{PDF}_n(\{z, z_2, \cdots, z_n\})\, dA(z_2)\cdots dA(z_n).$$
$$= \frac{1}{N} K_n(z,z).$$
$$\rho_n^{(2)}(z,w) := \int \mathrm{PDF}_n(\{z, w, z_3, \cdots, z_n\})\, dA(z_3)\cdots dA(z_n).$$
$$= \frac{1}{N(n-1)} \Big(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2 \Big).$$

Taking ∂_z on the first equation:

$$\partial\rho_n^{(1)}(z) = \int \Big(-NQ'(z) + \sum_{j=2}^{n} \frac{1}{z-z_j}\Big)\rho_n(\{z, z_2, \cdots, z_n\}) \prod_{j=2}^{n} dA(z_j)$$

$$= -NQ'(z)\rho_n^{(1)}(z) + (n-1)\int \frac{dA(w)}{z-w}\rho_n^{(2)}(\{z, w, z_3, \cdots, z_n\}) \prod_{j=3}^{n} dA(z_j)$$

$$= -NQ'(z)\rho_n^{(1)}(z) + \frac{1}{N}\int \frac{dA(w)}{z-w}\Big(K_n(z,z)K_n(w,w) - |K_n(z,w)|^2\Big).$$

Divide the whole equation by $\rho_n^{(1)}(z) = \frac{1}{N}K_n(z,z)$. Obtain the same equation for $\rho_{n+1}^{(1)}$ and take the difference of the two.

We obtain

$$\int \frac{|p_n(w)|^2 e^{-NQ(w)} dA(w)}{z-w} = \frac{1}{K_n(z,z)} \int \frac{|K_n(z,w)|^2 dA(w)}{z-w}$$
$$+(\text{terms with } \partial\rho_n^{(1)}(z))$$

Asymptotics of kernel

Theorem [Riser] For ellipse case, $Q(z) = |z|^2 - t\mathrm{Re}(z^2)$,

$$|K_n(z,w)|^2 = \frac{N}{\pi} e^{-N|z-w|^2}(1 + \mathcal{O}(N^{-\infty})),$$

when z and w are both inside the ellipse and sufficiently close to each other.

Proof) Based on the Christoffel-Darboux identity:

$$\frac{1}{N}\partial_{\overline{w}}\left(K_n(z,w)\, e^{\frac{N}{2}(|z|^2+|w|^2-2z\overline{w})}\right) = \sqrt{\frac{n}{N}}\frac{t\, p_n(z)\, p_{n-1}(w) - p_{n-1}(z)\, p_n(w)}{\sqrt{h_n h_{n-1}}\sqrt{1-t^2}} \mathrm{e}^{\frac{N}{2}(-2z\overline{w}+t\mathrm{Re}(z^2)+t\mathrm{Re}(w^2))}.$$

When z and w are inside the bulk (and close to each other), the polynomials in the right hand side peak on the boundary.

QUESTION: For real analytic potential of the type:

$$Q(z) = |z|^2 + (\text{harmonic})$$

the kernel inside the bulk is asymptotically given by

$$|K_n(z,w)|^2 = \frac{N}{\pi} e^{-N|z-w|^2}(1 + \mathcal{O}(N^{-\infty})).$$

This observation shows that the term

$$\frac{1}{K_n(z,z)} \int \frac{|K_n(z,w)|^2 dA(w)}{z-w}$$

and $\partial \rho_n^{(1)}$ are both **exponentially small in** N inside the bulk of the ellipse.

Let us come back to $\widehat{C}_n$ and use the expansion with:

$$\widehat{p}_n(z) = \sqrt{\rho\psi'(z)}\left(1 + \frac{1}{N}\Psi(z) + \dots\right)$$

and we define $\Phi(z)$ such that

$$\frac{\rho|\psi'(w)|\,\Phi(w)}{z-w}dA(w) := \oint_{\partial K}\left(\frac{\kappa^2}{12} + \frac{3\kappa}{8}\partial_{\mathbf{n}} + \frac{1}{8}\partial_{\mathbf{n}}^2\right)\frac{\rho|\psi'(w)|}{z-w}dA(w).$$

Using Plemelj-Sokhotski relation, we get, at the second order in $1/N$, the following identity:

$$\left[\widehat{C}_n(z)|_{\text{in}} - \widehat{C}_n(z)|_{\text{out}}\right]_{1/N} = -\sqrt{\frac{2\pi^3}{N}}\frac{\sqrt{\rho\,\psi'(z)}}{\psi(z)}\left(\overline{\Psi(z)} + \Phi(z)\right).$$

Therefore we get the following **analytic-anti-analytic decomposition problem**:

$$\Phi(z) = -\overline{\Psi(z)} + \sqrt{\frac{N}{2\pi^3}} \frac{\psi(z)}{\sqrt{\rho\,\psi'(z)}} \widehat{C}_n(z)|_{\mathrm{out}}$$

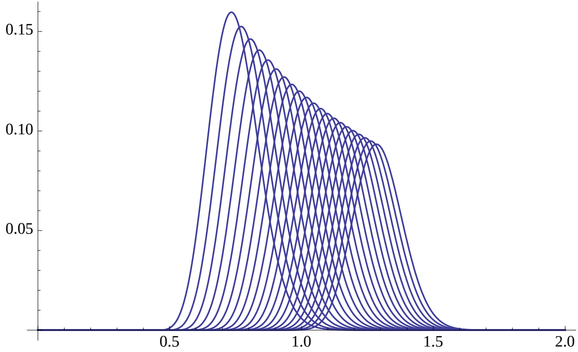
outside the set K.

This is WienerHopf decomposition on the Schottky double.

The end.

Kernel calculation using the sum

Plot of $|p_j|^2 e^{-NQ}$ along the major axis of the ellipse, for j from 10 to 30 for $N = 30$.



Since each $|p_j(z)|^2 e^{NQ(z)}$ spreads over $1/\sqrt{N}$, and since the center moves with the velocity $1/N$, at a single point there are $\sqrt{N}$ of the polynomials that contribute (upto exponentially small correction) to the density (and kernel).

One can calculate the kernel by (upto exponentiall correction)

$$\sum_{j=n_0-N^{1/2+\epsilon}}^{n_0+N^{1/2+\epsilon}} |p_j(z)|^2 e^{-NQ(z)} = \sum_{j=n_0-N^{1/2+\epsilon}}^{n_0+N^{1/2+\epsilon}} \exp\left[N\Psi_{-1} + \Psi_0 + \frac{1}{N}\Psi_1 + \dots\right]$$

Above, n_0 is chosen such that p_{n_0} is centered at z.

Each term Ψ_j is a function of the set K hence of the time T. And it has the taylor expansion in T:

$$\Psi_j = \Psi_j(T_0) + \frac{j-n_0}{N}\dot{\Psi}_j(T_0) + \frac{(j-n_0)^2}{2N^2}\ddot{\Psi}_j(T_0) + \dots$$

Poisson summation formula

One can perform the summation using Poisson summation formula: defining $r = j - n_0$

$$\sum_{r=-\infty}^{\infty} \exp\left(-\frac{A_1}{N}r^2 + A_2 r\right)\left[1 + \frac{A_3}{N^2}r^3 + \frac{A_4}{N^3}r^4 + \frac{1}{2}\frac{A_3^2}{N^4}r^6\right]$$

$$= \sqrt{4\pi}\alpha\, e^{\alpha^2 A_2^2}\left\{1 + i\frac{A_3}{N^2}\alpha^3 H_3(i\alpha A_2) + \frac{A_4}{N^3}\alpha^4 H_4(i\alpha A_2) - \frac{A_3}{2N^2}\alpha^6 H_6(i\alpha A_2)\right\}$$

Unpublished calculation by Bertola and McLaughlin shows that the following can be obtained by direction summation using only the leading asymptotics of polynomials.

$$\lim_{n,N\to\infty}\frac{1}{N}K_n\left(z_0+\frac{\xi}{\sqrt{N}},z_0+\frac{\eta}{\sqrt{N}}\right)$$
$$=\begin{cases}\frac{1}{\pi}\mathrm{e}^{-\frac{1}{2}|\xi-\eta|^2}\mathrm{e}^{\mathrm{i}\Im(\xi\overline{\eta})+\mathrm{i}\sqrt{N}\,\Im(\overline{z_0}(\xi-\eta))} & \text{(bulk, } \textit{Berman}'08\text{)}\\ \text{(the same)}\times\frac{1}{2}\mathrm{erfc}\left(\frac{1}{\sqrt{2}}(\xi\overline{n}+\overline{\eta}n)\right) & \\ \text{(boundary, Ameur-Kang-Makarov '1?)} & \end{cases}$$

$$\xi\overline{n}+\overline{\eta}n=\xi_\perp+\eta_\perp+\mathrm{i}(\xi_\parallel-\eta_\parallel).$$

$$\mathrm{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-t^2}\mathrm{d}t.$$

Further correction

Taking any boundary point $z_0 \in \partial K$, we define the zooming normal coordinate $y \in \mathbb{R}$ by

$$z = z_0 + \frac{y}{\sqrt{N}}\mathbf{n}.$$

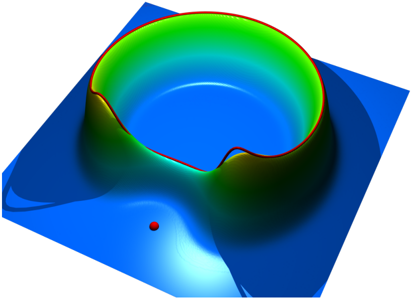
Then the following is true on a smooth part of ∂K:

$$\rho_n\left(1 + \frac{y}{\sqrt{N}}\right) = \frac{1}{2\pi}\mathrm{Erfc}(\sqrt{2}y) + \frac{1}{\sqrt{N}}\frac{\kappa(z_0)}{3\sqrt{2}\pi^{3/2}}(y^2 - 1)\,\mathrm{e}^{-2y^2} + \mathcal{O}\left(\frac{1}{N}\right).$$

From kernel to orthogonal polynomial

If we use the second assumption **(A2)** then the correction terms of the density in each order or $1/N$ must vanish. Using Poisson summation formula, this gives another recursive method to obtain higher order corrections of OP (work in progress with Roman Riser).

PLOT OF $|p_n|^2 e^{-NQ}$



THANK YOU FOR YOUR ATTENTION