Moments of Traces for Circular β-ensembles

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This is a joint work with Sho Matsumoto

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Outline

- Moments for Haar Unitary Matrices (D.E. Thm)
- Background for Circular β-Ensembles
- Moments for Circular β-Ensembles
- Proof by Jack Polynomials
- CLT

1. Moments for Haar Unitary Matrices

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  2) $\Gamma_n \stackrel{d}{=} Y(Y^*Y)^{-1/2}$

▶ Theorem (Diaconis and Evans: 2001)

(a) $a = (a_1, \cdots, a_k)$, $b = (b_1, \cdots, b_k)$ with $a_j, b_j \in \{0, 1, 2, \cdots\}$. For $n \geq \sum_{j=1}^{k} j a_j \vee \sum_{j=1}^{k} j b_j$,

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$$\mathbb{E}\left[\prod_{j=1}^{k} (Tr(U_n^j))^{a_j} \overline{(Tr(U_n^j))^{b_j}}\right] = \delta_{ab} \prod_{j=1}^{k} j^{a_j} a_j!$$

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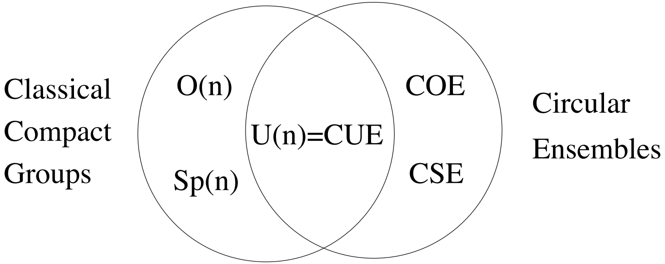
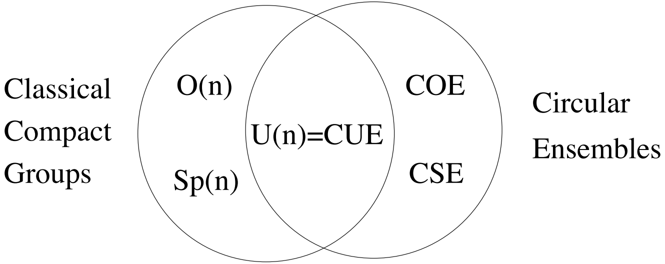
(b) For j and k,

$$\mathbb{E}\left[Tr(U_n^j)\overline{Tr(U_n^k)}\right] = \delta_{jk} \cdot j \wedge n.$$



Circular Ensembles and Haar-invariant Matrices from Classical Compact Groups



Circular Ensembles and Haar-invariant Matrices from Classical Compact Groups

Diaconis (2004) believes there is a good formula for COE and CSE

- Probability density function
  $e^{i\theta_1}, \cdots, e^{i\theta_n}$: eigenvalues of Haar-invariant unitary matrix.

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2. Background for Circular β-Ensembles

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$$f(\theta_1, \cdots, \theta_n | \beta) = Const \cdot \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta$$

  $\beta > 0$, $\theta_i \in [0, 2\pi)$

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- This model: *circular β-ensemble* by physicist Dyson for study of nuclear scattering data. Matrix model by Killip & Nenciu (04)

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Entries of COE : roughly $\mathbb{C}N(0, 1)$ (but dependent) (Jiang, JMP09)

Moments for Circular β-Ensembles

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Let M_n be COE. By elementary check

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- This suggest: moments for general β-ensemble depend on n

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$\lambda = (3, 2, 2) : |\lambda| = 7, m_2(\lambda) = 2,\ m_3(\lambda) = 1, l(\lambda) = 3,$

$$p_\lambda = (\sum_i \lambda_i^3) \cdot (\sum_i \lambda_i^2)^2$$

$\alpha > 0$, $K \geq 1$, $n \geq 1$, define

$$A = \left(1 - \frac{|\alpha - 1|}{n - K + \alpha}\,\delta(\alpha \geq 1)\right)^K$$

$$B = \left(1 + \frac{|\alpha - 1|}{n - K + \alpha}\,\delta(\alpha < 1)\right)^K$$

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Let $\theta_1, \cdots, \theta_n \sim f(\theta_1, \cdots, \theta_n | \beta)$, $\alpha = 2/\beta$.

- $Z_n = (e^{i\theta_1}, \cdots, e^{i\theta_n})$,
- $p_\mu(Z_n) = p_\mu(e^{i\theta_1}, \cdots, e^{i\theta_n})$

Theorem

(a) *If* $n \geq K = |\mu|$, *then*

$$A \leq \frac{\mathbb{E}\left[|p_\mu(Z_n)|^2\right]}{\alpha^{l(\mu)} z_\mu} \leq B$$

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If $\mu \neq \nu$ *and* $n \geq K = |\mu| \vee |\nu|$, *then*

$$\left|\mathbb{E}\left[p_\mu(Z_n)\overline{p_\nu(Z_n)}\right]\right| \leq \max\{|A-1|, |B-1|\} \cdot \alpha^{(l(\mu)+l(\nu))/2}(z_\mu z_\nu)^{1/2}$$

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(c) $\exists\, C = C(\beta)$ *s.t.* $\forall m \geq 1,\ n \geq 2$

$$\left|\mathbb{E}\left[|p_m(Z_n)|^2\right] - n\right| \leq C \frac{n^3 2^{n\beta}}{m^{1\wedge\beta}}$$

Take $\beta = 2$, then $A = B = 1$. We recover

- Theroem (Diaconis and Evans: 2001)
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Corollary

$\forall\, \beta > 0,$

$$(a) \quad \lim_{n\to\infty} \mathbb{E}\left[p_\mu(Z_n)\overline{p_\nu(Z_n)}\right] = \delta_{\mu\nu}\left(\frac{2}{\beta}\right)^{l(\mu)} z_\mu;$$

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$$(b) \quad \lim_{m\to\infty} \mathbb{E}\left[|p_m(Z_n)|^2\right] = n \quad \textit{for any } n \geq 2.$$

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$\mu \neq \nu : K = |\mu| \vee |\nu|$. *If* $n \geq 2K$, *then*

$$(a) \quad \left| \frac{\mathbb{E}[|p_\mu(Z_n)|^2]}{\alpha^{l(\mu)} z_\mu} - 1 \right| \leq \frac{6|1-\alpha|K}{n};$$

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Exact formula is given next

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Proofs by Jack Polynomial

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$$\int_{[0,2\pi)^n} J_\lambda^{(\alpha)}(Z_n) J_\mu^{(\alpha)}(\bar{Z}_n) \prod_{1 \le p < q \le n} |e^{i\theta_p} - e^{i\theta_q}|^{2/\alpha} \, d\theta_1 \cdots d\theta_n$$
$$= \delta_{\lambda\mu} \cdot \delta(l(\lambda) \le n) \cdot \text{ explicit const}$$

Write

$$J_\lambda^{(\alpha)} = \sum_{\rho:|\rho|=|\lambda|} \theta_\rho^\lambda(\alpha) p_\rho$$

$$p_\rho = \sum_{\lambda:|\lambda|=|\rho|} \Theta_\rho^\lambda(\alpha) J_\lambda^{(\alpha)}$$

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For $|\mu| = |\nu| = K$,

$$\mathbb{E}\left[p_\mu(Z_n)\overline{p_\nu(Z_n)}\right] = \sum_{\lambda\vdash K:\, l(\lambda)\leq n} \Theta_\mu^\lambda(\alpha)\Theta_\nu^\lambda(\alpha)\mathbb{E}(J_\lambda^{(\alpha)}\overline{J_\lambda^{(\alpha)}})$$

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- relationship between $\theta_\rho^\lambda(\alpha)$ and $\Theta_\rho^\lambda(\alpha)$

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$$\begin{aligned}&\mathbb{E}\left[p_\mu(Z_n)\overline{p_\nu(Z_n)}\right]\\ =\;& \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda\vdash K:\, l(\lambda)\leq n} \frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} \mathcal{N}_\lambda^\alpha(n)\end{aligned}$$

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\end{aligned}$$

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$$\mathcal{N}_\lambda^\alpha(n) = \prod_{(i,j)\in\lambda} \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i}$$

Young diagram

Main proof:

- play $C_\lambda(\alpha)$
- play $\mathcal{N}_\lambda^\alpha(n)$
- use orthogonal relations of $\theta_\mu^\lambda(\alpha)$

- Examples

$$\mathbb{E}[|p_1(Z_n)|^4] = \frac{2n\alpha^2(n^2 + 2(\alpha-1)n - \alpha)}{(n+\alpha-1)(n+\alpha-2)(n+2\alpha-1)}$$

▶ Examples

$$
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\mathbb{E}[|p_1(Z_n)|^4] &= \frac{2n\alpha^2(n^2+2(\alpha-1)n-\alpha)}{(n+\alpha-1)(n+\alpha-2)(n+2\alpha-1)} \\
&= \begin{cases} \frac{8(n^2+2n-2)}{(n+1)(n+3)}, & \text{if } \beta = 1 \\ 2, & \text{if } \beta = 2 \\ \frac{2n^2-2n-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4 \end{cases}
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= \quad & \frac{2\alpha^2(\alpha-1)n}{(n+\alpha-1)(n+2\alpha-1)(n+\alpha-2)} \\
= \quad & \begin{cases} \frac{8}{(n+1)(n+3)}, & \text{if } \beta = 1 \\ 0, & \text{if } \beta = 2 \\ \frac{-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4 \end{cases}
\end{aligned}
$$

Similar results also hold for Macdonald polynomials

Central Limit Theorem

Theorem

$(e^{i\theta_1}, \cdots, e^{i\theta_n})$: *any* β*-circular ensemble.*
$g(z) = \sum_{k=0}^{m} c_k z^k$
$X_n := \sum_{j=1}^{n} g(e^{i\theta_j})$.
Then $X_n - \mu_n \to \mathbb{C}N(0, \sigma^2)$ *where*

$$\mu_n = nc_0 \ \ \text{and} \ \ \sigma^2 = \frac{2}{\beta} \sum_{k=1}^{m} k|c_k|^2.$$

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$$\mu_n = nc_0 \ \ \text{and} \ \ \sigma^2 = \frac{2}{\beta} \sum_{k=1}^{m} k|c_k|^2.$$

Next consider $g(z) = \sum_{k=0}^{\infty} c_k z^k$ for $\beta = 1, 4$; $\quad \beta = 2$ by D. E.

Theorem

$Z_n := (e^{i\theta_1}, \cdots, e^{i\theta_n})$: *CSE* *(*$\beta = 4$*)*
$\{a_j, b_j\}$: $\sum_{j=1}^{\infty} (j \log j)(|a_j|^2 + |b_j|^2) < \infty$
$\sigma^2 := \sum_{j=1}^{\infty} j(|a_j|^2 + |b_j|^2)$

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Then, $\sum_{j=1}^{\infty} \left(a_j \, tr(Z_n^j) + b_j \, \overline{tr(Z_n^j)} \right) \to U + iV$

where $(U, V) \sim N_2(\mathbf{0}, \Sigma)$,

$$\Sigma = \frac{1}{4} \begin{pmatrix} \sum_{j=1}^{\infty} j|a_j + \bar{b}_j|^2 & 2 \cdot \mathrm{Im}(\sum_{j=1}^{\infty} j a_j b_j) \\ 2 \cdot \mathrm{Im}(\sum_{j=1}^{\infty} j a_j b_j) & \sum_{j=1}^{\infty} j|a_j - \bar{b}_j|^2 \end{pmatrix}.$$

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Proposition

$\beta = 4$. $\exists\, K > 0$ *s.t.* $\mathbb{E}[|\sum_{j=1}^{n} e^{im\theta_j}|^2] \leq Km\log(m+1)$ *for* $m \geq 1$, $n \geq 2$.

$$\mathbb{E}[|\sum_{j=1}^{n} e^{im\theta_j}|^2] = \frac{1}{4}m^2 \sum_{\lambda \vdash m : l(\lambda) \leq n} \frac{(\theta^{\lambda}_{(m)})^2}{C_\lambda} \mathcal{N}_\lambda(n)$$

where

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$$\theta_{(m)}^{\lambda}(\alpha) = \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} (\frac{1}{2}(j-1) - (i-1)),$$

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$$\begin{aligned} C_\lambda(\alpha) &= \prod_{(i,j)\in\lambda}\{(\frac{1}{2}(\lambda_i-j)+\lambda'_j-i+1) \\ &\qquad \times(\frac{1}{2}(\lambda_i-j)+\lambda'_j-i+\frac{1}{2})\}, \end{aligned}$$

Main contribution: $\lambda = (r, s, 1^{m-r-s})$

$$\frac{(\theta^{\lambda}_{(m)})^2}{C_\lambda} = \frac{(m-r-s+1)}{(m+r-s+1)(m+r-s)(m-r+s)(m-r+s-1)} \times \frac{2^{2r-2s}[(r-1)!]^2(2r-2s+1)\cdot(2s-2)!}{[(s-1)!]^2(2r-1)!}.$$

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$$\frac{1}{K}\cdot\frac{m}{\sqrt{(n-r+1)(n-s+1)}} \le \mathcal{N}_\lambda(n) \le K\cdot\frac{m}{\sqrt{(n-r+1)(n-s+1)}}.$$

if $m \ge n$. Done!

Thanks for your patience!