Universality results for the Cauchy-Laguerre chain matrix model

Thomas Bothner

Centre de recherches mathématiques, Université de Montréal

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This talk discusses joint work (BB 14 [6]) with Marco Bertola on the Cauchy matrix chain, the space $\mathcal{M}_+^p(n)$, $p, n \in \mathbb{Z}_{\geq 2}$ of p-tuples $(M_1, \ldots, M_p)$ of $n \times n$ positive - definite Hermitian matrices with joint probability density function

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This four step program has been successfully completed for the Hermitian one-matrix model, i.e. $p = 1$:

- Joint probability density on eigenvalues, for $M \in \mathcal{M}(n), U : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{d}\mu(M) \propto e^{-\operatorname{tr} U(M)}\mathrm{d}M \;\rightsquigarrow P(\{x_j\}_1^n)\mathrm{d}^n x = \frac{1}{Z_n}\Delta(X)^2 e^{-\sum_{j=1}^n U(x_j)}\prod_{j=1}^n \mathrm{d}x_j$$

with Vandermonde $\Delta(X) = \prod_{j<k}(x_j - x_k)$. (PR 60 [18], D 62 [10])

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- Determinantal reduction for the ℓ-point correlation function

$$\mathcal{R}^{(\ell)}(\{x_j\}_1^\ell) = \frac{\ell!}{(n-\ell)!}\int_{\mathbb{R}^{n-\ell}} P(\{x_j\}_1^n) \prod_{j=\ell+1}^n \mathrm{d}x_j = \det\left[\mathbb{K}_{11}(x_i, x_j)\right]_{i,j=1}^\ell$$

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$$\mathbb{K}_{11}(x,y) = e^{-\frac{1}{2}U(x)}e^{-\frac{1}{2}U(y)}\sum_{k=0}^{n-1}\pi_k(x)\pi_k(y)\frac{1}{h_k}$$

  and monic orthogonal polynomials $\{\pi_k\}_{k\geq 0}$

$$\int_{\mathbb{R}} \pi_n(x)\pi_m(x)e^{-U(x)}\,\mathrm{d}x = h_n\delta_{nm}. \quad \text{(D 70 [11])}$$

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$$\frac{1}{n}\mathbb{K}_{11}(x,x)\mathrm{d}x\rightharpoonup\mathrm{d}\mu_V(x)\quad\text{as}\quad n,N\to\infty:\quad\frac{n}{N}\to 1$$

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$$\lim_{n\to\infty} \frac{1}{n\rho_V(x^*)}\mathbb{K}_{11}\left(x^* + \frac{x}{n\rho_V(x^*)}, x^* + \frac{y}{n\rho_V(x^*)}\right) = K_{\sin}(x,y)$$

with $K_{\sin}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$ (regular bulk universality).

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$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}}\mathbb{K}_{11}\left(x^* \pm \frac{x}{(cn)^{2/3}}, x^* \pm \frac{y}{(cn)^{2/3}}\right) = K_{\mathrm{Ai}}(x,y)$$

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3. For $U(x) = NV(x) - a\ln x$ with $a > -1$ and $x > 0$ we have, (KV 03 [14])

$$\lim_{n\to\infty} \frac{1}{(cn)^2} \mathbb{K}_{11}\left(\frac{x}{(cn)^2}, \frac{y}{(cn)^2}\right) = K_{\mathrm{Bess},a}(x,y)$$

with $K_{\mathrm{Bess},a}(x,y) = \frac{J_a(\sqrt{x})\sqrt{y}J_a'(\sqrt{y}) - J_a(\sqrt{y})\sqrt{x}J_a'(\sqrt{x})}{2(x-y)}$ (hard edge universality).

- Cauchy matrix chain (BGS 09 [4]) reduced to spectral variables (MS 94 [16])

$$\mathrm{d}\mu(M_1,\ldots,M_p)\propto\frac{e^{-\mathrm{tr}\sum_{j=1}^{p}U_j(M_j)}}{\prod_{j=1}^{p-1}\det(M_j+M_{j+1})^n}\rightsquigarrow P(\{x_{1j}\}_1^n,\ldots,\{x_{pj}\}_1^n)=\frac{1}{\mathcal{Z}_n}$$

$$\times\,\Delta(X_1)\Delta(X_p)e^{-\sum_{m=1}^{p}\sum_{j=1}^{n}U_m(x_{mj})}\prod_{\alpha=1}^{p-1}\det\left[\frac{1}{x_{\alpha j}+x_{\alpha+1,k}}\right]_{j,k=1}^{n}\prod_{j=1}^{p}\prod_{\ell=1}^{n}\mathrm{d}x_{j\ell}$$

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$$\times \int_{\mathbb{R}_+^{n-\ell_1}}\cdots\int_{\mathbb{R}_+^{n-\ell_p}} P(\{x_{1j}\}_1^n,\ldots,\{x_{pj}\}_1^n)\prod_{j=1}^{p}\prod_{m_j=\ell_j+1}^{n}\mathrm{d}x_{jm_j}$$

$$= \det \begin{bmatrix} \boxed{\begin{matrix} \mathbb{K}_{11}(x_{1r}, x_{1s}) \\ 1\leq r\leq \ell_1, 1\leq s\leq \ell_1 \end{matrix}} & \cdots & \boxed{\begin{matrix} \mathbb{K}_{1p}(x_{1r}, x_{ps}) \\ 1\leq r\leq \ell_1, 1\leq s\leq \ell_p \end{matrix}} \\ \vdots & \ddots & \vdots \\ \boxed{\begin{matrix} \mathbb{K}_{p1}(x_{pr}, x_{1s}) \\ 1\leq r\leq \ell_p, 1\leq s\leq \ell_1 \end{matrix}} & \cdots & \boxed{\begin{matrix} \mathbb{K}_{pp}(x_{pr}, x_{ps}) \\ 1\leq r\leq \ell_p, 1\leq s\leq \ell_p \end{matrix}} \end{bmatrix}_{(\sum_1^p \ell_i)\times(\sum_1^p \ell_i)} ,$$

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with correlation kernels

$$\mathbb{K}_{j\ell}(x,y) = e^{-\frac{1}{2}U_j(x)-\frac{1}{2}U_\ell(y)}\mathbb{M}_{j\ell}(x,y), \quad \mathbb{M}_{p1}(x,y) = \sum_{\ell=0}^{n-1}\phi_\ell(x)\psi_\ell(y)\frac{1}{h_\ell}$$

and the remaining kernels are (suitable) transformations of $\mathbb{M}_{p1}(x,y)$.

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and the remaining kernels are (suitable) transformations of $\mathbb{M}_{p1}(x,y)$. The latter is constructed with the help of monic (Cauchy) biorthogonal polynomials $\{\psi_k,\phi_k\}_{k\ge 0}$

$$\iint_{\mathbb{R}_+^2}\psi_n(x)\phi_m(y)\eta_p(x,y)\mathrm{d}x\mathrm{d}y = h_n\delta_{nm}$$

with weight function on $\mathbb{R}_+^2$, (case $p=2$ as "limit")

$$\eta_p(x,y) = \int_0^\infty\cdots\int_0^\infty \frac{e^{-U_1(x)}}{x+\xi_1}\left(\frac{e^{-\sum_{j=2}^{p-1}U_j(\xi_{j-1})}}{\prod_{j=1}^{p-3}(\xi_j+\xi_{j+1})}\right)\frac{e^{-U_p(y)}}{\xi_{p-2}+y}\prod_{j=1}^{p-2}\mathrm{d}\xi_j.$$

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  2. With jump for $z \in \mathbb{R}\backslash\{0\}$

$$\Gamma_+(z) = \Gamma_-(z)\begin{pmatrix} 1 & e^{-U_1(z)}\chi_+ & 0 & & & \\ 0 & 1 & e^{-U_2(-z)}\chi_- & 0 & & \\ & 0 & 1 & e^{-U_3(z)}\chi_+ & & \\ & & 0 & 1 & \ddots & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

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$$\Gamma_+(z) = \Gamma_-(z)\begin{pmatrix} 1 & e^{-U_1(z)}\chi_+ & 0 & & & \\ 0 & 1 & e^{-U_2(-z)}\chi_- & 0 & & \\ & 0 & 1 & e^{-U_3(z)}\chi_+ & & \\ & & & & \ddots & \\ & & 0 & 1 & & \\ & & & \ddots & & \\ & & & & & 1 \end{pmatrix}$$

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The RHP is uniquely solvable iff $(\psi_n(z), \phi_n(z))$ exists, moreover (BB 14 [6])

$$\mathbb{M}_{j\ell}(x,y) = \frac{(-)^{\ell-1}}{(-2\pi\mathrm{i})^{j-\ell+1}}\left[\frac{\Gamma^{-1}(w;n)\Gamma(z;n)}{w-z}\right]_{j+1,\ell}\Bigg|_{\substack{w=x(-)^{j+1}\\ z=y(-)^{\ell-1}}}$$

This is in sharp contrast to the Itzykson-Zuber model.

We confine ourselves first to

$$U_j(x) = NV_j(x), \quad \forall\, j: \lim_{x\downarrow 0} \frac{V_j(x)}{|\ln x|} = \kappa_j > 0, \quad \lim_{x\to+\infty} \frac{V_j(x)}{\ln x} = +\infty$$

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$$\mathcal{Z}_n = \iint_{\mathbb{R}_+^n\times\mathbb{R}_+^n}\frac{\Delta^2(X)\Delta^2(Y)}{\prod_{j,k=1}^n(x_j+y_k)}e^{-N\sum_{j=1}^n(V_1(x_j)+V_2(y_j))}\mathrm{d}X\mathrm{d}Y$$

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with the energy functional (here $W_1(z) = V_1(z), W_2(z) = V_2(-z)$)

$$\begin{aligned}\mathcal{E}(\nu_1,\nu_2) = \sum_{j=1}^2\left[\iint \ln|s-t|^{-1}\mathrm{d}\nu_j(s)\mathrm{d}\nu_j(t) + \int W_j(s)\mathrm{d}\nu_j(s)\right] \\ -\iint \ln|s-t|^{-1}\mathrm{d}\nu_1(s)\mathrm{d}\nu_2(t); \quad \int\limits_{\mathbb{R}_+}\mathrm{d}\nu_1(s) = 1 = \int\limits_{\mathbb{R}_-}\mathrm{d}\nu_2(s).\end{aligned}$$

We are naturally lead to the minimization problem, i.e. vector equilibrium problem

$$E^{W_1,W_2} = \inf_{\substack{\mu_1 \in \mathcal{M}^1[0,\infty) \\ \mu_2 \in \mathcal{M}^1(-\infty,0]}} \mathcal{E}(\mu_1, \mu_2).$$

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Moreover the shifted resolvents $y_1 = -R_1 + \frac{1}{3}(2W_1' + W_2'), y_3 = R_2 - \frac{1}{3}(W_1' + 2W_2'),$ $y_2 = -(y_1 + y_3)$ *with*

$$R_j(z) = \int (s-z)^{-1} \mathrm{d}\mu_j^{W_j}(s),$$

are the three branches of the cubic

$$y^3 - R(z)y - D(z) = 0. \quad \text{(spectral curve)}$$

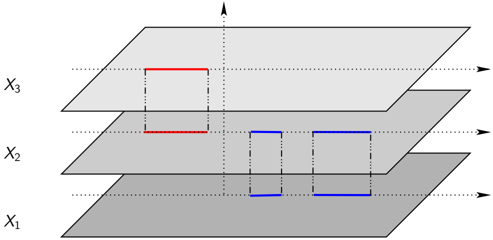
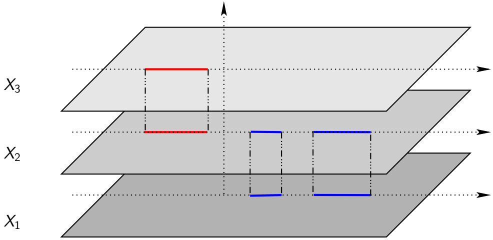


Figure 1 : The Hurwitz diagram for a typical three sheeted covering of $\mathbb{CP}^1$. The support of $\mu_1^{W_1}$ on the left in red and for $\mu_2^{W_2}$ on the right in blue. This corresponds to the situation $p = 2$ and $\lim_{x\downarrow 0} \frac{V_j(x)}{|\ln x|} > 0$ in place.



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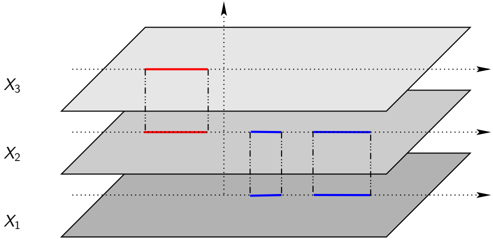
Near the branch points, i.e. edges, the densities $\rho_j(s)$ of $\mathrm{d}\mu_j^{W_j}(s) = \rho_j(s)\mathrm{d}s$ vanish like square roots, in the interior they are positive (generically!).



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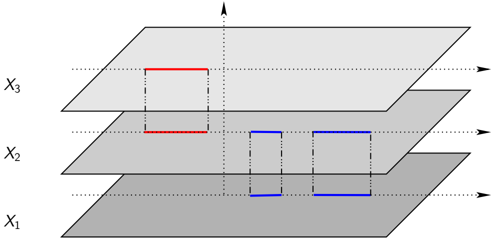

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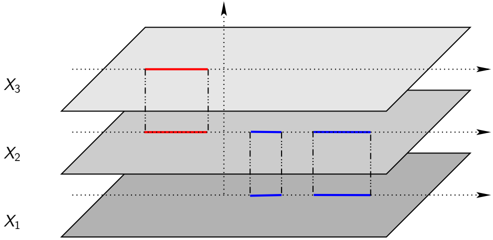
For non-screening potentials, the supports in Theorem 1 may contain the origin,



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For non-screening potentials, the supports in Theorem 1 may contain the origin, thus leading to a higher order branch point at the origin and a singular density

$$\rho_j(s) = \mathcal{O}\left(|s|^{-\frac{p}{p+1}}\right), \quad s \to 0. \qquad \text{"new" universality class}$$

The non-screening effect appears for instance for the classical Laguerre weights

$$U_j(x) = NV_j(x) - a_j \ln x, \quad a_j > -1: \ a_{k\ell} = \sum_{j=k}^{\ell} a_j > -1; \quad \lim_{x\to+\infty} \frac{V_j(x)}{\ln x} = +\infty$$

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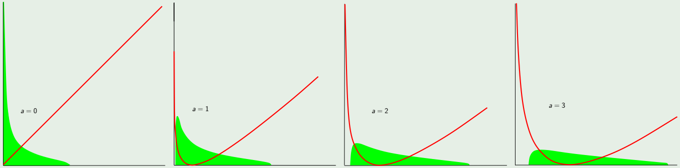

Figure 2 : The potentials $V_j(x)$ are shown in red for different choices of the parameter $a \geq 0$. In green the density of the measure $\rho_1(x)$.

Definition

Let $a_j, b_j \in \mathbb{C}$ and $0 \leq m \leq q, 0 \leq n \leq p$ be integers. The Meijer-G function is defined through the Mellin-Barnes integral formula

$$G_{p,q}^{m,n}\left(\begin{smallmatrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{smallmatrix} \middle| \zeta\right) = \frac{1}{2\pi\mathrm{i}} \int_L \frac{\prod_{\ell=1}^{m} \Gamma(b_\ell + s)}{\prod_{\ell=m}^{q-1} \Gamma(1 - b_{\ell+1} - s)} \frac{\prod_{\ell=1}^{n} \Gamma(1 - a_\ell - s)}{\prod_{\ell=n}^{p-1} \Gamma(a_{\ell+1} + s)} \zeta^{-s} \, \mathrm{d}s$$

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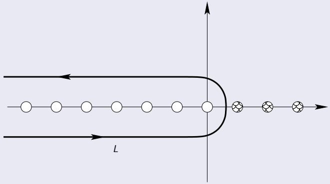

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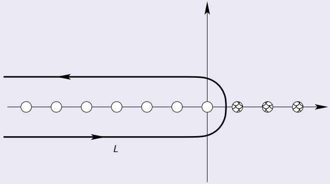

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These special functions have appeared recently in the statistical analysis of singular values of products of Ginibre random matrices (AB 12 [1], AKW 13 [2], KZ 13 [15]).

Conjecture (BB 14 [6])

For any $p \in \mathbb{Z}_{p\geq 2}$, there exists $c_0 = c_0(p)$ and $\{\eta_j\}_1^p$ which depend on $\{a_j\}_1^p$ such that

$$\lim_{n\to\infty} \frac{c_0}{n^{p+1}} n^{\eta_\ell - \eta_j} \mathbb{K}_{j\ell}\left(\frac{c_0}{n^{p+1}}\xi, \frac{c_0}{n^{p+1}}\eta\right) \propto \mathcal{G}_{j\ell}^{(p)}\left(\xi, \eta; \{a_j\}_1^p\right)$$

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$$\mathcal{G}_{j\ell}^{(p)}\left(\xi,\eta;\{a_j\}_1^p\right) = \int_L\int_{\widehat{L}} \frac{\prod_{s=0}^{\ell-1}\Gamma(u-a_{1s})}{\prod_{s=\ell}^{p}\Gamma(1+a_{1s}-u)} \frac{\prod_{s=j}^{p}\Gamma(a_{1s}-v)}{\prod_{s=0}^{j-1}\Gamma(1-a_{1s}+v)} \frac{\xi^v\eta^{-u}}{1-u+v}\frac{\mathrm{d}v\,\mathrm{d}u}{(2\pi\mathrm{i})^2}$$

$$+ \sum_{s\in\mathcal{P}\cup\{0\}} \operatorname*{res}_{v=s} \frac{\prod_{s=0}^{\ell-1}\Gamma(1+v-a_{1s})}{\prod_{s=\ell}^{p}\Gamma(a_{1s}-v)} \frac{\prod_{s=j}^{p}\Gamma(a_{1s}-v)}{\prod_{s=0}^{j-1}\Gamma(1+v-a_{1s})} \frac{\xi^v\eta^{-v}}{(-)^j\xi-(-)^\ell\eta}$$

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Theorem (BB 14 [6])

The conjecture holds for $p = 2,3$ and potentials $U_j(x) = Nx - a_j \ln x$.

1. No rigorous potential theoretic foundation for non-screening situation! We work with explicit spectral curves, i.e. start from Hurwitz diagram and verify a posteriori that the "guess" was correct. For $p = 3$:

$$y^4 - \frac{z-2}{2z}y^2 + \frac{(3z-4)(3z-8)^2}{432z^3} = 0 \qquad (3)$$

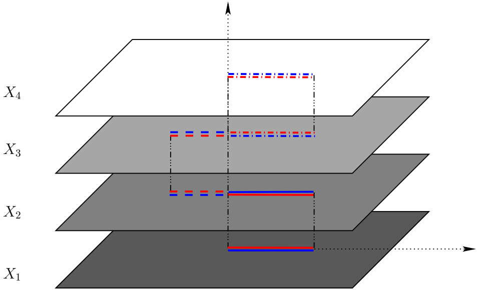

Figure 4 : The four sheeted Riemann surface corresponding to (3).

Why not (yet) universality theorem for arbitrary p? II

2. We construct the relevant parametrices explicitly for arbitrary $p \in \mathbb{Z}_{\geq 2}$, i.e. in particular the model problem at the origin is solved with the help of Meijer-G functions

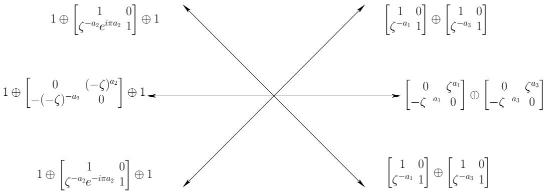

Figure 5 : The local model problem at the origin, situation $p = 3$.

but the error analysis becomes more involved for $p \geq 4$.

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