Standardization of upper-semicontinuous processes applications in Extreme Value Theory

Anne Sabourin\textsuperscript{1}  Johan Segers\textsuperscript{2}

\textsuperscript{1}LTCI, CNRS, Télécom ParisTech, Université Paris-Saclay (FR)
\textsuperscript{2}Université catholique de Louvain (BE)

2016/05, Fields Institute, Toronto Dependence, Stability and Extremes workshop.
This talk

- Standardization of stochastic processes
- Upper-semicontinuous processes
- Applications in extreme value theory (EVT) (this is the initial motivation)
Standardizing stochastic processes: why?

Collection of random variables $\xi = (\xi_s)_{s \in \mathcal{D}}$, $\mathcal{D}$ finite or compact $\subset \mathbb{R}^p$.
Margins: $F_s(x) = \mathbb{P}(\xi_s \leq x)$.

Clément’s talk this morning: Max-stable continuous process.

- Choose a (nice) target cdf $\Phi$: (uniform, Fréchet(1), Pareto(1), ...)
- Standardization map $U : \xi \mapsto \xi^* = \left( \Phi^{-1}(F_s(\xi_s)) \right)_{s \in \mathcal{D}}$.

$\xi$ is max-stable $\iff$ $\xi^*$ is simple max-stable and the margins are max-stable.

Work-flow for **model construction** / **statistical inference**:
Choose a model for $\xi^*$, fit the standardized data to/simulate from it, then apply the inverse standardization map.

**Under which conditions can we do that?**
Finite case $\mathbb{D} = \{1, \ldots, d\}$: standardizing makes sense

$\Phi = \mathcal{U}_{[0,1]}$; $\xi^* = \mathcal{U}(\xi) = (F_1(\xi_1), \ldots F_d(\xi_d))$.

**Sklar’s theorem:**

(I) For all \{ copula $C$ + margins $(F_j, 1 \leq j \leq d)$ \},

$\exists F$ a $d$-variate cdf with margins $(F_j)$ and copula $C$.

(II) Every cdf $F$ may be decomposed this way, i.e.

$\exists C: \mathbf{F}(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$.

$L(\xi)$ is characterized by \{ $L(\xi^*)$ + margins $F_s$ \}.
Continuous processes: standardizing makes sense

- $(C(\mathbb{D}, \mathbb{R}), \|\cdot\|_\infty)$, continuous functions $\mathbb{D} \to \mathbb{R}$.
- “continuous process”: a random continuous function, i.e. a measurable map $\Omega \mapsto C(\mathbb{D}, \mathbb{R})$.
- $\mathcal{L}(\xi)$ characterized by the fidis $\mathcal{L}(\xi_{s_1}, \ldots, \xi_{s_d})$ → back to the $d$-variate case.

\[
\mathcal{L}(\xi) \quad \text{characterized by} \quad \left( \begin{array}{c} \mathcal{L}(\xi^*) \\ \text{fidi copulas } C_{s_1, \ldots, s_d}(\cdot) \end{array} \right) + \text{margins } F_s.
\]

Question: Similar decomposition with semicontinuous processes?
Why care about semicontinuity? 

Truncating rainstorms outside a random closed patch (Schlather (2002), Huser and Davison (2014))

→ long-range independence (in space) of very heavy rainstorms (difficult to achieve without truncation).

Rain\( (s) = X \epsilon(s) 1_B(s) \)

→ Upper semicontinuous (uscm) process
(Pointwise) maximum of truncated rainstorms

\[ \xi(s) = \max_{i} X_i \varepsilon_i(s) 1_{B_i}(s) \]

\[ B_j \quad B_k \quad B_l \]

\[ X_l \varepsilon_l(s) \quad X_k \varepsilon_k(s) \quad X_j \varepsilon_j(s) \]

\[ \rightarrow \text{Again, usc process.} \]
Semicontinuity in EVT

Theory  
for **continuous** processes: well established.  

Practice  
some **semicontinuous** models are also used (‘truncated storms’, Voronoï fields) Schlather (2002), Davison and Gholamrezaee (2011), Huser and Davison (2014), Robert (2013)

Question  
Does standard EVT still apply to semicontinuous processes, and how?
Upper / Lower semicontinuous functions and processes

Roots:

- in variational analysis and random set theory:
  Choquet (47), Matheron (75), Norberg (86, 87), Salinetti and Wets (86),
  Rockafellar and Wets (98), Molchanov (2005), ...

- Mentioned occasionally in extreme-value analysis
  Vervaat (1981, 1988), Norberg (1987), Resnick and Roy (1991), ...

EVT for semicontinuous processes so far:

- Mostly restricted to simple max-stable processes

- Open problems (to our knowledge)
  Standardization, domains of attraction (asymptotics for maxima),
  Parallels with multivariate / continuous EVT, vague convergence (law of excesses), Statistical inference!
Upper semicontinuous functions

$(\mathbb{D}, d)$ a compact metric space. (Think $\mathbb{D} = [0, 1]$.)

A function $f : \mathbb{D} \to \overline{\mathbb{R}}$ is **upper semicontinuous** (*usc*) if

$$\forall s \in \mathbb{D}, \quad f(s) = \lim_{\varepsilon \to 0} \sup_{t: d(s, t) \leq \varepsilon} f(t).$$

this is equivalent to

$$\forall y \in \mathbb{R}, \quad A = \{s : f(s) \geq y\} \text{ is closed.}$$

**USC($\mathbb{D}$) = \{ $f : \mathbb{D} \to [-\infty, +\infty] : f$ is upper semicontinuous\}**
Semicontinuous functions: uniform topology inadequate

Locations of discontinuities don’t match exactly: no proximity

Try hypo-topology!
Key: identify a function with its hypograph

The hypograph of $f : \mathbb{D} \to \overline{\mathbb{R}}$ is a subset of $\mathbb{D} \times \mathbb{R}$:

$$\text{hypo } f = \{(s, x) \in \mathbb{D} \times \mathbb{R} : x \leq f(s)\}$$

Clearly, a function can be reconstructed from its hypograph:

$$f(s) = \sup\{x \in \mathbb{R} : (s, x) \in \text{hypo } f\}$$
Upper semicontinuous function  \iff  closed hypograph

\begin{align*}
f & \text{usc, hypo } f \text{ closed} & \quad f & \text{not usc, hypo } f \text{ not closed}
\end{align*}

\begin{align*}
f & \text{is upper semicontinuous} & \iff & \text{hypo } f \text{ is closed in } \mathbb{D} \times \mathbb{R}
\end{align*}

\[
\text{USC}(\mathbb{D}) \sim \text{HYPO}(\mathbb{D}): \text{family of closed hypographs } \subset \mathcal{F}.
\]
Fell topology on the family of closed sets

Painlevé/Kuratowski/Fell topology on $\mathcal{F} = \mathcal{F}(\mathbb{D} \times \mathbb{R})$

subbase: \[
\begin{align*}
\mathcal{F}_G &= \{ F \in \mathcal{F} : F \cap G \neq \emptyset \}, \quad G \text{ open}, \\
\mathcal{F}^K &= \{ F \in \mathcal{F} : F \cap K = \emptyset \}, \quad K \text{ compact}
\end{align*}
\]

Base for the Fell topology: \[
\{ \mathcal{F}^K_{G_1,\ldots,G_n} = \mathcal{F}^K_{G_1} \cap \ldots \cap \mathcal{F}^K_{G_n} \}.
\]
Hypo-topology

Topology on $\text{USC}(\mathbb{D})$: trace of Fell’s topology onto $\text{HYPO}(\mathbb{D})$. Open sets $= \{U \cap \text{HYPO}(\mathbb{D}), U \in \mathcal{F}\}$

N.B. : $(\text{USC}(\mathbb{D}), \text{HYPO}(\mathbb{D}))$ is compact, metric!
Hypo and pointwise convergence are different

\[ \frac{1}{n}, \frac{2}{n} \]
Upper semicontinuous process $= \text{random }usc\text{ function}$

By definition, an $usc$ process is a random element in $USC(\mathcal{D})$, i.e. a map

$$\xi : (\Omega, \mathcal{A}, \mathbb{P}) \to (USC(\mathcal{D}), HYPO(\mathcal{D})).$$
The law of an usc process is *not* determined by its fidis

\[ \xi(s) = X1_{\{U=s\}} \]

\(D = [0, 1], \quad U \sim \text{Uniform}(0, 1), \quad X \text{ any random variable } \Omega \to \mathbb{R}^+\)

\(\forall (s_1, \ldots, s_k) \in [0, 1]: \xi(s_1) = \cdots = \xi(s_k) = 0 \text{ a.s., although } \xi \neq 0\)
The law of an *usc* process: determined by the capacity functional

Capacity functional of a random closed set $F$:

$$T_F(K) = \Pr(F \cap K \neq \emptyset), \quad K \text{ compact.}$$

For an *usc* process:

$$1 - T_\xi(K) = \Pr(\text{hypo } \xi \cap K = \emptyset), \quad K \subset \mathbb{D} \times \mathbb{R} \text{ compact}$$
Max-stable processes

**Definition: usc max-stable process**

An usc process $\xi$ with non-degenerate margins is **max-stable** if $\forall n$ there exist functions $\alpha_n > 0$ and $\beta_n$ such that, for $\xi_1, \ldots, \xi_n \overset{iid}{\sim} \xi$,

\[
\bigvee_{i=1}^{n} \xi_i \overset{d}{=} a_n \xi + b_n
\]

- The margins $(G_s)_{s \in \mathbb{D}}$ are necessarily max-stable (one-to-one $(x^-_G, x^+_G) \rightarrow (0, 1)$).

- Definition implicitly assumes that the r.h.s. is a usc process.

- **Simple max-stable**: Fréchet(1) margins $\Phi(x) = 1_{x>0} e^{-1/x}$, then $\alpha_n(s) = n, \beta_n(s) = 0$;

Can we ‘reduce’ to the simple max-stable case?
Sklar’s theorems for max-stable processes?

Questions

Sklar I  Given a simple max-stable process $\xi^*$, and max-stable margins $G_s$, $s \in \mathbb{D}$, is the stochastic process

$$\{\xi_s = G_s^{-}(\Phi(\xi^*_s))\}_{s \in \mathbb{D}}$$

a (max-stable) usc process?

Sklar II  Given a max-stable process $\xi$, $\exists$? a simple max-stable usc process $\xi^*$ such that

$$\xi \overset{d}{=} \{G_s^{-} \circ \Phi(\xi^*_s)\}_{s \in \mathbb{D}} \text{ in USC}(\mathbb{D}) \ ?$$
Standardization to simple max-stable processes: not always possible

\[ \xi(1) = X \vee Y \]

\[ \xi(s) = X \]

\[ \xi(s) = Y \]

\[ X, Y \overset{i.i.d.}{\sim} \Phi (\text{Fréchet}(1)) \]

Standardization to Fréchet(1) requires halving \( \xi(1) \): no longer \( usc \)
An admissible class of transformations for usc processes

- $\mathcal{U} = \text{family of functions } U : \mathbb{D} \times [-\infty, +\infty] \rightarrow [-\infty, +\infty] \text{ s.t.}$
  - (a) For every $s, x \mapsto U(s, x)$ is non-decreasing, right-continuous.
  - (b) For every $x, s \mapsto U(s, x)$ is usc
    (think $U(s, x) = F(s, x)$ for now)

- For $U \in \mathcal{U}$, define the mapping
  $$U^* : z \in \text{USC}(\mathbb{D}) \mapsto U^*(z) := \{U(s, z(s))\}_{s \in \mathbb{D}}.$$

- Let $\mathcal{U}^* = \{U^* : U \in \mathcal{U}\}$.

**Proposition: usc preserving transformations**

Every $U^* \in \mathcal{U}^*$ is a hypo-measurable mapping from $\text{USC}(\mathbb{D})$ to itself.

**Lemma: Composition**

If $U, V \in \mathcal{U}$, then $U \circ V : (s, x) \mapsto U(s, V(s, x))$ belongs to $\mathcal{U}$
Examples of usc-preserving maps

- Fix \( y \in \text{USC}(\mathbb{D}) \). Then
  \[ U_\lor : (s, x) \mapsto x \lor y(s) \text{ and } U_\land : (s, x) \mapsto x \land y(s) \text{ belong to } \mathcal{U}. \]
  Associated maps: \( U_\lor^* : z \mapsto z \lor y \) and \( U_\land^* : z \mapsto z \land y \).

- If \( a : \mathbb{D} \to (0, \infty) \) is continuous, then \( U_a^* : (s, x) \mapsto a(s) x \in \mathcal{U} \).
  If \( b : \mathbb{D} \to \mathbb{R} \) is usc, then \( U_b^+ : (s, x) \mapsto x + b(x) \in \mathcal{U} \).
  Associated maps: \( U_a^* : z \mapsto az \) and \( U_b^+ : z \mapsto z + b \).

---

**Lemma (the right-continuous inverse is usc-preserving)**

\( \xi = (\xi(s) : s \in \mathbb{D}) \) an usc process, \( \xi(s) \in [-\infty, \infty] \), \( F_s(x) = \mathbb{P}(\xi(s) \leq x) \).
Define

\[
F_s^\rightarrow(p) = \sup\{y \in \mathbb{R} : F_s(y) \leq p\}, \quad (s, p) \in \mathbb{D} \times [0, 1].
\]

Then \( U_\xi : (s, x) \mapsto F_s^\rightarrow((x \lor 0) \land 1) \) belongs to \( \mathcal{U} \).
Building a non-standard usc process from a standard one

Necessary and sufficient condition: on marginal distribution functions.

- **Z**: an usc process with standard uniform margins.
- 
  \((F_s)_{s \in \mathbb{D}}\): a family of cdf’s, right-continuous inverses \(F_s^{-1}(\cdot)\).
- Define a stochastic process \(\xi: \xi(s) = F_s^{-1}((Z(s) \lor 0) \land 1)\)

**Proposition (Sklar I for usc processes)**

The following are equivalent:

1. \(\xi\) is an usc process (with margins \(F_s\)).
2. For every \(p \in [0, 1]\), the function \(s \mapsto F_s^{-1}(p)\) is usc.
Building a max-stable $\xi$ from a simple max-stable $\xi^*$

- Let $\xi^*$ be a simple max-stable \textit{usc} process, Fréchet(1) margins $\Phi$.
- Let $G_s(\cdot)$, $s \in \mathbb{D}$ be GEV distributions, $G_s^{-1}$: right-inverse.
- Define a stochastic process $\xi$: $\xi_s = G_s^{-1}(\Phi(\xi^*_s))$, $s \in \mathbb{D}$.

**Proposition (à la Sklar I for max-stable usc processes)**

The following are equivalent

1. $\xi$ is an \textit{usc} process (with margins $G_s$).
2. $\forall p \in [0, 1]$, the function $s \mapsto G_s^{-1}(p)$ is \textit{usc}.

In such a case $\xi$ is max-stable with norming functions $a_n$, $b_n$ determined by the margins $G_s$. 
Sklar II: Can every $\xi$ be represented this way?
Not clear without additional assumptions on the margins

$\xi$: an usc process, margins $F_s$, right inverses $F_s^{-\to}$.

à la Sklar II for usc processes

Suppose the following two conditions hold:

(a) For every $s \in \mathbb{D}$, $F_s(\cdot)$ has no atoms in $[-\infty, \infty]$.

(b) For every $x \in \mathbb{R} \cup \{+\infty\}$, the function $s \mapsto F_s(x)$ is usc.

Then, indeed:

(i) $Z: Z(s) = F_s(\xi(s))$ is an usc process with uniform margins.

(ii) $\tilde{\xi}: \tilde{\xi}(s) = F_s^{-\to}(Z(s))$ is an usc process, and

$$\forall s \in \mathbb{D}, \text{ almost surely }, \tilde{\xi}(s) = \xi(s).$$

In particular, $\tilde{\xi}$ and $\xi$ have identical fidis.

! same fidis $\neq$ same distribution in USC($\mathbb{D}$)!
Regularity properties of a usc process with GEV margins

\[ \xi: \text{a usc process with GEV margins } G_s(\cdot) = G(\cdot, \theta(s)), \]
\[ \theta(s) = (\mu(s), \sigma(s), \gamma). \]

**Question:** Are the sufficient conditions for standardization met?

**Answer:**

(a) For every \( s \in \mathbb{D} \), \( G_s(\cdot) \) has no atoms in \([-\infty, \infty]\).

(b) For every \( x \in \mathbb{R} \cup \{+\infty\} \),

The function \( s \mapsto G_s(x) \) is usc

\[ \iff \theta \text{ is continuous.} \]
Standardizing a max-stable \(usc\) process

\(\xi\): an \(usc\) process with \(GEV\) margins \(G_s(\cdot) = G(\cdot, \theta(s))\).

\(à\ la\ Sklar\ II\ for\ usc\ processes\ with\ GEV\ margins\)

If the \(GEV\) parameter \(\theta : \mathbb{D} \rightarrow \Theta\) is continuous, then:

(a) \(\xi^* : \xi^*(s) = -\frac{1}{\log G_s(\xi(s))}\) is an \(usc\) process with Fréchet(1) margins.

(b) \(\tilde{\xi} : \tilde{\xi}(s) = G_s^{-1}(\Phi(\xi^*(s)))\) is an \(usc\) process and, with proba. 1,

\[
\forall s \in \mathbb{D}, \quad \tilde{\xi}(s) = \begin{cases} \xi(s) & \text{if } G_s(\xi(s)) < 1, \\ \infty & \text{if } G_s(\xi(s)) = 1. \end{cases}
\]

If in addition, \(\sup_{s \in \mathbb{D}} G_s(\xi(s)) < 1\) (a.s.), then:

with probability 1, \(\xi = \tilde{\xi}\) and the following are equivalent:

(i) The \(usc\) process \(\xi\) is max-stable.

(ii) The \(usc\) process \(\xi^*\) is simple max-stable.
Conclusion

- Hypo-topology is well-adapted to extremes of \textit{usc} processes.

- Standardization of \textit{usc} processes with GEV margins is possible if the marginal \textit{c.d.f.’s} are continuous w.r.t. space variable.

- Max-stability is preserved if the upper end-point is almost never reached anywhere.

  First step towards statistically grounded modeling within classical EVT framework.

Further topics:

Standardization in the \textbf{max-domain of attraction} is possible too, and the limit is \textbf{max-stable} under mild conditions (in progress).


