Invariance (Symmetry) Principles and Conservation Laws (the next 3 classes)

- Invariance and Operators in Quantum Mechanics (Perkins 3.1)
- Translations and Rotations (Perkins 3.2)
- Parity (Perkins 3.2 and Griffiths 4.6)
- Charge Conjugation (Perkins 3.7 and Griffiths 4.7)
- Groups, Angular Momentum, and Spin (Griffiths 4.1 - 4.4)
- Flavor Symmetries (Isospin) (Griffiths 4.5)
- Neutral Kaon Mixing and CP Violation (Griffiths 4.8 and Perkins 7.14)
Operators in Quantum Mechanics

\[ i \frac{\partial}{\partial t} \psi_3(t) = H \psi_3(t) \]
\[ \psi_3(t) = T(t, t_0) \psi_3(t_0) \]
\[ T(t, t_0) = \exp \left[ -i (t-t_0) H \right] \]
\[ T^{-1} = T^* = \exp \left[ i (t-t_0) H \right] \]
so, \[ \psi_3^*(t) = \psi_3(t) T^{-1}(t, t_0) \]

**Schrodinger picture:**
\[ \Psi(t) = \int \psi_3^*(t) Q(t) \psi_3(t) \, dU \]

**Heisenberg picture:**
\[ Q(t) = \int \psi_3^*(t_0) Q(t) \psi_3(t_0) \, dU \]

\[ \Rightarrow \psi_3^*(t) Q(t) \psi_3(t) = \psi_3^*(t) Q(t_0) \psi_3(t_0) \]
\[ = \psi_3^*(t) T^{-1} \alpha T \psi_3(t) \]
\[ \Rightarrow Q(t) = T^{-1} \alpha T \]
Invariance and Conservation

\[ i \frac{dQ}{dt} = i \frac{dT}{dt} \left[ T^{-1}Q_tT \right] = \]
\[ = i \frac{dT}{dt} Q_t + i T^{-1}Q_t \frac{dT}{dt} \]

Recall \( T = \exp \left[ -i (\text{commutator terms}) \right] \)
\( T^{-1} = \exp \left[ i (\text{commutator terms}) \right] \)

So \( i \frac{dQ}{dt} = i(iHT)Q_t + iT^{-1}Q_t(-iHT) \)
\[ = -(HT^{-1}Q_tT + T^{-1}Q_tHT) \]
\[ = -H\mathcal{Q} + \mathcal{Q}H \]
\[ = \mathcal{Q}, H \]

\[ i \frac{dQ}{dt} = [Q, H] \quad \text{assuming that} \quad \frac{\partial Q}{\partial t} = 0 \]

\( Q \) commutes with \( H \iff \mathcal{Q} \) is conserved
Translation in Quantum Mechanics

Translation by an infinitesimal $\delta x$

$\psi' (x + \delta x) = \psi (x) + \delta x \frac{d\psi}{dx}$

$= (1 + \delta x \frac{d}{dx}) \psi (x) = D \psi (x)$

Where $D = (1 + \delta x \frac{d}{dx})$

recall $\psi = e^{i(kx - \omega t)}$

$-i\delta x \psi = h\psi$

$-i\delta x \rightarrow p$; $\frac{d}{dx} \rightarrow ip$

so $D = (1 + ip \hat{p})$

For a finite translation $\Delta x = n \delta x$

$D = \lim_{n \rightarrow \infty} \left(1 + i\delta x \hat{p}\right)^n = e^{i\Delta x \hat{p}}$

$D$ is unitary; $D^* D = D' D = 1$

$P_x$ is the generator for the operator $D$

of space translations
Translational Invariance and Conservation of Linear Momentum

\[ D = e^{i \Delta x \frac{\partial}{\partial x}} \left( 1 + i \delta x \frac{\partial}{\partial x} \right) \]

If \( H \) is independent of space translations, then

\[ [D, H] = 0 \]
\[ [e^{i \Delta x \frac{\partial}{\partial x}}, H] = 0 \]
\[ [\lim_{\Delta x \to 0} (1 + i \delta x \frac{\partial}{\partial x}), H] = 0 \]
\[ [(1 + i \delta x \frac{\partial}{\partial x}), H] = 0 \]
\[ [\frac{\partial}{\partial x}, H] = 0 \]

Translation invariance \( \Rightarrow \) conservation of linear momentum
Rotation in Quantum Mechanics

completely analogous to translations

\[ R = 1 + \delta \theta \frac{\partial}{\partial \theta} \]

recall \[ J_z = -i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i \frac{\partial}{\partial \theta} \]

(appendix C, Perkins)

\[ R = 1 + i \delta \theta J_z \]

finite rotation \[ \Delta \theta = \theta \delta \theta \]

\[ R = \lim_{\delta \theta \to 0} \left( 1 + i \delta \theta J_z \right)^n \]

\[ \text{if } \quad [R, H] = 0 \]

\[ [\lim_{\delta \theta \to 0} (1 + i \delta \theta J_z)^n, H] = 0 \]

\[ [J_z, H] \]
Parity in Quantum Mechanics

Spatial inversion of coordinates
\((x, y, z) \rightarrow (-x, -y, -z)\)

\[ P \psi(\vec{r}) = \psi(-\vec{r}) \]

\[ P^2 \psi(\vec{r}) = P \left[ P \psi(\vec{r}) \right] = P \psi(-\vec{r}) = \psi(-(-\vec{r})) = \psi(\vec{r}) \]

\[ P^2 = 1 \]

eigenvalues \(= \pm 1\)

(i) \(\psi(x) = \cos x\), \(P \psi(x) = \cos(-x) = \cos x = \psi\)

\[ P \psi = \psi \]

(ii) \(\psi(x) = \sin x\), \(P \psi(x) = \sin(-x) = -\sin x = -\psi\)

\[ P \psi = -\psi \]

(iii) \(\psi(x) = \cos x + \sin x\), \(P \psi = \cos x - \sin x\)

\[ \mp 1 \psi(x) \]

\[ [P, H] = 0 \quad \Rightarrow \quad H(\vec{r}) = H(-\vec{r}) \]
Parity in Spherically Symmetric Systems

\[ \psi(r, \theta, \phi) = \chi(r) \psi_{n}^{m}(\theta, \phi) \]
\[ = \chi(r) \sqrt{\frac{2}{\pi} \frac{(n+m)!}{(n-m)!}} e^{im\phi} \]

Under inversion \( (r, \theta, \phi) \to (-r, \pi - \theta, \pi + \phi) \)
\[ e^{im\phi} \to e^{-im\phi} = e^{-im\pi} e^{im\phi} = (-1)^{m} e^{im\phi} \]
\[ \mathcal{P} \psi_{n}^{m}(\cos \theta) \to (-1)^{d+m} \mathcal{P} \psi_{n}^{m}(\cos \theta) = (-1)^{d+m} \psi_{n}^{m}(\cos \theta) \]
\[ \psi = \psi' = \chi(r) \sqrt{\frac{2}{\pi} \frac{(n+d)!}{(n-d)!}} e^{im\phi} \]
\[ = \chi(r) \sqrt{\frac{2}{\pi} \frac{(n+m)!}{(n-m)!}} e^{-im\phi} \]
\[ = (-1)^{d+m} (-1)^{m} \psi = (-1)^{d+m} \psi \]
Parity as a Quantum Number

-parity is a multiplicative quantum $\hat{\eta}$

- $\eta = \eta_0 \eta_0 \eta_0$

- $\hat{p}\eta = p (\eta_0 \eta_0 \eta_0)$
  - $\eta_0 (-\hat{p}) \eta_0 (-\hat{p}) \eta_0 (-\hat{p})$
  - $(p \eta_0) (p \eta_0) (p \eta_0)$

-if each of $a, b, c$ have definite parity, then $\eta$ has parity

\[ \text{parity is conserved in strong and electromagnetic interactions} \]

\[ \pi \quad J^p = 1^+ \]

\[ p = \pi^+ \pi^- \]

\[ \pi^0 \quad J^p = 0^+ \]

\( p (\pi^+ \pi^- m\eta) = -1 \)

\( J(p) = 1, \quad J(\pi) = 0 \)

\( f = 1 \quad \text{so} \)

\( (-1)^d p(\pi) p(\pi) \)

\( (-1) (-1) (-1) = -1 \quad \checkmark \)
The Tau-Theta Puzzle

The tau and the theta were observed experimentally to have the same lifetimes and the same masses, but opposite parities. At the time, parity was assumed to be a symmetry of nature. In that case, the tau and the theta could not be decays of the same particle.

These are weak decays of what is now called the $K^+$. We now understand that the strong interaction is invariant under parity inversion, but the weak interaction is not.
Parity Non-Conservation
Parity Eigenstates

\[ p(s) = s \quad \text{scalar} \]
\[ p(\phi) = -\phi \quad \text{pseudo-scalar} \]
\[ p(\vec{\nu}) = -\vec{\nu} \quad \text{(polar) vector} \]
\[ p(\vec{\alpha}) = \vec{\alpha} \quad \text{axial vector} \]

Examples:

\[ p(\vec{r}) = -\vec{r} \quad \text{position is a vector} \]
\[ p(\vec{p}) = -\vec{p} \quad \text{momentum is a vector} \]
\[ \vec{p} (\vec{L}) = \vec{L} \quad \text{angular momentum is an axial vector} \]
\[ \vec{p} (\vec{S}) = \vec{S} \quad \text{spin is an axial vector} \]
\[ \vec{p} (\vec{E}) = -\vec{E} \quad \text{electric field is a vector} \]
\[ \vec{p} (\vec{B}) = -\vec{B} \quad \text{magnetic field is an axial vector} \]
\[ \vec{F} = q (\vec{E} + \vec{v} \times \vec{B}) \quad \text{is a vector} \]