Calculus of Variations

Monge–Ampère equations and Bellman functions: The dyadic maximal operator

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Received 26 February 2008; accepted 28 February 2008

Presented by Jean-Pierre Kahane

Abstract

We find explicitly the Bellman function for the dyadic maximal operator on $L^p$ as the solution of a Bellman partial differential equation of Monge–Ampère type. This function has been previously found by A. Melas (2005) in a different way, but it is our partial differential equation-based approach that is of principal interest here. Clear and replicable, it holds promise as a unifying template for past and current Bellman function investigations.

Résumé


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1 L. Slavin’s research supported in part by the NSF (grant DMS-0701254).
2 A. Stokolos’ research supported in part by a grant from the Faculty Research and Development Fund of the College of Liberal Arts and Sciences of DePaul University.
3 V. Vasyunin’s research supported in part by RFBR (grant 05-01-00925).

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doi:10.1016/j.crma.2008.03.003
1. Introduction

For a locally integrable function $g$ on $\mathbb{R}^n$ and a set $E \subset \mathbb{R}^n$ with $|E| \neq 0$, let $\langle g \rangle_E = \frac{1}{|E|} \int_E g$ be the average of $g$ over $E$. Let $p > 1$ and $q > 1$ be conjugate exponents, i.e. $p^{-1} + q^{-1} = 1$. Let $\varphi$ be a nonnegative locally $L^p$-function on $\mathbb{R}^n$. Fix a dyadic lattice $D$ on $\mathbb{R}^n$ and consider the dyadic maximal operator:

$$M\varphi(x) = \sup_{I \ni x; I \in D} \langle \varphi \rangle_I.$$ 

Following F. Nazarov and S. Treil [2], we define the Bellman function for $M\varphi$,

$$B(f, F, L) = \sup_{0 \leq q \in L_p(\mathbb{R}^n)} \left\{ \langle (M\varphi)^p \rangle_Q : \langle \varphi \rangle_Q = f; \langle \varphi^q \rangle_Q = F; \sup_{R \ni Q} \langle \varphi \rangle_R = L \right\}. \quad (1)$$

Observe that $B$ is independent of $Q$ and well-defined on the domain: $\Omega = \{(f, F, L): 0 < f \leq L; f^p \leq F \}$. Finding $B$ will, among other things, provide a sharp refinement of the Hardy–Littlewood–Doob maximal inequality

$$\|M\varphi\|_p \leq q\|\varphi\|_p. \quad (2)$$

In [2], the authors show that $B(f, F, L) \leq q^p F - pq f L^p - 1 + p L^p$, which implies (2). A. Melas in [1], using deep combinatorial properties of the operator $M$ and without relying on the Bellman PDE, finds $B$ explicitly. In contrast, we develop a boundary value problem of Monge–Ampère type that $B$ must satisfy (assuming sufficient differentiability) and solve it, producing the function from [1]. Our approach has been used as the foundation of several recent Bellman function results. We first restrict our attention to the one-dimensional case and then show that the Bellman function does not depend on dimension.

2. Finite-differential and differential properties of $B$

Let $Q$ be an interval and $Q_-, Q_+$ its left and right halves, respectively. Let $(f_\pm, F_\pm) = (f_{Q_\pm}, F_{Q_\pm}), (f, F) = ((f_-, F_-) + (f_+, F_+))/2$. Taking suprema in the identity

$$\langle (M\varphi)^p \rangle_Q = \frac{1}{2} \langle (M\varphi)^p \rangle_{Q_-} + \frac{1}{2} \langle (M\varphi)^p \rangle_{Q_+},$$

over all $\varphi$ with appropriate averages, we obtain:

$$B(f, F, L) \geq \frac{1}{2} B(f_-, F_-, \max \{f_-, L\}) + \frac{1}{2} B(f_+, F_+, \max \{f_+, L\}). \quad (3)$$

Any function $B$ satisfying this pseudo-concavity property on $\Omega$ will be a majorant of the true Bellman function. The following theorem phrases this condition in a differential form:

**Theorem 2.1.** Let $z = (f, F)$. Assuming sufficient smoothness on the Bellman function $B$, condition (3) holds for all admissible triples if and only if:

$$\det \left( \frac{\partial^2 B}{\partial z^2} \right) = 0, \quad B_{ff} \leq 0, \quad B_L \geq 0 \text{ on } \Omega; \quad 2B_{fL} + B_{LL} \leq 0, \quad B_L = 0 \text{ when } f = L. \quad (4)$$

3. Homogeneity, boundary value problem, solution

We reduce the order of the PDE in (4) by using the multiplicative homogeneity of $B$: $B(f, F, L) = L^p B(f/L, F/L, 1)$, i.e. $L^p G(x, y)$, where $x = f/L, y = F/L^p$. In addition, $F = f^p$ only for functions that are constant on $Q$, so $B(f, f^p, L) = L^p$, meaning $G(x, x^p) = 1$. Coupling this with the first and the last conditions in (4), we get a boundary value problem for $G$ on the domain $\{(x, y) | 0 < x \leq 1; x^p \leq y\}$:

$$G_{xx} G_{yy} = G_{xy}^2; \quad G(x, x^p) = 1; \quad p G(1, y) = G_x(1, y) + p y G_y(1, y). \quad (5)$$

We look for the solution of the Monge–Ampère equation (5) in the general parametric form:

$$G(x, y) = tx + f(t)y + g(t); \quad x + f'(t)y + g'(t) = 0. \quad (6)$$
Fix a value of \( t \), i.e. fix one of the straight-line trajectories in (6). Let \((u(t), u^p(t))\) be the point where that trajectory intersects the lower boundary \( y = x^p \). We have:

\[
G(u, u^p) = tu + f(t)u^p(t) + g(t) = 1; \quad u(t) + f'(t)u^p(t) + g'(t) = 0.
\]

Differentiating the first equation and using the second one, we get, after some algebra, \( f = -t/(pu^{p-1}), \ g = 1 - tu/q \). Assume now that the trajectory intersects the right boundary \( x = 1 \) at the point \((1, v(t))\). Then \( G(1, v) = t + f + g \). On the other hand, parametrization (6) implies \( G_x = t, \ G_y = f(t) \) and so the second boundary condition in (5) becomes \( G(1, v) = 1/p + f \). This gives \( g = -t/q \), allowing us to express \( t = q/(u - 1) \). Simplifying, we obtain a complete solution of the form (6):

\[
G(x, y) = \frac{y}{u^p}; \quad x - \frac{qu - 1}{qu^p} y - \frac{1}{q} = 0.
\]  

(7)

In terms of the original variables, we get a Bellman function candidate near the boundary \( f = L \):

\[
B(f, F, L) = Fu^{-p}(f/L, F/L^p).
\]  

(8)

4. From the candidate to the true function

4.1. Condition \( B \geq B \)

One can readily verify that the rest of conditions (4) are satisfied by the candidate (8). Therefore, property (3) holds and one can perform the Bellman induction: take any nonnegative function \( \varphi \in L^p_0(\mathbb{R}^n) \) and an interval \( Q_0 \subset D \). For an interval \( Q \subset Q_0 \), \( Q \subset D \), let \( X_Q = (f_Q, F_Q, L_Q) \) with \( f, F \), and \( L \) defined as in (1). Then

\[
B(f_Q, F_Q, L_Q) \geq \frac{1}{2} B(X_{Q_0}) + \frac{1}{2} B(X_{Q_0}^+) \geq \frac{1}{|Q_0|} \sum_{Q \subset Q_0, |Q| = 2^{-n}|Q_0|} |Q| |B(X_Q)| \geq \frac{1}{|Q_0|} \sum_{Q \subset Q_0, |Q| = 2^{-n}|Q_0|} |Q|L^p_Q \geq \frac{1}{|Q_0|} \sum_{Q \subset Q_0, |Q| = 2^{-n}|Q_0|} |Q| \left( \sup_{R \supset Q} \langle \varphi \rangle_R \right)^p \rightarrow \langle (M\varphi)^p \rangle_{Q_0}, \quad \text{as } n \to \infty.
\]  

(9)

Here we have used that \( B \geq L^p \). Taking supremum on the right over all \( \varphi \) with the above \( X_{Q_0} \) we get \( B \geq B \).

4.2. Condition \( B \leq B \)

To get the reverse inequality, we need to construct, for every point \((f, F, L) \in \Omega \), a sequence of nonnegative functions on \((0, 1), \{\varphi_n\}_n \), so that

\[
\lim_{n \to \infty} \langle (M\varphi_n)^p \rangle_{(0, 1)} \geq B(f, F, L).
\]

To do this, we use the trajectories \( t = \text{const} \) of the Monge–Ampère equation from Section 3. In the original variables, this gives:

\[
f = \frac{L}{q} + AF.
\]  

(10)

On the boundary \( f = L \) going along these trajectories yields the extremal sequence

\[
\varphi_n(t) = \begin{cases} \alpha_nL, & 0 < t < 2^{-n}, \\ \varphi_n(2^kt - 1), & 2^{-k} < t < 2^{-k+1}, k = 2, \ldots, n, \\ \beta_n\varphi_n(2t - 1), & \frac{1}{2} < t < 1. \end{cases}
\]  

(11)

The definition is understood recursively, whereby the function is defined on a portion of \((0, 1)\), then on the same portion of the remaining part, and so on. The numbers \( \alpha_n \) and \( \beta_n \) are chosen so that \( \langle \varphi_n \rangle_{(0, 1)} = L \) and \( \langle \varphi_n^p \rangle_{(0, 1)} = F \). This means

\[
\frac{1}{2^n} \alpha_n + \frac{1}{2} \beta_n = \frac{1}{2^n} + \frac{1}{2}, \quad \frac{1}{2^n} \alpha_n^p + \frac{1}{2} \beta_n^p \frac{F}{L^p} = \left( \frac{1}{2^n} + \frac{1}{2} \right) \frac{F}{L^p}.
\]

One can show that $\alpha_n M \varphi_n \geq \varphi_n$ and $\alpha_n \to u(1, F/L)$. Thus, we have
\[
\lim_{n \to \infty} \left\langle (M \varphi_n)_{(0,1)} \right\rangle = \lim_{n \to \infty} \frac{1}{\alpha_n} \left\langle \varphi_n \right\rangle_{(0,1)} = \lim_{n \to \infty} \frac{F}{\alpha_n} = F u^{-p}(1, F/L) = B(L, F, L),
\]
which gives $B(L, F, L) \geq B(L, F, L)$.

On the boundary $F = f^p$ the situation is simple: here the only test functions are constants and so $B(f, f^p, L) = B(f, f^p, L) = L^p$. Having constructed the extremal sequences on the two boundaries, we get the extremal sequence at any point $(f, F, L)$ with $f > L/q$ as their weighted dyadic rearrangement built along the unique extremal trajectory of the form $(10)$ passing through the point.

One observes, however, that trajectories (10) cannot be used with $A < 0$, since they then would intersect the “forbidden” boundary $f = 0$. (It is forbidden because, for a nonnegative function, $f = 0$ implies $F = 0$.) In fact, in the region $0 < f < L/q$, no trajectory can lean either to the left or to the right (the forbidden boundary to the left, the existing extremal trajectory $f = L/q$ to the right). We conclude two things: the trajectories are vertical in this region and the candidate (8) no longer works there. However, this is quickly rectified: If $f < L/q$, no trajectory can lean either to the left or to the right (the forbidden boundary to the left, the existing extremal trajectory $f = L/q$ to the right). We conclude two things: the trajectories are vertical in this region and the candidate (8) no longer works there. However, this is quickly rectified: If $G(x, y) = a(x)y + b(x)$, then $G(x, x^p) = 1$ implies that $G(x, y) = 1 + a(x)(y - x^p)$. Now $G_{xx}G_{yy} - G_{xy}^2 = -(a'(x))^2 = 0$, and $G(1/q, y) = q^p y$ implies that $a(x) = q^p$. Thus we get the unique two-piece Bellman function candidate:
\[
B(f, f^p, L) = \begin{cases} F u^{-p}(f/L, F/L), & L < qf, \\ L^p q + F f^{-p}(F - f^p), & L \geq qf. \end{cases}
\]

(In the notation of (1), $u^{-p}(x, y) = \omega_p((px - p + 1)y)^p)$.) This $B$ still satisfies (3). Therefore, Bellman induction (9) works. We now need an extremal sequence proving that $B \geq B$ in the region $L \geq qf$. There is a unique extremal trajectory passing through each point of the region. However, the trajectory is vertical and so intersects the boundary of $\Omega$ at a single point; as a result we cannot use a weighted average of boundary extremal sequences like we just did for the region $L > qf$. We deal with it by tilting the trajectory slightly to the right, which produces a (distant) second boundary point, at the boundary $f = L$. This lets us use the extremal sequence $\varphi_n$ from (11), while simultaneously reducing the tilt. Namely, fix $(f, F, L)$ and $k \geq 1$. Define $\gamma_k$ and $F_k$ so that $L - \gamma_k = 2^k(F - \gamma_k)$ and $F_k - \gamma_k = 2^k(F - \gamma_k)$. (Observe that $\gamma_k \to f$ and $F_k \to \infty$.) Using (11), form a sequence $\{\varphi_{k, n}\}_{n=1}^\infty$ with $\varphi_{k, n}(0,1) = L$ and $\varphi_{k, n}(0,1) = F_k$, so that $(M \varphi_{k, n})_{(0,1)} \to B(L, F_k, L)$, as $n \to \infty$. Let,
\[
\varphi_{k, n}(t) = \begin{cases} \varphi_{k, n}(2^k t), & 0 < t < 2^{-k}, \\ \gamma_k, & 2^{-k} < t < 1, \\ 2L - f, & 1 < t < 2. \end{cases}
\]

Direct computation shows that $\langle \varphi_{k, n} \rangle_{(0,1)} = f$, $\langle \varphi_{k, n} \rangle_{(0,1)} = F$, and $\langle \varphi_{k, n} \rangle_{(0,2)} = L$. Then
\[
\langle (M \varphi_{k, n})_{(0,1)} \rangle \geq L^p(1 - 2^{-k}) + 2^{-k}(M \varphi_{k, n})_{(0,1)} \xrightarrow{n \to \infty} L^p(1 - 2^{-k}) + 2^{-k}B(L, F_k, L) \xrightarrow{k \to \infty} L^p + (F - f^p)u^{-p}(1, \infty) = L^p + q^p(F - f^p).
\]

5. Several dimensions

It turns out that the Bellman function (1), (12) is dimension-free. Fix a dyadic cube $Q$ and let $Q_1, \ldots, Q_{2^n}$ be its dyadic offspring. Then
\[
B(2^{-n} \sum_{k=1}^{2^n} z_k, L) \geq 2^{-n} \sum_{k=1}^{n} B(z_k, \max\{f_k, L\}).
\]

Therefore, we can run the induction (9) to prove that $B \geq B$. The other direction is shown by a trivial modification of the one-dimensional maximizing sequences. A similar argument can be used to show that the same Bellman function works for the maximal operator on trees, the setting of choice in [1].

References
