Quiz 5: Black Body Radiation. Debye Model.

1. Assuming that the Earth and the Moon are black bodies, at approximately the same distance from the Sun, estimate the relative correction to the Moon's temperature due to radiation from Earth versus that from the Sun alone. The average distance between the Earth and the Moon is about 60 times the radius of Earth.

   Hint: $J = \sigma T^4$ (Stephan-Boltzmann law for the total emission from a black body).

   Solution

   Temperature T_M of the Moon due to radiation from Sun alone

$$\sigma T_S^4 \frac{4\pi R_S^2}{4\pi d^2} \pi R_M^2 = \sigma T_M^4 4\pi R_M^2$$
$$T_M^4 = \frac{R_S^2}{4d^2} T_S^4$$

   Temperature $\widetilde{T}_M$ of the Moon due to radiation from both Earth and Sun

$$\left(\sigma T_S^4 \frac{4\pi R_S^2}{4\pi d^2} + \sigma T_E^4 \frac{4\pi R_E^2}{4\pi l^2} \right) \pi R_M^2 = \sigma \widetilde{T}_M^4 4\pi R_M^2$$

   whereof

$$\frac{\widetilde{T}_M^4}{T_M^4} = 1 + \frac{T_E^4 R_E^2 d^2}{T_S^4 R_S^2 l^2}$$

   Temperature of Earth is the same as temperature of Moon

$$T_E^4 = \frac{R_S^2}{4d^2} T_S^4$$

   so that

$$\frac{\widetilde{T}_M}{T_M} = \left(1 + \frac{R_E^2}{4l^2} \right)^{1/4} \approx 1 + \frac{R_E^2}{16 l^2} \approx 1 + \left(\frac{1}{240} \right)^2$$

   that is correction is $\sim 1.7x10^{-3}\%$.

2. Using the Debye density of states, calculate the zero-point energy

$$\Delta\varepsilon_0 = \sum_{\alpha}^{\omega_m} \frac{1}{2}\hbar\omega_\alpha$$

for a solid with N atoms and express it in terms of the Debye temperature.

Solution

$$3N = V\int_0^{\omega_m} \mathcal{N}(\omega)\, d\omega = \frac{3V}{(2\pi)^3}\int_0^{\omega_m} \frac{4\pi\omega^2}{\overline{u}^3} d\omega = \frac{V\omega_m^3}{2\pi^2\overline{u}^3}$$

$$\mathcal{N}(\omega) = \frac{3V\omega^2}{2\pi^2\overline{u}^3}$$

$$\Delta\varepsilon_0 = V\int_0^{\omega_m}\left(\frac{1}{2}\hbar\omega\right)\mathcal{N}(\omega)\, d\omega = \frac{3V\hbar\omega_m^4}{16\pi^2\overline{u}^3} = \frac{3V\Theta\omega_m^3}{16\pi^2\overline{u}^3} = \frac{9N}{8}\Theta$$

3. The lattice free energy in the Debye approximation is given by

$$F = N\varepsilon_0 + N\upsilon T\left\{3\log\left[1-\exp\left(-\frac{\Theta}{T}\right)\right] - D\left(\frac{\Theta}{T}\right)\right\}$$

where

$$D(x) = \frac{3}{x^3}\int_0^x \frac{z^3 dz}{\exp z - 1}$$

Find the energy E In terms of D only, not derivatives) and consider specifically the limit of $T \gg \Theta$.

Solution

$E = F - T\partial F/\partial T$

$$\begin{aligned} E - N\varepsilon_0 &= -N\upsilon T^2 \frac{\partial\left\{3\log\left[1-\exp\left(-\frac{\Theta}{T}\right)\right] - D\left(\frac{\Theta}{T}\right)\right\}}{\partial T} \\ &= N\Theta\upsilon \frac{\partial\left\{3\log\left[1-\exp(-x)\right] - D(x)\right\}}{\partial x} \\ &= N\Theta\upsilon\left[3\frac{\exp(-x)}{1-\exp(-x)} - D'(x)\right] = N\Theta\upsilon\left[\frac{3}{\exp x - 1} - D'(x)\right] \end{aligned}$$

but

$$D'(x) = -\frac{9}{x^4}\int_0^x \frac{z^3 dz}{\exp z - 1} + \frac{3}{\exp x - 1} = -\frac{3D}{x} + \frac{3}{\exp x - 1}$$

whereof

$$E = N\varepsilon_0 + 3TN\upsilon D\left(\frac{\Theta}{T}\right)$$

$T \gg \Theta$ corresponds to $x \ll 1$ and

$$D(x) \simeq \frac{3}{x^3}\int_0^x \frac{z^3 dz}{z} = 1$$

whereof

$$E \simeq N\varepsilon_0 + 3TN\upsilon$$