Mesoscopic Fluctuations Of Orbital Magnetic Response In Level-Quantized Metals

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We evaluate the distribution function of mesoscopic fluctuations of orbital magnetic response in finite-size level-quantized metal particles and Aharonov-Bohm rings for temperatures smaller than the mean level spacing. We find a broad distribution with the reduced moments much larger than the mean. For strong spin-orbit interaction we find very long tails due to thermal activation of large effective moments of the electrons at the Fermi level.

I. INTRODUCTION

The question of mesoscopic fluctuations of orbital magnetic response has been extensively studied within the perturbation theory approach which, generally, holds when the temperature and/or level broadening is larger than the mean level spacing $\Delta$. In 2D (which we only consider here), the variance of the magnetic moment was predicted to be

$$\langle \delta M^2 \rangle \sim \mu_B^2 (k_F \ell)^2 \left( \frac{\phi}{\phi_0} \right)^2 \ln \left( \frac{E_c}{T^*} \right) \sim \mu_B^2 (k_F \ell)^2 \left( \frac{\mu_B H}{\Delta} \right)^2 \ln \left( \frac{E_c}{T^*} \right)$$

where

$$T^* = \max \{ T, \tau_H^{-1} \}$$

with the following notations: $\mu_B$ is Bohr magneton, $k_F$ is the magnitude of the Fermi wave vector, $\ell$ is the electron mean-free-path, $\phi$ is the flux through the sample or the Aharonov-Bohm (AB) flux for rings, $\phi_0 = 2\pi/e$ is the flux quantum in units where $\hbar = c = 1$, $E_c \sim D/L^2$, where $D = v_F \ell/2$ is the diffusion coefficient, and $L$ the sample size (typically a ring or disk circumference), and

$$\tau_H^{-1} \sim E_c \left( \frac{\phi}{\phi_0} \right)^2 \sim \frac{\langle \mu_B H \rangle^2}{\Delta} (k_F \ell)$$

The logarithmic dependence changes its form when

$$\phi \approx \phi_c \sim \phi_0 \left( \frac{T}{E_c} \right)^{1/2}$$

The significance of the flux scale $\phi_c$ is such that at $T \sim \Delta$ it determines the scale of transition from Gaussian Orthogonal (GOE) to Gaussian Unitary (GUE) Ensemble. It is also the scale of linear response in the problem. (While eq. (1) indicates that, aside from logarithmic corrections, the linear response regime extends to $\phi \sim \phi_0$ for the fluctuations, it is obtained in the so called ”zero-mode” approximation [1] for systems with simply connected geometries. However, the higher modes may need to be summed up to correctly evaluate the fluctuations [2]. Consequently, unless specifically stated otherwise, we shall limit our consideration to $\phi < \phi_c$ where the response is linear both for the fluctuations and the mean, for simply connected and AB geometries.)

1In an earlier work, Altshuler and Spivak [3] considered current fluctuations in SNS junctions with broken time-reversal symmetry. This problem is equivalent to the persistent current fluctuations in AB rings. If the time-reversal symmetry is not broken and the Cooperon contribution is taken into account, their result would be equivalent to Ref. [1], including the logarithmic term.

2Insofar as the exact numerical coefficient in eq. (1), both Refs. [1] and [3] used the Euler-Maclaurin method for Matsubara summation with a resulting integral over the continuous variable $x = mT/E_c$, $dx = T/E_c \ll 1$. However, for small $x$ a more accurate procedure must be used. It is explained in Ref. [3] for the mean response (Appendix II) and can be trivially extended to fluctuations.
The average magnetic moment was predicted to be \( \langle M \rangle \sim \frac{\mu_B \phi_{\text{magnetic}}}{\phi_0} \), \( \phi < \phi_c \)

\[ \mu_B \frac{\phi_m}{\phi_0}, \phi > \phi_c \] (5)

which, for \( T \sim \Delta \), becomes

\[ \langle M \rangle \sim \mu_B (k_F \ell) \frac{\phi_m}{\phi_0} \sim \mu_B \frac{\mu_B H}{\Delta} (k_F \ell), \phi < \phi_c \]

\[ \mu_B \frac{\phi_m}{\phi_0} \sim \mu_B \frac{\mu_B H}{\Delta}, \phi > \phi_c \] (6)

Comparing (6) and (1) on the scale \( \phi < \phi_c \), we see that \( \langle M \rangle \) and \( \delta M^2 \) are of the same order of magnitude. While numerically the latter is larger (1), it is clear that the diamagnetic response is much less likely than the paramagnetic response.

In this work we address the regime in which level-quantization becomes important, namely, \( T \ll \Delta \). In this case, the perturbation theory approach, which uses the level density correlation function and standard thermodynamics, is no longer applicable and one needs to use a single electron picture in conjunction with thermal occupancy of a two-level system. In previous papers, we considered the mean response of GOE and Gaussian Symplectic Ensemble (GSE), the latter being the case for strong spin-orbit (SO) interaction. We have argued the for GOE the mean response can be formulated in terms of a single-electron van Vleck response at the Fermi level (5) and for GSE in terms of the effective magnetic moments/persistent currents of electrons in the last occupied (Fermi) state (6). Here, we apply these approaches to mesoscopic fluctuations.

II. GOE

As explained in Ref. (5), the mean orbital response can be understood in terms of the van Vleck (vV) response that involves virtual transitions from the last occupied level \( \varepsilon_i \) to the first unoccupied level \( \varepsilon_f \) of the Fermi sea

\[ \epsilon_{\text{vV}} = \frac{\langle i | \widehat{M}_z | f \rangle^2 H^2}{\varepsilon_i - \varepsilon_f} = \frac{\widehat{M}_{if}^2 H^2}{\varepsilon_i - \varepsilon_f} \] (7)

where \( \widehat{M}_z \) is the magnetic moment along the magnetic field \( H \) (perpendicular to the sample). There are three possible sources for non-self-averaging (fluctuations) based on this picture. The first two, based on eq. (7), are the fluctuations of \( \widehat{M}_{if} \) and the distribution of the level spacing at the Fermi level (that is, the distribution of the values in the denominator of (7)). We expect the latter to be dominant and will neglect the former. The third, which may lead to occasional diamagnetism, is the system-dependant nature of the Fermi-sea cancellation between the diamagnetic and paramagnetic contributions to the total magnetic response (5), (7); it is not studied here.

The mean value of the vV response is obtained as follows. \( \widehat{M}_{if} \) can be found using the semiclassical approximation (and, more precisely, using the result for the magnetic dipole absorption) (5) and is given by

\[ \langle \widehat{M}_{if} \rangle^2 \sim \mu_B^2 (k_F \ell) \] (8)

In what follows, we will use a dimensionless measure \( x \) for the level spacing at the Fermi level,

\[ \varepsilon_f - \varepsilon_i = x \Delta \] (9)

Averaging with the GOE (5) distribution function for the nearest energy levels, we find (5)

\[ \bar{\epsilon}_{\text{vV}} = -s \langle \widehat{M}_{if} \rangle^2 H^2 \frac{\pi}{2\Delta} \int_0^\infty \exp \left( -\frac{\pi x^2}{4} \right) dx = -\frac{\pi v}{2} \langle \widehat{M}_{if} \rangle^2 H^2 \] (10)

\[ \sim -|\chi_L| H^2 A (k_F \ell) \sim -\tau_H^{-1} \] (11)

Here \( s \) is the level degeneracy (\( s = 2 \), on the account of spin), \( v = s \Delta^{-1} \) is the density of levels, \( \chi_L \) is the Landau susceptibility (5) and \( A \) is the sample area. As was pointed out earlier, it is assumed that \( \tau_H^{-1} \ll \Delta \), with the opposite limit corresponding to GUE.
To consider the fluctuations, we notice that the second order perturbation theory is used in derivation of (7). Consequently, the limit of its applicability \( \epsilon_{vV} \) is, per (5) and (8),

\[
\epsilon_{vV} \sim \frac{\tau_H^{-1}}{x} \ll x \Delta
\]

that is

\[
x \gg \left( \frac{\tau_H^{-1}}{\Delta} \right)^{1/2} \text{ or } \epsilon_f - \epsilon_i \gg \left( \frac{\tau_H^{-1}}{\Delta} \right)^{1/2}
\]

An important observation about the above conditions is that the range of linear response is sample-specific and depends on the Fermi level spacing of a particular particle. On the other hand, when describing the statistical distribution of the magnetic energies, it is only meaningful to consider their values at a given field (or a range of fields).

An estimate of the fluctuation can be obtained by substituting the r.h.s. of (13) into eq. (7)

\[
(14)
\]

Below we will show that (14) corresponds to the result for the reduced higher cumulants for the distribution of magnetic energies and is due to the systems with small level spacing at the Fermi level.

Since the reduced moments grow with the order of the moment, the cumulants and the moments should be of the same order of magnitude and it is sufficient to evaluate the latter. We begin with the evaluation of the variance/second moment of the \( vV \) response

\[
\bar{e}^2 \sim \frac{\pi}{2} \tau_H^{-2} \int \frac{dx}{x} \exp \left( -\frac{\pi x^2}{4} \right) \sim \tau_H^{-2} \ln \left( \frac{\Delta}{\tau_H^{-1}} \right)
\]

(omitting subscript ”vV”). The higher moments and the reduced moments can be evaluated similarly and are given by

\[
\bar{e}^n \sim \frac{\pi}{2} \tau_H^{-n} \int \left( \frac{\tau_H^{-1}}{\Delta} \right)^{1/2} x^{-n+1} dx \exp \left( -\frac{\pi x^2}{4} \right)
\]

\[
n \sqrt{\bar{e}^n} \sim \left( \tau_H^{-1} \Delta \right)^{1/2} \left( \frac{\tau_H^{-1}}{\Delta} \right)^{1/n} \rightarrow \left( \tau_H^{-1} \Delta \right)^{1/2}
\]

in agreement with the estimate (14). Alternatively, eqs. (13)-(17) can be evaluated using the distribution function for the fluctuations which is found as

\[
P(\epsilon) = \frac{\pi}{2} \int_0^{\left( \frac{\tau_H^{-1}}{\Delta} \right)^{1/2}} d\epsilon \delta \left( \epsilon - \frac{\tau_H^{-1}}{x} \right) x \exp \left( -\frac{\pi x^2}{4} \right) = \frac{\pi}{2} \tau_H^{-2} \epsilon^3 \exp \left( -\frac{\pi \epsilon^2}{4} \right), \quad \epsilon < \left( \tau_H^{-1} \Delta \right)^{1/2}
\]

with the case \( \epsilon > \left( \tau_H^{-1} \Delta \right)^{1/2} \) (or, equivalently, \( x < \left( \tau_H^{-1} / \Delta \right)^{1/2} \)) addressed below.

A comparison should be made between (13) and the perturbative result (6). First, assuming for the latter that \( T \sim \Delta \) (the limit of applicability of the perturbative approximation), we find that the corresponding energy fluctuation is

\[
\bar{e}_{pert}^2 \sim \tau_H^{-2} \ln \left( \frac{E_c}{\Delta} \right)
\]

The difference in log factors is because (13) is obtained in a two-level approximation, whereas (19) takes into account contribution of levels within \( E_c \) of the Fermi level; otherwise, the results are in qualitative agreement. (The mean values \( \bar{e}_{vV} \) are also in agreement (6)). Although the higher moments had not been evaluated perturbatively, it is expected that \( n \sqrt{\epsilon_{pert}^2} \sim \tau_H^{-1} \) and are smaller than obtained in a two level picture. This should be anticipated since the two-level response is very sensitive to the variations of the energy level spacing at the Fermi level.

For \( x < \left( \tau_H^{-1} / \Delta \right)^{1/2} \) the perturbation theory evaluation of \( \epsilon_{vV} \) is no longer valid. This applies to the fraction \( \sim \tau_H^{-1} / \Delta \) of all systems with sufficiently small Fermi level spacing. There are two possible approaches to this case that yield, essentially, the same result. In the first approach, the Fermi level and the first unoccupied state can be
viewed as effectively, a doubly degenerate state with respect to the magnetic field perturbation. Using the secular equation \( \text{[10]} \), we find that the magnetic field will split the levels in the amount

\[
\sqrt{|\hat{M}_f|^2} H^2 \sim (\tau_H^{-1} \Delta)^{1/2}
\]  
(20)

(The second order term is still given by eq. \( \text{[7]} \) but with the final state \( f \) being the next unoccupied state at a distance greater than \((\tau_H^{-1} \Delta)^{1/2}\) from the Fermi level.). Alternatively, one can argue that such systems are effectively in the GUE regime. The perturbative expression for the mean value is

\[
\langle \tilde{G}_{\text{pert}} (\text{GUE}) \rangle \sim (\tau_H^{-1} \Delta)^{1/2}
\]  
(21)

which is obtained from the second eq. \( \text{[8]} \) using the substitution \( \Delta \rightarrow (\tau_H^{-1} \Delta)^{1/2} \), where the latter is the upper bound of the level spacing in the systems that have effectively undergone the GOE \( \rightarrow \) GUE transition.

The physical interpretation of the above results may be as follows. In a particular systems, the linear response regime is determined by the Fermi level spacing in this system, as per eq. \( \text{[12]} \).

\[
M \sim \mu_B H (k_F \ell) \frac{\Delta}{\pi \Delta}
\]  
(22)

As a result of linear response, the magnetic moment approaches the value

\[
M \sim \mu_B (k_F \ell)^{1/2}
\]  
(23)

with little change for larger fields (until \( \tau_H^{-1} \) approaches \( \Delta \) - see below). The statistical distribution of the response implies the need to consider the same range of fields for all systems. The mean magnetic moment is given by

\[
\langle M \rangle \sim \frac{\mu_B^2 H (k_F \ell)}{\Delta} \frac{\tau_H^{-1} \Delta}{\mu_B (k_F \ell)^{1/2}}
\]  
(24)

The slowly decaying distribution function

\[
P (\epsilon) \propto \frac{\tau_H^{-2}}{\epsilon^3}, \quad \tau_H^{-1} < \epsilon < (\tau_H^{-1} \Delta)^{1/2}
\]  
(25)

is due to systems with small Fermi level spacing, where the magnetic moment may become as large as \( \text{[23]} \).

At \( \tau_H^{-1} > \Delta \), the single-level ansatz is no longer valid and the results of Ref. \( \text{[1]} \) (as described in the Introduction) should be applied instead; at \( \tau_H^{-1} \approx \Delta \) the latter are consistent with the present results.

III. GSE

It was argued \( \text{[1]} \), \( \text{[6]} \) that in a GSE the electrons at the Fermi level have a magnetic moment

\[
M \sim \mu_B (k_F \ell)^{1/2}
\]  
(26)

(which is, incidentally, of the same order of magnitude as given by \( \text{[23]} \)) induced by the spin-orbit interactions. For a system with even number of electrons, the magnetic moments of the two Fermi-level electrons cancel each other. A system with an odd number of electrons, on the other hand, should have a permanent magnetic moment \( \text{[6]} \) and a Curie-like susceptibility. This is in incomplete analogy with the purely spin magnetism \( \text{[2]} \), the sole difference being that \( M = \mu_B \) in the latter case. Consequently, we refer to Ref. \( \text{[6]} \) for the discussion of the distribution function of the susceptibility obtained in the two- and three-level approximation for the even- and odd-electron systems respectively; in both cases, one finds long tails due to thermal activation of the magnetic moments.

It should be noted that the single-electron magnetic moment \( \text{[26]} \) can be attributed to the large electron \( g \)-factor, \( g \sim (k_F \ell)^{1/2} \), induced by the spin-orbit interaction \( \text{[6]} \), \( \text{[3]} \), \( \text{[4]} \). Furthermore, it was argued \( \text{[3]} \), \( \text{[4]} \) that the latter can fluctuate by as much as the order of magnitude. We point out, however, that in the temperature regime \( T \ll \Delta \) considered here, the thermal activation effects should be dominant as they may produce much larger fluctuations.
IV. DISCUSSION

The two-level picture considered here predicts the large fluctuations of the orbital magnetic response due to the possibility of small level separations at the Fermi level. Obviously, the possible observation of these effects imposes restrictions on the temperature and the magnetic field. Moreover, such restrictions are sample specific since the magnetic energy and the temperature must be compared with a particular Fermi level separation of the nearest states. (See, for instance, eqs. (12) and (13)). On the other hand, a statistical description of the fluctuations implies a common (range of) magnetic field and temperature for all systems. Consequently, while in a given system one can observe the linear response regime due to the vV energy (7) by sufficiently reducing the magnetic field, the Fermi-level spacing in this system may be smaller than the magnetic energy corresponding to the field at which all systems are analyzed. The latter is critical for understanding of the effective cut-offs in the analysis of the distribution function.

The issues that remain to be understood are the fluctuations of the matrix elements $|\bar{M}_{if}|^2$ and the details of the cancellation between the van Vleck paramagnetism and the precession diamagnetism over the Fermi sea. These will require further, largely numerical, studies. Also, our formalism should be applicable, with minor changes, to orbital magnetism of integrable systems, such as a rectangle with incommensurate sides [15]. We hope to address this problem in a future work.

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