Scaling in quantum chaos: A study of quantum dots

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We study the level statistics of the quantum spectrum of a harmonic oscillator with biquadratic corrections. The distribution of interlevel spacings along the lines of constant scaled energy, whose values determine the classical dynamics, is obtained numerically. A transition from the Poisson to the Wigner character of the distribution is observed with the increase of the scaled energy. Using a quasiclassical approximation and assuming ergodic chaos we derive the level correlation function along the lines of constant scaled energy by employing the trace formula and keeping only the diagonal term. Our calculation yields a smoothed out tail of a Gaussian ensemble and is based on the exact cancellation of two ratios: the ratio of the mean level density along the line of constant scaled energy to that at a fixed biquadratic coupling and the ratio of the action to the period of a long periodic orbit. We further argue that the Wigner distribution will be found along any smooth line crossing the spectrum in the region of classical chaos. Applicability of our findings to quantum-dot structures at semiconductor interfaces is discussed.

I. INTRODUCTION

This work is motivated by recent experiments on zero-dimensional structures—quantum dots. They are obtained in semiconductor heterojunctions by lateral confinement of electrons in the two-dimensional (2D) electron gas by a “lattice” of finite-size gates. Despite the square shape of the gates, the confining potential in the region of the 2D electron gas under the gate is nearly parabolic due to electrostatic smoothing, as confirmed by experiment. Properties of parabolic quantum dots have been extensively discussed by theorists. These investigations were largely inspired by their nature as a mesoscopic solid-state object with near perfect symmetry (cylindrical), akin to atoms. In fact, this symmetry forms the centerpiece around which all previous theories were built.

It is clear, however, that the parabolic confinement is only an approximation whose validity is borne out by several factors, the most important being the distance of the electron layer from the gated region. Furthermore, the experiments have probed only the low-lying states while it is likely that higher up the biquadratic terms should become more prominent in defining the spectrum. Therefore, we are interested in the spectrum of the Hamiltonian

\[ H = \frac{1}{2m}(p_x^2+p_y^2)+\frac{m\omega^2}{2}(x^2+y^2)+bx^2y^2+\frac{c}{4}(x^2+y^2)^2, \]

which had previously been studied with respect to level statistics both for purely conceptual reasons and as pertaining to the phenomenological nuclear models. Here, we extend this analysis in two directions. First, we analyze the distribution of interlevel spacings along the lines of constant scaled energy whose values determine the classical dynamics. Second, we offer a semiclassical derivation of level correlations along such lines in the spirit of Hannay and Ozorio de Almeida and Berry. It revolves around following the variations of scaled \( \tilde{H} \) at a fixed scaled energy. Finally, in the Appendix, we briefly address the applicability of this model to a solid-state object, a quantum dot.

In what follows, we shall concentrate on a simplified version of Eq. (1) by setting \( c = 0 \). Clearly, even for a small \( b \), the biquadratic term will be comparable to the parabolic potential sufficiently far from the center or, equivalently, for a sufficiently high energy. In this sense the model is expressly nonperturbative, leading to chaotic classical dynamics and Wigner-Dyson statistics of the quantum spectra. This statement can be formalized by scaling the momenta and lengths in a fashion that eliminates all parametric dependencies in Eq. (1), as discussed below in more detail. The transition to chaos has been demonstrated by Pullen and Edmonds and Bolotin et al. The latter have also formulated the criterion for transition to irregular behavior. It states that the critical energy can be approximated by the minimum of the potential energy on the line of zero Gaussian curvature of the potential surface and yields the critical energies close to the values obtained in numerical simulations.

It should be noted that our problem is similar to the problem of hydrogenic states in magnetic field near the ionization threshold, which had been extensively researched in the past. Since the latter can be mapped to an equivalent problem of a 2D harmonic oscillator with anharmonic corrections (sixth-order terms), the critical energy can be also found from the above criterion and yields a value close to the numerical result. A slight modification of the argument is required, compared to our case, because the (regularized) Hamiltonian found in the mapping procedure is fixed at two, while the frequency of the oscillator is not fixed and is related to the "true" energy.

In Refs. 5 and 6 the quantum spectrum of the Hamiltonian (1) was examined at the fixed values of \( b \). It was demonstrated that for sufficiently high energies the distri-
bution of interlevel spacings is well fitted by the Wigner distribution. However, for the purpose of describing the transition to Wigner-Dyson (WD) statistics, one should consider level statistics along the lines of constant scaled energies because the latter determine the classical behavior of the system. Research in this direction has been undertaken by Wintgen and Friedrich for the hydrogenic states in a magnetic field. Here, we address a similar question for the Hamiltonian (1).

Despite the realization that the level statistics has to be investigated along the lines of constant scaled energy, no attempt has been made to explain why the statistics along these lines should become WD in the classically chaotic region of the spectrum. On the other hand, it has been understood that the variation along these lines should be described in terms of scaled $\tilde{h}$. Indeed, scaling of the momenta and coordinates introduces a scaled version of $\tilde{h}$ as a function (here) of $b$, $\tilde{h}(b)$. Considering the intersections of lines $E(b)$ with the line of a constant scaled energy $\tilde{E}$, one finds in a quasiclassical approximation that the number of such intersections scales as $N(\tilde{E}) \sim \tilde{h}^{-1}$. Consequently, the interlevel separation should be measured in terms of $\tilde{h}$. Using $\tilde{h}^{-1}$ as such a measure also achieves automatic folding of the spectrum along the line of constant scaled energy, that is, a constant level density. Folding of the spectrum was roughly approximated in Refs. 5 and 6 by considering narrow energy intervals.

In view of the above, the quasiclassical derivation of the level correlation function, based on the trace formula and the diagonal approximation, must be modified to account for the variations of $\tilde{h}$ rather than energy, along the lines of constant scaled energy. This is accomplished by reworking the formalism of Refs. 7 and 8 in terms of the scaled variables. Physically, it means fixing the position in the energy spectrum and observing the levels as they move through that position as $\tilde{h}$ is changed. This picture is consistent with the notion that in this class of problems the relative magnitude of $\tilde{h}$ is uniquely defined by all the other parameters of the system. In Sec. II we provide such a derivation and also argue that the Wigner distribution will be found along any smooth line crossing the spectrum in the region of classical chaos. In Sec. III we present the results of the numerical evaluation of the distribution function of interlevel spacings along the lines of constant scaled energy. Towards that end, we performed matrix diagonalization using the harmonic-oscillator basis on a finite grid and verified its accuracy against the spectral method, which is independent of the choice of a specific basis. For sample runs, both yield close energy eigenvalues. The approach of Ref. 10 is briefly discussed in connection with this evaluation.

II. THEORY

We begin with the simplified version of the Hamiltonian (1),

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{m \omega^2}{2} (x^2 + y^2) + bx^2 y^2,$$

and rescale the coordinates and momenta as

$$(x, y) = (x, y) l, \quad (p_x, p_y) = (p_x, p_y) ml,$$

where

$$l = \sqrt{m \omega^2 / b}$$

is the distance where the harmonic and biquadratic terms become comparable. Introducing the quantum parameter

$$b_0 = \frac{m^{\frac{3}{2}} \omega^3}{\tilde{h}},$$

we obtain a scaled, parameter-independent version of the Hamiltonian (1') as follows:

$$\tilde{H} = \tilde{h} \omega b_0 \left( \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + \frac{1}{2} [x^2 + y^2] + x^2 y^2 \right) = \tilde{h} \omega b_0 \tilde{H}.$$  

The value of the scaled energy $\tilde{E}$ determines the classical behavior of the system described by the Hamiltonian $\tilde{H}$, the transition to irregular (chaotic) dynamics taking place at around $\tilde{E} = \frac{1}{2}$. This well-established result is found on the basis of numerical investigation of the Poincaré surface of section and the analytical argument of Bolotin et al., which relates the change of sign of the Lyapunov exponents to the line of zero Gaussian curvature of the potential-energy surface (the scaled variant is obtained from Ref. 6 by setting $b = 1$). In the remainder of this work we will investigate level correlations in the quantum spectrum of the Hamiltonian (1') along the lines of constant scaled energy $\tilde{E}$. In other words, we study the correlations of the intersection points of the quantum levels $E(b)$ with the lines $\tilde{E} = \text{const}$.

Clearly, scaling in Eq. (2) introduces a scaled version of the Planck constant,

$$\tilde{h} = \frac{b}{b_0} \ll 1,$$

where the latter inequality points away from the extreme quantum limit. Consequently, the lines of constant scaled energy are given by

$$E = \tilde{h} \omega b_0 \tilde{E} = \tilde{h} \omega \tilde{E} \tilde{h}^{-1},$$

in accordance with Eq. (5), so that tracing such lines amounts to the change of $\tilde{h}^{-1}$. In what follows we shall assume a quasiclassical approximation and describe the quantum energy spectrum in statistical terms. In this limit, the mean number of energy levels below a given energy is given by

$$\langle N(E, b) \rangle = \frac{1}{(2\pi \tilde{h})^2} \int (d'p)(d'q) \theta(H - E)$$

$$= \frac{1}{(2\pi \tilde{h})^2} \int (d'p)(d'q) \theta(\tilde{H} - \tilde{E})$$

$$= \langle N(\tilde{E}, \tilde{h}) \rangle,$$

where the second equation is a corollary of Eqs. (2)–(6). Applying the integral in the right-hand side to the Hamiltonian in Eq. (5), we find that
\[ \langle N \rangle = 3 \tilde{N}^{-2} E^2 F(\tfrac{1}{2}, \tfrac{1}{2}, 3, -4E), \] (9)

where \( F \) is the hypergeometric function. On a line of constant scaled energy, \( \tilde{E} = \text{const} \), for instance, Eq. (9) yields \( \langle N \rangle \propto (E/\hbar \omega)^3 \), while for a fixed \( b \), or equivalently \( \tilde{N} \), \( \langle N \rangle \) has an additional energy dependence as the last argument of the hypergeometric function. A natural definition of the scaled density of levels would be

\[ \tilde{d}_h = \frac{\partial N}{\partial \tilde{E}}, \] (10)

which, as can be easily understood from Eq. (7), relates to the usual definition of the level density as follows:

\[ \frac{1}{\hbar \omega} \tilde{d}_h = d \langle E, b \rangle = \frac{\partial N(E, b)}{\partial E}. \] (11)

For the mean level density the latter reduces to

\[ \langle d \langle E, b \rangle \rangle = \frac{1}{(2\pi \hbar)} \int \langle d^f \rangle (d^f \{ \theta (H - E) \rangle, \] (12)

while

\[ \langle \tilde{d}_h \rangle = \frac{1}{(2\pi \hbar)} \int \langle d^f \rangle (d^f \{ \theta (\tilde{H} - \tilde{E}) \rangle, \] (12')

which follows from Eqs. (8), (10), and (11).

In classical dynamics the value of scaled energy determines whether the behavior of the system will be regular or chaotic.\textsuperscript{5,6} This classical behavior is reflected in the structure of the quantum spectrum.\textsuperscript{11} Therefore, it is desirable to investigate the level statistics along the lines of constant scaled energy.\textsuperscript{10} This, however, requires a definition of the density of levels, which reflects that the latter is a function of \( \tilde{h} \) on such lines. The definition employed here reads as

\[ \tilde{d}_h = \frac{\partial N}{\partial \tilde{N}}. \] (10')

This particular definition is convenient because the resulting mean level density,

\[ \langle \tilde{d}_h \rangle = \frac{\langle N \rangle}{\langle \tilde{N} \rangle} = \frac{\langle N \rangle}{\langle \tilde{N} \rangle} = \frac{1}{(2\pi \hbar)} \int \langle d^f \rangle (d^f \{ \theta (\tilde{H} - \tilde{E}) \rangle, \] (13)

is constant along an \( \tilde{E} = \text{const} \) line, which automatically attains the folding of the spectrum. Consequently, in a numerical analysis of the level distribution one need not worry about the systematic dependence of the level density on energy.\textsuperscript{11}

We now turn to the analysis of the fluctuations of the level density around its mean. This is described by the contribution of the periodic orbits expressed in terms of the scaled variables,\textsuperscript{11}

\[ \tilde{d}_h^{\text{scat}} = \frac{1}{\hbar} \sum_{\text{periodic orbits}} \tilde{A}_i \exp \left( \frac{i S_i}{\hbar} \right), \] (14)

where \( \tilde{A}_i(\tilde{E}) \) and \( S_i(\tilde{E}) \) are the amplitude and the action of the periodic orbit \( i \), respectively. The central question here is the relationship between the periodic-orbit contributions to the densities (10) and (10'). Without offering a rigorous proof, we argue that for the evaluation of the correlation function in the ergodic approximation below, it is sufficient to assume that the periodic-orbit contribution (and hence the total level density) along the lines of constant scaled energy are rescaled to follow the rescaling of the respective mean level densities, as prescribed by Eqs. (12') and (13),

\[ \langle \tilde{d}_h \rangle = \frac{1}{\hbar} \frac{\langle N \rangle}{\langle \tilde{N} \rangle} \frac{\partial \langle N \rangle}{\partial \tilde{E}}, \] (15)

so that

\[ \tilde{d}_h^{\text{scat}} = \frac{1}{\hbar} \frac{\langle N \rangle}{\langle \tilde{N} \rangle} \frac{\partial \langle N \rangle}{\partial \tilde{E}} \frac{\partial \langle N \rangle}{\partial \tilde{E}}. \] (16)

It must be emphasized that the latter equation is true only in some generic average sense that will be further expanded on below. Combining Eqs. (14) and (15), we obtain the following expression for the periodic-orbit contribution to the level density along the lines of constant scaled energy;

\[ \tilde{d}_h^{\text{scat}} = \frac{\langle N \rangle}{\langle \tilde{N} \rangle} \frac{\partial \langle N \rangle}{\partial \tilde{E}} \frac{\partial \langle N \rangle}{\partial \tilde{E}} \sum_i \tilde{A}_i \exp \left( \frac{i S_i}{\hbar} \right). \] (16)

Using Eq. (16), we find that the correlation function of the fluctuations of the level density around its mean is given by

\[ \langle \tilde{d}_h^{\text{scat}}(\tilde{h} - f) + \Delta \tilde{h} - f \rangle \tilde{d}_h^{\text{scat}}(\tilde{h} - f) \rangle \] (17)

In deriving Eq. (17) we have made a diagonal approximation,\textsuperscript{8} which accounts only for the interference between the time-reversed paths along the same periodic orbit. It is known that this approximation is equivalent to the perturbative expansions.\textsuperscript{13,14} Scaling the action and the amplitude as

\[ (\tilde{A}, \tilde{S}) \rightarrow (\tilde{A}, \tilde{S}) \frac{1}{\hbar} \frac{\hbar}{\hbar} \frac{f(\tilde{h} - f)^{1/2}}{f(\tilde{h} - f)^{1/2}}, \] (18)

we find, following Ref. 7, the following expression:

\[ \langle \tilde{d}_h^{\text{scat}}(\tilde{h} - f + \Delta \tilde{h} - f) \rangle \tilde{d}_h^{\text{scat}}(\tilde{h} - f) \rangle \] (19)

Using now the Hannay–Orozio de Almeida sum rule,\textsuperscript{7}

\[ \sum_i \tilde{A}_i \theta(|S_i| - |\tilde{S}_i|) = \frac{1}{\tilde{S}^2} \left( \frac{\partial \langle N \rangle}{\partial \tilde{E}} \right)^2, \] (20)
which follows from the assumption of ergodic chaos and the ensuing relationship between the action and the period of long periodic orbits,

\[ \frac{\mathcal{S}}{T} = f \left( \frac{\langle N \rangle}{\delta(N)} \right), \]

we finally derive

\[ \langle dE^{\text{car}}(\vec{\hbar}^{-f} + \Delta \vec{\hbar}^{-f}) dE^{\text{car}}(\vec{\hbar}^{-f}) \rangle = \frac{1}{2\pi^2} \frac{1}{(\Delta \vec{\hbar}^{-f})^2}, \]

a smoothed out tail of the level correlation function of a Gaussian ensemble, consistent with the WD level statistics. The novelty of Eq. (22) relative to previous derivations is that it describes the correlations along the lines of constant scaled energy.\(^7\)\(^8\)

Notice that Eq. (21) is rather “featureless” in that it contains no information on the discreteness of the spectrum or even on the mean interlevel spacing. The latter dependence can be evaluated in the diffusive models using a nonperturbative technique\(^5\)\(^6\) based on the nonlinear \(\sigma\) model. At this time, however, there exists no transparent passage from the results of Ref. 15 to the derivation founded on the trace formula (13). Clearly, the diagonal approximation is a major culprit and the interference between different long periodic orbits must be accounted for. Otherwise, Eq. (22) is based to a large degree on the assumption of uniform coverage of the energy surface by long periodic orbits in the case of ergodic chaos. But the same assumption also leads to Eq. (8), which helps appreciate the lack of features in Eq. (22). A good account of this problem can be found in Ref. 8 on p. 274. It can only be resolved by adopting a nonperturbative approach in \(\vec{\hbar}\).

The latter will not, of course, affect the use of quasiclassical approximation—taking classical actions along periodic orbits—which is perturbative in \(\vec{\hbar}\). Physically it means that although one could not account for low-lying quantum states or the exact positions of the levels, an exact statistical description is possible in certain parts of the spectrum based on the classical dynamics of the system. Notice also that Eq. (21) was obtained\(^1\) by substituting the time average along the periodic orbits by a phase-space average, justified by the ergodic coverage of the energy surface by such orbits. Such an approximation leads to a “generic” result, which does not discern between different orbits. Since the periodic-orbit contribution describes the deviations from the mean density, the surmise of Eq. (15) can be, accordingly, justified in the same average generic sense.

Based on the above considerations it is now straightforward to realize why the same tail of a Gaussian ensemble is found in the evaluation of the correlations at a given value of \(\hbar\) as well.\(^9\) Moreover, a simple generalization is possible for a cut of the spectrum by any smooth line \(E(\vec{\hbar})\). Indeed, defining for simplicity the interlevel separation along this line in terms of \(\hbar^{-f}\) (it can be done because the correlation function obtained in this approximation is insensitive to the folding of the spectrum, as is seen from the derivation above), we find that the factor relating the level density along this line to the level density at a constant \(\hbar\),

\[ dN = \frac{\partial N}{\partial E} \left| \frac{dE}{\hbar^{-f}} + \frac{\partial N}{\partial \hbar^{-1}} \right| \frac{d\hbar^{-1}}{E}, \]

(23)

cancels out the factor in the power of the exponent relating it to the period of a long orbit,

\[ e^{i(\Delta E^f + S \Delta \hbar^{-1})} = e^{i(T \Delta E^f + S \Delta \hbar^{-1})} \]

\[ = \exp \left[ iT \hbar^{-1} \left| \frac{dE}{d\hbar^{-1}} + \frac{1}{\hbar^{-1}} \frac{\partial \langle N \rangle}{\partial E} \right| \Delta \hbar^{-1} \right]. \]

(24)

As a result we once again arrive at Eq. (22), where \(f = 1\) in this particular case.

As a final remark we would like to point out that Eqs. (12)' and (14) are derived by starting with the quantum-mechanical expression for the level density,

\[ d(E) = \sum_n \delta(E - E_n), \]

with subsequent use of the semiclassical approximation for the Green's function.\(^5\) On the other hand, a first-principle derivation of the periodic-orbit contribution to the level density along the lines of constant scaled energy would be based on the formula

\[ d(\hbar) = \frac{\partial N}{\partial \hbar^{-f}} = \frac{\partial}{\partial \hbar^{-1}} \sum_n \delta(\bar{E} - E_n), \]

(25')

where \(\bar{E} = \text{const}\), and \(E_n\) are the eigenenergies of the Hamiltonian \(\hat{H}\) in Eq. (5) sensitive to the change of the scaled Planck constant \(\vec{\hbar}\). Unfortunately, it is not presently clear how to implement this approach and we were unable to make tangible progress in this direction.

III. NUMERICAL SIMULATIONS

Our numerical procedure involves the diagonalization of the Hamiltonian \((\hbar')\) on the grid in the \((E,b)\) plane. This enabled us to find the intersections of the energy levels with the lines of constant scaled energy, \(\bar{E} = \frac{5}{4}, \frac{3}{4}, \frac{7}{4}, \frac{9}{4}\), and \(\frac{11}{4}\). We have identified 1141, 986, 827, 679, and 533 such intersections, respectively, inside the area defined by \(0 \leq E \leq 149\) and \(0 \leq b \leq 0.03\), where \(E\) and \(b\) are expressed in the units of \(\hbar\omega\) and \(b_0\), respectively. We used 500 \(b\) values, with a step of \(10^{-4}\). Our findings are summarized in Figs. 1 and 2 and Table I. Figure 1 shows the dependence \(\ln(\langle N \rangle)\) vs \(\ln(E)\) along the lines of constant scaled energy. Notice that according to Eq. (7), \(E\) and \(\vec{\hbar}^{-1}\) are interchangeable along such lines since \(E = \bar{E} \vec{\hbar}^{-1}\).
FIG. 1. $\ln(N)$ vs $\ln(E^2/\hbar^2)$ along the lines of constant scaled energy (a) $E = \frac{1}{2}$, (b) $E = \frac{5}{4}$, (c) $E = \frac{7}{4}$, (d) $E = \frac{9}{4}$, and (e) $E = \frac{11}{4}$. 
semiclassical analysis of the preceding section implies a straight line with the slope 2 and an offset found from Eq. (9). Notice that due to the $C_{4v}$ symmetry of the Hamiltonian, we evaluate the eigenvalues in only one of the subspaces and, consequently, the coefficient in Eq. (9) must be changed to $1/16$ (see next paragraph). Table I provides a cumulative view of the fitting results vs. that expected from theory. In Fig. 2, we plot the histograms of the nearest-neighbor interlevel spacings. They are fitted by the Brody distribution and the resulting $q's$ are listed. Clearly, a transition from the Poisson to Wigner distribution, the limiting distributions of the Brody distribution with $q = 0$ and $q = 1$, respectively, is taking place.

Based on the symmetry of the Hamiltonian, the wave function can be divided into four subgroups according to the eigenvalues of the rotation operator: $f(r)e^{i(4L + n)\varphi}$, where $L = 0, \pm 1, \pm 2, \ldots$ and $n = 1, 2, 3, 4$. Considering the reflection operator, the $L = 0$ subgroup can be further separated into $f(r)\sin(4L\varphi)$ and $f(r)\cos(4L\varphi)$. We used the latter in our simulations. Notice, however, that in evaluating $\langle N \rangle$ we assigned $1/2$, instead of 1, to each intersection with $L = 0$ since there is no corresponding sine contribution.

It is clear from Fig. (1) and Table I that our numerical procedure is quite accurate. We used a harmonic-oscillator basis set and retained only about a third of the lowest eigenvalues. For some $b's$ and ranges of energy we have verified our evaluation vs. the spectral method.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$E$ & \multicolumn{2}{c|}{Slope} & \multicolumn{2}{c|}{Intercept} \\
& Expected & Best fitted & Expected & Best fitted \\
\hline
$\frac{3}{4}$ & 2.000 & 2.007 & -2.934 & -3.005 \\
$\frac{5}{4}$ & 2.000 & 2.041 & -3.000 & -3.269 \\
$\frac{7}{4}$ & 2.000 & 2.018 & -3.052 & -3.242 \\
$\frac{9}{4}$ & 2.000 & 2.025 & -3.095 & -3.352 \\
$\frac{11}{4}$ & 2.000 & 2.019 & -3.132 & -3.388 \\
\hline
\end{tabular}
\caption{Comparison of the best-fitted slope and intercept in Fig. 1 vs analytical results given by Eq. (9) (see text).}
\end{table}
FIG. 2. Histogram: the distribution function of the nearest-levels spacings. Solid line: the best-fitted Brody distribution. Short-dashed and long-dashed lines: Poisson and Wigner distributions, respectively. (a)–(e) correspond to the same scaled energies as in Fig. 1. Parameter $q$ varies as follows: (a) $q = 0.22$, (b) $q = 0.29$, (c) $q = 0.45$, (d) $q = 0.59$, and (e) $q = 0.61$; $q = 0$ corresponds to the Poisson distribution and $q = 1$ to the Wigner distribution.
which is based on the solution of a time-dependent Schrödinger equation and, as such, does not involve a choice of a particular basis. We found an excellent match with our results based on matrix diagonalization. In this regard, we would like to point out that there exists another procedure, which proposes to evaluate the eigenvalues directly on the lines of constant scaled energy. We contend that scaling of variables used in that approach strips the harmonic-oscillator set of its special role, because the operator whose eigenvalues are to be determined no longer has semblance to the harmonic-oscillator Hamiltonian. Therefore, an optimization of the basis set is called for, and, to the best of our understanding, leads to a nonorthogonal set. Diagonalization on a grid, used in this work, is a straightforward technique which we found to be totally satisfactory as we were able to trace all the energy levels within the boundaries specified in the previous paragraph.


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APPENDIX

This appendix addresses the potential in the quantum-dot structure. We assume that the gate is characterized by the gridlike potential, so that the Fourier representa-
tion is called for;

$$V(x, y, 0) = \sum_{k_1, k_2} V_{k_1, k_2} e^{i(k_1 x + k_2 y)}. \quad (A1)$$

Using the Poisson equation, we find the potential in the region underneath the gate,

$$V(x, y, z) = \sum_{k_1, k_2} V_{k_1, k_2} e^{-i(k_1 x + k_2 y)} e^{i(k_1 x + k_2 y)}. \quad (A2)$$

Clearly, specifics of the gate shape enter only through $V(x, y, 0)$. Consider, for instance, the circumstance of the square-shaped gates,

$$V(x, y, 0) = -V_0 N(x) N(y), \quad (A3)$$

where

$$N(x) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \left[ \theta \left( x + \frac{a}{2} + 2na \right) + \theta \left( x - \frac{a}{2} + 2na \right) \right]. \quad (A3')$$

Then, with the help of the identity

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imx} = \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m), \quad (A4)$$

we immediately obtain

$$N(x) = \frac{1}{2} + \sum_{l=-\infty}^{\infty} n_l e^{ilx}, \quad (A5)$$

where $n_l = \sin(l \pi r/2)/l \pi r$ and $x = x/a$.

Combining now the above equations and introducing the notation $\xi = \exp(-\xi)$, we find the following expression for the potential,

$$\frac{V(x, y, z)}{V_0} = -\frac{1}{4} - \sum_{l=1}^{\infty} \xi^n n_l \left[ \cos(l \tilde{x}) + \cos(l \tilde{y}) \right] - 4 \sum_{l_1, l_2=1}^{\infty} \xi \sqrt{l_1^2 + l_2^2} n_{l_1} n_{l_2} \cos(l_1 \tilde{x}) \cos(l_2 \tilde{y}). \quad (A6)$$

Obviously, the term $\propto \xi \quad [the \ l \ = \ 1 \ term \ of \ the \ first \ sum \ on \ the \ right \ hand \ side \ of \ Eq. \ (A6)] \ gives \ the \ largest \ contribution \ to \ the \ potential \ sufficiently \ far \ beneath \ the \ gate \ region. \ This \ means, \ however, \ that \ the \ variables \ effectively \ separate \ in \ the \ potential \ making \ it \ integrable \ (being \ a \ sum \ of \ two \ one-dimensional \ potentials). \ Notice \ that \ by \ expanding \ the \ cosine \ in \ this \ case, \ one \ obtains \ the \ potential \ given \ by \ Eq. \ (1) \ with \ b = -c/2. \ For \ the \ latter, \ the \ motion \ is \ known \ to \ be \ integrable \ when \ c \geq b \geq -c/2, \ but \ can \ be \ chaotic \ outside \ this \ domain \ of \ the \ parameter \ space \ for \ a \ sufficiently \ large \ energy. \ Notice \ that \ the \ line \ b = -c/2 \ maps \ to \ the \ line \ of \ the \ rotation \ of \ the \ coordinates \ by 45\degree.$$

Based on this analysis, one could hope to find the Wigner-Dyson spectrum only in the circumstance when the region of the electron gas is not too deep below the gate region. On the other hand, it cannot be too close to the gate either as the square hard-wall potential renders motion integrable as well. According to a recent review, experimenters are nearing the ability to control this distance. It should also be pointed out that our derivation ignores the interactions between the electrons in the dot. In that regard we would like to mention a numerical calculation which self-consistently takes into account the potential of the gate and the electrons in the dot. One should also consider the possibility that the potential of Eq. (2) with the parameters allowing for chaotic spectra, can be produced by controlling the shape of the gate. At any rate, we wish to emphasize that semiconductor quantum-dot structures appear to be a rather promising candidate for the study of classical-to-quantum correspondence in solid states in the systems other than billiards (hard-wall potentials).

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1For a review see D. Heitmann and J. P. Kotthaus, Phys. Today 46 (6), 56 (1993), and references therein.
11See, e.g., M. C. Gutzwiller, Chaos in Classical and Quantum


17D. Wintgen (private communication).
FIG. 2. Histogram: the distribution function of the nearest-levels spacings. Solid line: the best-fitted Brody distribution. Short-dashed and long-dashed lines: Poisson and Wigner distributions, respectively. (a)–(c) correspond to the same scaled energies as in Fig. 1. Parameter $q$ varies as follows: (a) $q = 0.22$, (b) $q = 0.29$, (c) $q = 0.45$, (d) $q = 0.59$, and (e) $q = 0.61$; $q = 0$ corresponds to the Poisson distribution and $q = 1$ to the Wigner distribution.
FIG. 2. (Continued).