LEVEL REPULSION IN INTEGRABLE SYSTEMS

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We provide evidence that level repulsion in semiclassical spectrum is not just a feature of classically chaotic systems, but classically integrable systems as well. While in chaotic systems level repulsion develops on a scale of the mean level spacing, regardless of the location in the spectrum, in integrable systems it develops on a much longer scale — such as geometric mean of the mean level spacing and the running energy in the spectrum for hard wall billiards. We show that at this scale level correlations in integrable systems have a universal dependence on the level separation, as well as discuss their exact form at any scale. These correlations have dramatic consequences, including deviations from the Poissonian statistics in the nearest level spacing distribution and persistent oscillations of the level number variance over an energy interval as a function of the interval width. We illustrate our findings on two specific models — rectangular infinite well and a modified Kepler problem — that serve as generic types of a hard wall billiard and a potential problem without extra symmetries. Our theory and numerical work are based on the concept of parametric averaging that allows sampling of a statistical ensemble of integrable systems at a given spectral location (running energy).

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1. Introduction

Properties of semiclassical spectra have been subject of intense scrutiny over the past half-century. The two cases of classically integrable systems without extra symmetries (degeneracy) and classically chaotic, ergodic systems—received the most attention.1,2 Chaotic systems are characterized by the Wigner–Dyson statistics of nearest level spacing and a weak logarithmic dependence of the level rigidity and level number variance on the width of the interval. Integrable systems were believed to exhibit Poissonian statistics of level spacing and saturation of the level rigidity and level number variance.3–7

Difficulty with the chaotic systems lies mainly in the fact that it is challenging to establish the relationship between the properties of the spectrum and underlying classical dynamics. Periodic orbit theory only recently developed some traction8
outside the perturbative regime. Exact description of spectral correlations was due to the random matrix theory (RMT) and the non-linear sigma model, which produced identical results. The former assumes averaging over the spectrum while the latter is based on the Anderson model of electron diffusion, believed to be equivalent to a generic chaotic, ergodic motion. Furthermore, the latter provides an alternative approach to ensemble averaging — at a given energy in the spectrum but for various realizations of disorder (disorder averaging).

Conversely, we argue that spectral properties of some integrable systems are amenable to both semiclassical (based on classical dynamics and the periodic orbit theory) and direct quantum-mechanical derivations. Such derivations provide proof of nearest level correlations. We argue that a repulsive term in the level correlation function leads to a small but distinctive deviation from the Poissonian spacing statistics. This is naturally related to the long established (numerically and analytically) fact that, following the initial linear increase with the interval width, the spectral rigidity and the level number variance achieved “saturation” for widths beyond a value proportional to the geometric mean of the mean level spacing and the center of the interval (the running position in the spectrum).

Such “saturation,” as already recognized in Ref. 3, clearly contradicts the Poissonian absence of correlations — as the latter would imply an uninterrupted linear growth of those two quantities. (Uninterrupted linear growth of spectral rigidity and level number variance, as well as the Poissonian statistics for the nearest level spacing, are recovered in the “semiclassical limit” of infinite energy; this is in contrast to classically chaotic systems where the repulsive term, level spacing distribution, spectral rigidity and level number variance do not depend on the running energy.)

While both the second order statistics and the form factor received much attention for rectangular billiards, the existence of a repulsive term in the level correlation function of integrable systems and its relationship to the fact that both the spectral rigidity and the level number variance exhibit complex oscillatory behaviors around their respective “saturation” values was first established in our previous works.

Our ability to observe the described effects numerically is based on the procedure of statistical averaging for integrable systems, which we call “parametric averaging.” Previous works performed averaging over spectrum wherein different positions of the running energy were deemed as different systems. As already mentioned, with the advent of “disorder averaging” for chaotic systems, it became possible to perform averaging for a given value of the running energy using different realizations of disorder as statistical ensemble. “Parametric averaging” allows us to achieve the same goal for integrable systems. Examples include averaging over the

\[ a \text{In Ref. 3, averaging over “several values” of the billiard’s aspect ratio was performed for the level rigidity, but not for the nearest level spacing; parametric ensemble averaging for integrable systems is rigorously defined in Ref. 17.} \]
aspect ratio in rectangular billiards and over the strength of the $r^{-2}$ term in the modified Kepler problem.

This paper is organized as follows. In Sec. 2, we first discuss general properties of the level correlation function. We then discuss a simple ansatz, for not too large level separation, that reveals a repulsive term in the level correlation function. We proceed to derive the simplest form of the nearest level spacing distribution by taking into account the repulsive term in the level correlation function. In Sec. 3, we present the specific forms of the level correlation function for the rectangular billiard and the modified Kepler problem. In Sec. 4, we show that our numerical results for the latter two models, obtained on the basis of the concept of parametric ensemble averaging, are in excellent agreement with our prediction for the deviation from the Poissonian statistics. We conclude in Sec. 5 with the summary of our results and discussion of the “semiclassical limit.”

2. Level Correlations in Integrable Systems

In what follows, it is assumed that the spectrum is flattened, so that the mean level density $\langle \rho \rangle$ and the mean level spacing $\Delta$ are constant (and are set to unity — that is, all energies are measured in the units of $\Delta$):\footnote{For reference purposes, $\Delta = 2\pi \hbar^2 / mA$ in a rectangular billiard, where $A$ is the area of the billiard and $m$ is the mass of the particle.}

$$\Delta = \langle \rho \rangle^{-1} = 1.$$  

Deviation from the mean, $\delta \rho(\epsilon) = \rho(\epsilon) - \langle \rho \rangle$, has an obvious property that $\langle \delta \rho(\epsilon) \rangle = 0$ and its equivalent

$$\int \delta \rho(\epsilon) d\epsilon = 0,$$

where the integral is taken over the spectrum.

Spectral correlations are characterized by the two-point correlation function

$$K(\epsilon_1, \epsilon_2) = \langle \delta \rho(\epsilon_1) \delta \rho(\epsilon_2) \rangle.$$  

It immediately follows from Eq. (2) that

$$\int K(\epsilon_1, \epsilon_2) d\epsilon_1 = \int K(\epsilon_1, \epsilon_2) d\epsilon_2 = 0.$$  

In cases when the level repulsion sets in on a scale that is much smaller than $\epsilon \gg 1$ (specifically, $\sim 1$ for chaotic systems and $\sqrt{\epsilon}$ for hard wall integrable billiards), the correlation function can be written as

$$K(\epsilon_1, \epsilon_2) = K(\epsilon, \omega)$$

with

$$\epsilon = \frac{\epsilon_1 + \epsilon_2}{2}, \quad \omega = \epsilon_2 - \epsilon_1.$$
Consequently, we also have
\[ \int K(\epsilon, \omega) d\omega = 0. \] (7)

The correlation function can further be separated into two terms\(^{15}\):
\[ K(\epsilon, \omega) = \delta(\omega) - K(\epsilon, \omega), \] (8)

where the first term corresponds to uncorrelated levels and the second term, which satisfies
\[ \int K(\epsilon, \omega) d\omega = 1 \] (9)
corresponds to level repulsion.

As we pointed out, derivation of \( K(\epsilon, \omega) \) from the periodic orbit theory\(^2,4\) and classical dynamics is a challenging problem for chaotic systems. The main reason for it is a breakdown of the “diagonal approximation,” in which interference between long periodic orbits is neglected. On the other hand, from RMT\(^3\) and the non-linear model,\(^{10}\) the form of \( K(\epsilon, \omega) \) is well-known. For the unitary ensemble, for instance, it is given by
\[ K(\epsilon, \omega) = \frac{\sin^2 \omega}{\pi \omega^2} \] (10)

which, as is the case for orthogonal and symplectic ensembles, is \( \epsilon \)-independent and indicates that level repulsion develops on the scale of mean level spacing.

Our main interest is the form of \( K(\epsilon, \omega) \) for integrable systems. Unlike chaotic systems, it does not reduce to a simple closed form. On the other hand, the diagonal approximation is valid, which allows us to obtain the correlation function from the periodic orbits theory for simple classical dynamics. Furthermore, in such cases, it is also straightforward to obtain \( K(\epsilon, \omega) \) directly from the knowledge of the quantum mechanical spectrum.

In the periodic orbit theory, the Fourier transform of \( K(\epsilon, \omega) \) is given by\(^4\) (in what follows, we set \( \hbar = 1 \))\(^c\)
\[ K(\epsilon, t) = \sum_j A_j^2 \delta(t - T_j), \] (11)

where \( A_j \) and \( T_j \) are amplitudes and periods of the periodic orbits at energy \( \epsilon \) and it is known that
\[ K(\epsilon, \infty) = \frac{1}{2\pi}. \] (12)

\(^c\)To maintain consistency with Ref. 15, we use the same letter for the form factor and its Fourier transform and distinguish the two by the argument.
Level Repulsion in Integrable Systems

It is clear that \( K(\epsilon, t) = 0 \) for \( t < T_{\text{min}} \), where \( T_{\text{min}} (\sim \epsilon^{-1/2} \text{ in billiard}) \) is the period of the shortest periodic orbit.\(^d\) Based on these limiting behaviors, previously, we introduced a simplified ansatz\(^d\)

\[
K(\epsilon, t) = \begin{cases} 
  1/2\pi, & t > T_{\text{min}} \\
  0, & t < T_{\text{min}} 
\end{cases}
\]  

which encapsulates the key element of level repulsion. An alternative derivation is shown in the Appendix A. Consequently we find\(^d\)

\[
K(\epsilon, \omega) = \delta(\omega) - \frac{\sin(\omega T_{\text{min}})}{\pi \omega}.
\]  

In specific cases of rectangular billiard and modified Kepler problem, this can also be obtained directly from Eqs. (20) and (24) below.\(^d\)\(^d\)

The common feature (which is trivially obvious for hard wall billiards) of integrable systems with no extra symmetries is that

\[
T_{\text{min}}^{-1} \propto \epsilon^{\delta} \gg 1 \tag{15}
\]

that is, level repulsion develops on a scale much larger than the mean level spacing (here \( \delta \) is a rational positive fraction). Nonetheless, it immediately affects the nearest level spacing distribution function via a relationship (see Appendix B for a discussion)\(^d\)\(^d\)

\[
P(s) = g(s) \exp \left[ - \int_0^s g(\omega) d\omega \right], \tag{16}
\]

where

\[
g(\omega) = 1 - K(\epsilon, \omega). \tag{17}
\]

Until now, \( K(\epsilon, \omega) \) had not been taken into consideration for integrable systems and the Poissonian distribution

\[
P_P(s) = \exp(-s) \tag{18}
\]

was obtained as a result. With the simplified ansatz, the distribution becomes

\[
P(s) = \left[ 1 - \frac{\sin(s T_{\text{min}})}{\pi s} \right] \exp \left[ -s + \frac{\text{Si}(s T_{\text{min}})}{\pi} \right] \tag{19}
\]

and, in particular,

\[
P(0) \approx 1 - \frac{T_{\text{min}}}{\pi} \tag{19'}
\]

where \( \text{Si} \) is the sine integral. This is the central result of the paper. As will be shown below, it is convincingly confirmed numerically for the rectangular hard wall billiard and a modified Kepler problem discussed in the next Section.

\(^d\)Dimensional \( T_{\text{min}} \) is measured in units of \( h/\Delta \).
3. Rectangular Billiard and Modified Kepler Problem

The level correlation can be explicitly evaluated for the rectangular billiard and the modified Kepler problem.\textsuperscript{15–17} They can also be studied numerically against the analytical results obtained using the correlation function.

We begin with the rectangular hard wall billiards.\textsuperscript{15} Without Balian-Bloch-like corrections,\textsuperscript{19} the level correlation function is found as

\[
K(\epsilon, \omega) = \frac{1}{\sqrt{\pi} \epsilon} \sum_{M_1, M_2 = 0}^{\infty} 4\delta_M \cos\left[\frac{4\pi}{\epsilon\sqrt{\frac{\alpha}{3}} \left(M_1^2 \alpha^{1/2} + M_2^2 \alpha^{-1/2}\right)}\right] \frac{\sqrt{M_1^2 \alpha^{1/2} + M_2^2 \alpha^{-1/2}}}{\sqrt{M_1^2 \alpha^{1/2} + M_2^2 \alpha^{-1/2}}} ,
\]

where \(\alpha\) is the aspect ratio of the rectangle (assumed to be close to unity, \(\alpha \sim 1\)) and

\[
\delta_M = \begin{cases} 
0, & \text{if } M_1 = M_2 = 0, \\
1/4, & \text{if one of } M_1, M_2 \text{ is zero} \\
1, & \text{otherwise}.
\end{cases}
\]

For the modified Kepler problem, described by the potential

\[
V(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}
\]

the relevant spectrum is that of a particle in a two-dimensional potential with hard wall potential in one direction and harmonic potential in the other and is given, in dimensionless units, by\textsuperscript{16–18}

\[
\epsilon_{p,l} = 2p\sqrt{2\beta} + l^2
\]

for which the level correlation function is given by\textsuperscript{17}

\[
K(\epsilon, \omega) = \sum_{M_r = 1}^{\infty} \left( \left[ \frac{M_r}{(2\beta/3\epsilon)^{1/4}} \right] + \frac{1}{4} \right) \frac{2\sqrt{2\beta}}{\pi^2 M_r} \sin \left( \frac{\pi M_r \omega (3\epsilon \sqrt{2\beta})^{1/4}}{} \right) ,
\]

where \(M_r\) is the radial winding number and \(\left\lfloor \right\rfloor\) is the floor function.

As explained in Refs. 14–17, averaging in Eqs. (20) and (24) is understood as parametric averaging, namely averaging over the aspect ratio for rectangles of the same area and over for the modified Kepler problem. This allows for ensemble averaging with the same running energy, which is different from averaging over the energy spectrum that “washes away” important features of these systems, such as oscillations of the level number variance.

Persistent oscillations of the level number variance follows directly from (20) and (24) and are in excellent agreement with numerical simulations for both systems.\textsuperscript{15,17} Averaging over oscillations gives a “saturation value,” which is twice that of the saturation rigidity.\textsuperscript{4} Existence of these saturation values is directly related to
the deviation of the nearest level distribution from Poissonian statistics and level repulsion in integrable systems.

Both (20) and (24) reduce to the form given by (14) for not too large $\omega$, up to $\sim T_{\min}^{-1}$. In view of existence of a number of short periodic orbits with periods close to $T_{\min}$, an obvious improvement on ansatz (13)–(14) is to replace $T_{\min}$ with $\tilde{T}_{\min} = c T_{\min}$, $c \sim 1$. For rectangular billiards, specifically, this yields

$$\tilde{T}_{\min} = c \frac{2 \pi^{1/2}}{\sqrt{\epsilon}}$$

while for Coulomb problem we find the situation is slightly more complicated due to existence of two frequencies (periods) - radial and angular — given by

$$\omega_r = (\sqrt{2} \beta \epsilon)^{1/3}, \quad \omega_\theta = \frac{\omega_r}{\gamma}, \quad \gamma \geq \left(\frac{2 \beta}{3 \epsilon}\right)^{1/3}.$$  \hspace{1cm} (26)

Consequently, we have $T_{\min} \rightarrow \tilde{T}_{\min}$, where using the floor function $\lfloor \gamma \rfloor$

$$\tilde{T}_{\min} = 2 \pi \omega_r^{-1} (c_1 + c_2 \lfloor \gamma \rfloor).$$  \hspace{1cm} (27)

### 4. Numerical Results

For rectangular billiards, we show results for the running energy $\epsilon \leq 10^4$ (in units of mean level spacing, as explained above) obtained for an ensemble of $3 \times 10^5$ samples specified by the values of aspect ratio $\alpha$ that are picked from a normal distribution centered at $\alpha_0 = 1$ with half width of 0.2. For the modified Kepler problem, we show results for obtained with $\epsilon \leq 3 \times 10^5$ samples specified by the values of $\beta$ that are picked from a normal distribution centered at $\beta_0 = 3 \times 10^6$ with half width $\beta_0/20$.

In Figs. 1 and 2 red line is the Poissonian (exponential) distribution, dotted line is the numerical calculation and green line is the best fit (with $\tilde{T}_{\min}$) using (19) for, respectively, rectangular billiards and modified Kepler problem. Deviation from the

![Fig. 1. Rectangular Billiards: Poissonian distribution (18) (red line), numerical calculation (dotted line) and best fit using (19) (green line).](image-url)
Poissonian statistics is quite striking in both systems and is in excellent agreement with the simplified ansatz (19).

In Figs. 3 and 4 — rectangular billiards and the modified Kepler problem, respectively — we investigate the following quantity:

\[ P^{(b)}(s) = \int_0^s P(x) \, dx \]  

(28)

as a function of running energy \( \epsilon \) for a value of level separation \( s \ll 1 \). For the Poissonian statistics, this trivially yields

\[ P_P^{(b)}(s) = \frac{1 - \exp(-s)}{s} \]  

(29)

which is \( \epsilon \)-independent. Using now (19) in (28), we find

\[ P^{(b)}(s) = 1 - \exp[-s + \text{Si}(s\tilde{T}_{\min})/\pi] \lesssim P_P^{(b)}(s) - \frac{\tilde{T}_{\min}}{\pi} \]  

(30)

Fig. 2. Modified Kepler Problem: Poissonian distribution (18) (red line), numerical calculation (dotted line) and best fit using (19) (green line).

Fig. 3. Rectangular Billiards: Poissonian distribution (29) (red line), numerical calculation (dotted line) and best fit of (30) (green line).
and, choosing a value of level separation $s = 0.01$, we compare the Poissonian value $P_P^{(s = 0.01)} = 0.995$ (red line) to the expression given by (30). The dotted line, as in Figs. 1 and 2, is the numerical calculation while the green line is a numerical fit using (30). Figures 3 and 4 provide an even more striking demonstration of the level repulsion and deviation from the Poissonian statistics.

We particularly emphasize the results shown in Fig. 4. The observed “jumps” — as insert pronouncedly demonstrates — occur at the values of the running energy $\epsilon$ where $\gamma$ becomes integer. But these are precisely the positions of the “jumps” in the saturation level rigidity,\textsuperscript{16,17} which firmly establishes (an already mentioned) connection between the short-range and long-range characteristics of the spectrum.

5. Conclusions

We demonstrated the existence of level repulsion in the semiclassical spectrum of classically integrable systems and dispelled the long-standing and widely accepted notion of the Poissonian distribution of nearest level spacing. We can account for the deviation from the Poissonian statistics using a simplified ansatz (14), which via (16) results in a distribution given by (19) and is applicable to any “generic” classical integrable system (that is integrable system without extra symmetries). Our results are convincingly confirmed numerically using ensembles of rectangular billiards and modified Kepler problem potentials, which ultimately validates our analytical approach (just as the use of (14) captures the key features of saturation of spectral rigidity\textsuperscript{15} first observed in Refs. 3 and 4).

For the latter two systems, strong level correlations are expressed by exact Eqs. (20) and (24), obtained both semiclassically and quantum mechanically,\textsuperscript{15–17} and reduce to ansatz (14) for shorter energy scales. These very correlations are also responsible for very unusual behavior of level rigidity and level number variance at large energy scales.\textsuperscript{15–17} Particularly striking effect, that intimately relates the short-range and long-range spectral characteristics, is the “jumps” in the nearest level spacing distribution observed in Fig. 4; remarkably they occur at the same
energies where jumps in the saturation level rigidity are observed\(^{17}\) and correspond
to classical periodic orbits.

It must be pointed out that in accordance with Eqs. (15) and (19), the nearest
level spacing distribution tends toward the Poissonian limit\(^{5,6}\) in the "semiclassical
limit," which corresponds to infinite running energy (position in the spectrum)
\(\epsilon \to \infty\).\(^{21}\) In the same limit, saturation of spectral rigidity
\(^{3,4}\) would not occur
since the saturation scale \(\propto T_{\min}^{-1} \to \infty\) by (15) and one would observe unrestricted
linear growth of spectral rigidity instead. This is in contrast to the classically chaotic
circumstance,\(^{1,10}\) where the level correlation function and, consequently, the nearest
level spacing distribution function, spectral rigidity and level number variance do
not depend on \(\epsilon\).

**Appendix A. Derivation of the Ansatz**

An integrable system has its action-angle variables \((J_i, w_i)\) with \(i = 1, 2, \ldots, m.\)
The density of levels is expressed by Green function
\[
\rho(E) = -\frac{1}{\pi} \text{Im} \left[ \int dw G(w, w; E) \right],
\]
where \(G\) can be expressed by Feynman propagator\(^{22}\)
\[
G(w, w; E) = -i \int_0^\infty dt \exp(iEt) F(w, w; t).
\]
Now we assume that \(G(w, w; E)\) does not depend on \(w.\) This means that each
periodic orbit covers the entire invariant torus, which is the case for both the rect-
angular billiard and the modified Kepler problem. We have
\[
\rho(E) = \frac{(2\pi)^m}{\pi} \text{Im} \left[ i \int_0^\infty dt \exp(iEt) F(0, 0; t) \right].
\]
The correlation function is expressed by the Feynman propagator as
\[
K(\epsilon_1, \epsilon_2) = \frac{(2\pi)^{2m}}{\pi^2} \text{Im} \left[ i \int_0^\infty dt_1 \exp(i\epsilon_1 t_1) F(0, 0; t_1) \right]
\times \text{Im} \left[ i \int_0^\infty dt_2 \exp(i\epsilon_2 t_2) F(0, 0; t_2) \right] - 1
\]
Now write \(F(0, 0; t_1) = e^{f(t_1)} F'(0, 0; t_1),\) where \(f\) and \(F'\) are real functions.
Then
\[
K(\epsilon_1, \epsilon_2) = \frac{(2\pi)^{2m}}{\pi^2} \int_0^\infty dt' \cos(\epsilon_1 t' + f(t')) \cos(\epsilon_2 t' + f(t')) F'^2(0, 0; t') - 1,
\]
where only the diagonal term \(t_1 = t_2 = t'\) is considered. This is consistent with the
diagonal approximation as \(F\) is a summation over periodic orbits semiclassically.
\[
\cos(\epsilon_1 t' + f(t')) \cos(\epsilon_2 t' + f(t')) = \frac{1}{2} [\cos(\omega t') + \cos(2\epsilon t' + f(t'))]
\]
(\ref{eq:cosproduct})
The second term is ignored as the integration of a quickly oscillatory term for $\epsilon \gg 1$ gives 0. Then we get

$$K(\epsilon, \omega) = \frac{(2\pi)^{2m}}{2\pi^2} \int_0^\infty dt' \cos(\omega t') F''(0, 0; t') - 1. \quad (A.7)$$

The spectral form factor is defined as

$$K(\epsilon, t) = \int_{-\infty}^{\infty} K(\epsilon, \omega) \exp(-i\omega t) d\omega. \quad (A.8)$$

We have

$$K(\epsilon, t) = \frac{(2\pi)^{2m}}{2\pi^2} \int_{-\infty}^{\infty} d\omega \int_0^\infty dt' \cos(\omega t')$$

$$\times F''(0, 0; t') \exp(-i\omega t) - \int_{-\infty}^{\infty} \exp(-i\omega t) d\omega \quad (A.9)$$

where the first term contains a periodic orbit of zero orbit length. Such a periodic orbit gives the average density of levels and cancels the second term. After the cancellation, $F'$ is left with $F''$ excluding periodic orbits with zero orbit length. The integration over $\omega$ gives two delta functions $\delta(t - t')$ and $\delta(t + t')$. It follows that $t' = t$ for $t > 0$ and

$$K(\epsilon, t) = (2\pi)^{2m-1} F''(0, 0; t) \quad (A.10)$$

which also works for negative $t$ due to time-reversal symmetry. We believe this simple formula captures central physics. Before $T_{\text{min}}$ the particle can not propagate back. We assume that after $T_{\text{min}}$, the particle uniformly propagates to any point in the invariant torus and the probability density to propagate back is $F''(0, 0; t) = (2\pi)^{-m}$. This implies Eq. (13).

**Appendix B. Derivation of Porter’s Formula**

We rederive Porter’s formula below. We define the complete two-point correlation function as

$$C(\epsilon_1, \epsilon_2) \equiv C(\epsilon, \omega) \equiv \langle \rho(\epsilon_1) \rho(\epsilon_2) \rangle \quad (B.1)$$

with $\epsilon$ and $\omega$ defined in Eq. (6). To find the nearest level spacing distribution $P(s)$, we divide the nearest neighbor spacing $s = \epsilon_2 - \epsilon_1$ into $M$ equal small intervals $\delta \epsilon$ and assume each small interval is statistically independent. Suppose there is a level at $\epsilon_1$, the probability not to have a level from $\epsilon_1$ to $\epsilon_1 + \delta \epsilon$ is $1 - C(\epsilon_1, \epsilon_1 + \delta \epsilon) \delta \epsilon$; the probability not to have a level from $\epsilon_1 + \delta \epsilon$ to $\epsilon_1 + 2\delta \epsilon$ is $1 - C(\epsilon_1, \epsilon_1 + 2\delta \epsilon) \delta \epsilon$; similar for other small intervals. Hence $P(s)$: The probability to have two levels at $\epsilon_1, \epsilon_2$ and not a level in the range $s$, is

$$P(s) = C(\epsilon_1, \epsilon_2) \prod_{m=1}^{M} [1 - C(\epsilon_1, \epsilon_1 + m\delta \epsilon) \delta \epsilon]. \quad (B.2)$$
Employing the approximation \( \exp[-C(\epsilon_1, \epsilon_1 + m\delta\epsilon)] \approx 1 - C(\epsilon_1, \epsilon_1 + m\delta\epsilon)\delta\epsilon \), Eq. (B.2) becomes Porter's formula

\[
P(s) = C(\epsilon_1, \epsilon_2) \exp \left[ - \int_{\epsilon_1}^{\epsilon_2} C(\epsilon_1, x) \, dx \right]. \quad (B.3)
\]

After applying the approximation \( C(\epsilon_1, x) \approx C(\epsilon - (x - \epsilon_1/2), \epsilon + (x - \epsilon_1/2)) \equiv C(\epsilon, x - \epsilon_1) \), we get

\[
P(s) = C(\epsilon, s) \exp \left[ - \int_0^s C(\epsilon, \omega) \, d\omega \right]. \quad (B.4)
\]

This is Eq. (16).

**References**