Describing-function theory for flow excitation of resonators

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A theory is presented for the mechanism by which resonators are excited by grazing flow. The theory allows prediction of oscillation characteristics for the range of Reynolds numbers, frequencies, and resonator amplitudes for which the acoustically excited mean flow rolls up into discrete vortices. The resonator-flow system is treated as an autonomous nonlinear system. Limit cycles of the system are found using describing-function analysis, in which each component of a nonlinear oscillating system is represented by an associated frequency-response function. This mathematical approach is shown to be a generalization of models in which the resonator and flow are considered parts of a feedback system. The theory's predictions for the frequencies of oscillation compare favorably with experiment. The results indicate that both "edge" and "resonator" feedback contribute to the mechanism of self-excited oscillations of the resonator-flow system.

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INTRODUCTION

Many nonlinear systems exhibit limit cycles, that is, they can undergo nearly-periodic oscillations in which the effects of the system's nonlinearity are balanced by the effects of the linear parts of the system. Such an oscillation is called "self-excited," since there is no external forcing which is independent of the motion. For a self-excited oscillation to occur, there must exist a stable, nearly periodic solution to the nonlinear system of equations which describes the motion of the physical system.

Self-excited flow oscillations are common examples of nonlinear systems which possess limit cycles. This is true for flows in many geometries (Rockwell, 1983; Blake and Powell, 1986). However, these flows are extremely difficult to model analytically. Workers who have attempted to model such flows from first principles have, because of the inherent complexity of the problem, been forced to make great simplifications. Although such models have been quite successful in characterizing the physical nature of self-excited flow oscillations, they have had less success in predicting quantitative characteristics of these oscillations such as the frequency and amplitude (Howe, 1981; Nelson *et al.*, 1983; Crighton, 1992).

Alternatively, the limit cycles can be understood in terms of a feedback process (Powell, 1961; Cremer and Ising, 1968). A few investigators have analyzed flow-resonator oscillations in this way. Cremer and Ising (1968), Elder (1973), and Yoshikawa and Saneyoshi (1980) have treated the case of jet excitation of organ pipes. [A review of Cremer and Ising's work is given in Mast (1993).] Elder and co-workers (Elder, 1978; Elder *et al.*, 1982) have also analyzed several cases of flow excitation of cavity resonators; in these studies the flow disturbances were treated using linear shear layer instability models and the oscillation amplitude was assumed to be limited by nonlinear orifice resistance. Parthasarathy *et al.* (1985) and Shakkotai *et al.* (1987) have put forth a completely linear model in which "eddy" sources were represented by time-delayed linear terms. However, although this model has some features similar to those of nonlinear feedback models, no linear model can fully describe the essentially nonlinear process involved in flow excitation of resonators.

The complicated process of resonator-flow oscillations may be understood in a mathematically satisfying way when the resonator-flow system is analyzed using concepts borrowed from nonlinear control theory (Slotine and Li, 1991). This approach was first hinted at by Powell (1961), and was put into more concrete form by Cremer and Ising (1968), who used a feedback model in their analysis of jet excitation of organ pipes. In the present paper, feedback descriptions of flow-resonator systems are shown to be complementary to analytic descriptions which employ describing-function analysis. Describing-function analysis is used here to analyze a flow-resonator system that is considerably different from that treated by Cremer and Ising.

In this paper, a new theory is presented for the self-excited oscillations of resonators subjected to grazing boundary-layer flow. Nonlinearity is assumed to be associated with the finite growth and saturation of flow disturbances rather than with nonlinear orifice resistance. The theory explicity treats the linear and nonlinear parts of the flow-resonator system using describing-function analysis. The describing function for the nonlinear part of the system is estimated for the case of a nonlinearly saturated flow disturbance by making use of experimental observations of flow-resonant systems. This describing function and a function describing the linear response of the resonator are used to determine the amplitude, frequency, and mean-flow characteristics for limit-cycle oscillations of the flow-resonator system.

I. DESCRIBING-FUNCTION ANALYSIS

In order to clarify the nature of the mathematical analysis presented in this paper, it is believed warranted to present a brief introduction to the techniques used. Since readers in the acoustical community may not be familiar with techniques and nomenclature used in nonlinear control theory, a few notes about describing-function analysis are given here.

Describing-function analysis is a method by which nonlinear oscillating systems can be modeled as a group of coupled elements, with each element represented by an associated frequency-response function. This technique is commonly used in control theory to calculate limit cycles of nonlinear systems—typically, the nonlinear part of the system is represented by one frequency response function and the linear part by another. In the nomenclature of control theory, the frequency response functions are called "describing functions" (Slotine and Li, 1991, pp. 157–190). These differ from linear transfer functions in that the describing functions themselves may be nonlinear functions of their argument: in other words, the output may depend nonlinearly on the input amplitude or phase.

In this section, a simple example of describing-function analysis is given in order to illustrate how this method is used to find limit cycles of nonlinear systems. The system to be used as an example is the unforced Van der Pol equation (Slotine and Li, 1991, p. 158):

$$\ddot{x}+\alpha(x^2-1)\dot{x}+x=0. \tag{1}$$

Equation (1) represents a one-degree-of-freedom oscillator which is negatively damped for amplitudes below one and positively damped for amplitudes above one.

For a system like that described by Eq. (1), it is reasonable to expect that a limit cycle may exist. This is so because Eq. (1) has the general form of an equation describing a forced linear oscillator:

$$\ddot{x}-\alpha\dot{x}+x=y. \tag{2}$$

Here the "forcing" function y is not external to the system, but depends on the amplitude of oscillation:

$$y=-\alpha x^2\dot{x}. \tag{3}$$

To seek a limit cycle, one first postulates an approximate solution of the form

$$x=A\cos(\omega t) \tag{4}$$

and substitutes the postulated solution into Eq. (2) with y given by Eq. (3). This results, after some trigonometric manipulation, in the equation

$$(1-\omega^2)\cos(\omega t)+\alpha\sin(\omega t)$$
$$=(A^2\alpha\omega/4)[\sin\omega t+\sin(3\omega t)]. \tag{5}$$

A solution to Eq. (1) will not be sinusoidal for finite values of α. However, it is possible to postulate a nearly-sinusoidal solution and to determine the amplitude of the fundamental component of this solution. Once an approximate solution is found, it can be checked in Eq. (1) to determine its range of validity.

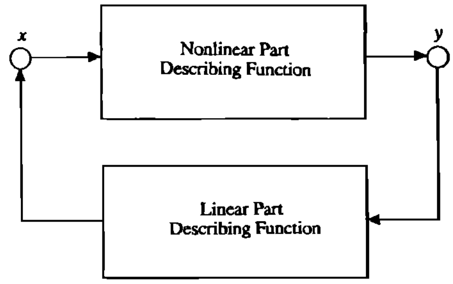

FIG. 1. Feedback loop illustrating the describing-function method for seeking limit cycles of Van der Pol's equation.

To seek the fundamental component of a nearly sinusoidal limit cycle of Eq. (1), the term of frequency 3ω in Eq. (5) is dropped. Then all terms are of frequency ω. The resulting equation can be thought of as the real part of the complex equation

$$(1-\omega^2)e^{-i\omega t}-i\alpha e^{-i\omega t}=(-i\omega A^2\alpha/4)e^{-i\omega t}, \tag{6}$$

or simply

$$\frac{-i\omega A^2\alpha}{4}\,\frac{1}{(1-\omega^2)-i\alpha}=1. \tag{7}$$

The criterion expressed in Eq. (7) is that the product of two describing functions equal one. Equation (7) can be written in the form

$$\frac{y(x)}{x}\,\frac{x(y)}{y}=1. \tag{8}$$

The first term here corresponds to the first term in Eq. (7); the second terms in the equations also correspond.

The first term of Eq. (8) is the describing function associated with the nonlinear part of the system; this describing function represents the fundamental part of the response of the nonlinear part of the system for the case of sinusoidal input. The second term describes the response of the linear part of the system to sinusoidal input (including the negative resistance which the oscillator incurs at small amplitudes). Both of these describing functions have the form of frequency-response functions.

These two parts of the system can be interpreted as comprising the feedback loop sketched in Fig. 1. The output x of the linear block is fed into the nonlinear block. The output y of the nonlinear block is then fed back into the linear block. Mathematically, these two operations amount to specifications that y is a function of x [as seen in Eq. (3)] and that x is in turn a function of y [as seen in Eq. (2)]. The criterion for a limit cycle of the feedback system shown in Fig. 1 is that the product of the two describing functions be equal to one; of course, this is just the condition stated in Eqs. (7) and (8).

Equation (7) has a solution for $\omega=1$ and $A=2$; this solution is the sought limit cycle. Thus there is a solution for x of the form

$$x\approx 2\cos(t). \tag{9}$$

This is precisely the limit cycle which is found when Eq. (1) is treated by averaging or by perturbation methods; the error associated with the approximations made is of order $O(\alpha^2)$ (Guckenheimer and Holmes, 1990, pp. 68–69).

Describing-function analysis is a conceptually simple way to determine the leading-order behavior of a nonlinear system. In the example just given, the system of interest can be conveniently studied by other analytic means. This is not the case, however, for the physical system of a flow-excited resonator; in this case, describing-function analysis provides analytic insight not otherwise available.

II. THEORY

In this section, the method of describing-function analysis is applied to develop a theory for flow excitation of resonators that can be described by lumped-element models. The scope of the model includes flow excitation of Helmholtz resonators (such as wine bottles) and of flexible-walled cavity resonators such as aneurysms in arteries (Mast, 1993; Mast and Pierce, 1995). The flow-resonator system is analyzed as an autonomous nonlinear system in which the nonlinearity is associated with saturation of disturbances in an unstable mean flow. The Mach number of the mean flow is assumed to be negligibly small, and all resonator dimensions are assumed to be much less than an acoustic wavelength.

A. Flow-excited resonators as a nonlinear system

Many resonators of size much smaller than a characteristic wavelength can be described as one-dimensional, lumped-element oscillators. The equation governing the motion of such an oscillator is

$$M\ddot{x}+R\dot{x}+Kx=F(x,t), \tag{10}$$

where x is the average displacement of the fluid in the opening (taken to be positive in the direction pointing into the cavity) and $F(x,t)$ is the force exerted on the fluid in the resonator's opening; it is proportional to the pressure p_{above} above the opening. The pressure p_{above} in general depends on the fluid displacement x as well as on the time t. Thus the system described by Eq. (10), composed of the resonator and the external flow, is nonlinear. One expects that, under certain conditions, the system may possess limit cycles. That is, there may exist nearly-sinusoidal solutions for the independent variable x.

Limit cycles of the system can be sought in a manner analogous to that used in Sec. I. That is, one first postulates a sinusoidal form for x (corresponding to the lowest-order Fourier component of the true oscillatory solution), and then determines the amplitude and frequency of the sinusoid. The fundamental component of a limit cycle proves to be a good approximation to the physically occurring limit cycle when the linear part of the system (in this case, the resonator) has a moderate amount of damping (Slotine and Li, 1991, pp. 164–165). Since only the fundamental component of the solution is sought, the left side of Eq. (10) can be replaced by its counterpart for constant-frequency oscillations.

The difficulty in such analysis is that the forcing function $F(x,t)$ is not explicitly known. However, even though $F(x,t)$ cannot be precisely specified, the effect of the nonlinearity of the system [completely contained in the function $F(x,t)$] can be determined using the method of describing-function analysis. For the purposes of this analysis, the describing function characterizing the nonlinear part of the system can be approximated using empirically observed relationships between the oscillatory flow $q=A\dot{x}$ and the forcing function $F(x,t)$.

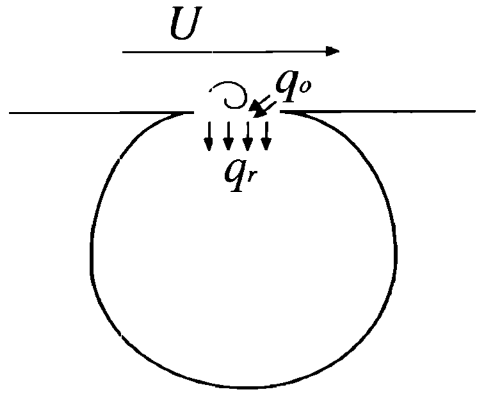

FIG. 2. Sketch of the resonator-flow system, with breakdown of the flow in the opening into the "orifice" flow q_o and the "resonator" flow q_r.

For reasons of simplicity and generality, it is desirable to carry out the describing-function analysis in terms of nondimensional quantities. In the present paper, the quantities used are the ratios of two volume velocities. Two such describing functions are derived. The first, termed the "forward gain function," describes the nonlinear interaction between the oscillatory flow in the resonator's neck and the unstable mean flow. The second, the "backward gain function," describes how flow disturbances in turn force the resonator.

These two describing functions are chosen such that their product is identically one under the conditions for which a limit cycle occurs. This is ensured by specifying the two functions so that when an oscillation is occurring, the two functions are reciprocals of one another.

The volume velocity in the neck of the resonator is split into two parts, the "orifice" volume velocity q_o and the "resonator" volume velocity q_r. Each is taken to be positive in the direction pointing into the resonator cavity. Both are purely oscillatory quantities. The flow q_o is the flow that would occur in the absence of any resonator for a given pressure disturbance across the orifice. The flow part designated by q_r is simply the rest of the flow:

$$q_r=q_{\mathrm{total}}-q_o. \tag{11}$$

When the resonator is sounding near its natural frequency, q_r is approximately equal to q_{total}. A schematic of this breakdown of the flow field is shown in Fig. 2.

The forward gain function yields the "orifice" volume velocity q_o associated with a specified input "resonator" volume velocity q_r. This describing function is associated with the flow disturbance caused by a given oscillation of the flow in the resonator's neck; thus for the purposes of deriving this function the resonator flow q_r is taken to be given. The forward gain function is then given by $q_o(q_r)/q_r$.

The backward gain function is associated with the response of the resonator to a given flow disturbance. For the purposes of calculating this function, the orifice flow q_o (defined to be proportional to the pressure associated with the flow disturbance) is taken to be given and the resulting resonator flow q_r is calculated; the ratio of the two is the (nondimensional) describing function. The backward gain function is simply $q_r(q_o)/q_o$.

For a limit cycle of the system to exist, both q_r and q_o must be finite. Since each is a function of the other, a criterion analogous to Eq. (8) must be satisfied for a limit cycle to occur:

$$\left(\frac{q_o(q_r)}{q_r}\right)\left(\frac{q_r(q_o)}{q_o}\right)=1. \tag{12}$$

Below, the two describing functions are derived and the procedure for seeking limit cycles of the flow-resonator system is outlined. A dispersion relation for the frequency of oscillation is then derived and a mathematically simple procedure is given for finding numerical roots of the feedback equation describing the limit cycles of the system.

B. Forward gain function

First, the describing function corresponding to the nonlinear interaction between the resonator and the unstable mean flow is derived. This describing function is termed the "forward gain function."

The flow external to the resonator opening is assumed to take the form of an unstable shear layer. Ronneberger (1980) noted that an unstable shear layer exposed to a sinusoidal pressure disturbance will roll up into discrete vortices for Strouhal numbers St_θ such that

$$St_\theta=\omega\theta/U_\infty>0.15, \tag{13}$$

where θ is the momentum thickness of the shear layer and ω is the radial frequency of the flow disturbance. (The vortices are shed at the frequency of the applied pressure disturbance.) As the analysis in the present paper assumes that the flow disturbance takes the form of discrete vortices, it is valid only for Strouhal numbers that satisfy Eq. (13). For Strouhal numbers within this range and for high Reynolds number flows, the details of the boundary layer characteristics are not important to the physical mechanism of oscillation.

Furthermore, Bruggeman and co-workers (Bruggeman, 1987; Bruggeman *et al.*, 1991) have shown that in the case of shear-layer flow past a resonator's opening, for a given resonator geometry, the disturbance source strength can be assumed constant (for constant frequency and varying amplitude of the acoustic field that excites the shear layer) if the system is sounding near the resonator's natural frequency and if the flow oscillations are of "moderate" amplitude. The amplitude is considered moderate if the velocity u' associated with the flow disturbance is within the range

$$O(10^{-3})\leq u'/U_\infty\leq O(10^{-1}). \tag{14}$$

Within this amplitude range the disturbance flow q_o is proportional to the square of the mean flow velocity, U^2 (Cremer and Ising, 1968). For a linear resonator response, this also implies that the pressure amplitude of the resonator oscillations will be proportional to U^2. This proportionality has been observed by many experimental investigators in studies of flow excitation of resonators (Blokhintsev, 1945; Cremer and Ising, 1968; Shakkotai *et al.*, 1987; Bruggeman, 1987; Bruggeman *et al.*, 1991). The constant of proportionality, however, depends on the specific geometry of the resonator and flow (Bruggeman, 1987). The fact that the disturbance flow is of constant amplitude means that the initially growing flow disturbances are saturated in amplitude at the time of their interaction with the downstream edge of the resonator. This saturation is a nonlinear effect, and renders linear shear layer theories invalid for oscillation amplitudes in this range. The present paper assumes that the amplitudes of oscillation will be within this range.

When a pressure oscillation associated with the sounding of a resonator is present, the frequency of the vortex shedding will be equal to the frequency of that pressure oscillation, as observed by Ronneberger (1980). The vortex shedding also occurs with a specific phase relationship to the pressure oscillations associated with the resonator. This phase relationship, along with the amplitude relationship stated above, provides the information necessary to estimate the describing function associated with the nonlinear part of the flow-resonator system, as is done below.

In a given cycle, the shedding of the vortex from the upstream edge occurs when the triggering (acoustic) velocity first becomes positive, directed into the resonator (Bruggeman, 1987). For a rigid-walled Helmholtz resonator, this corresponds to the point in the cycle when the oscillating pressure inside the resonator reaches its minimum value, since the cavity pressure is

$$\dot{p}_{\text{cav}}=\rho c^2 q_{\text{neck}}/V_{\text{cav}} \tag{15}$$

(Pierce, 1989, p. 338). This implies that the cavity pressure will be at a minimum at the instant the vortex is shed from the upstream edge; this is exactly the phase relationship between cavity pressure and vortex shedding that was observed by Nelson *et al.* (1981) in an experimental study of a flow-excited Helmholtz resonator.

After being shed, the vortex convects downstream at a speed close to half the mean-flow velocity (Nelson *et al.*, 1981; Bruggeman, 1987; Hirschberg *et al.*, 1989). This convection speed is approximately equal to the average of the flow velocities above and below the opening.

The flow disturbance initially grows as it convects downstream. Since the flow disturbance grows spatially while it is exposed to the exciting acoustic wave, it is at its largest when in the vicinity of the downstream edge of the cavity (Ronneberger, 1980; Nelson *et al.*, 1981; Bruggeman, 1987). Also, the vortical disturbance is a much more efficient source of sound when in the vicinity of a boundary (Curle, 1955). For the purposes of approximating the form of the source, the downstream edge can be modeled as a rectangular corner. Furthermore, as shown by the calculations of Conlisk and Rockwell (1981), a single vortex approaching a corner will not cause a significant pressure fluctuation until it reaches the immediate vicinity of the corner and the presence of the boundary causes the vortex to decelerate. One concludes the effect of the flow disturbance on the resonator can be idealized as that of an acoustic source located at the downstream edge of the opening. This is a common approximation used in cavity-tone problems, and has been found in many cases to provide good agreement with experiment (Heller and Bliss, 1975; Tam and Block, 1978; Blake and Powell, 1986). The form of the source used in the present development is an oscillating volume velocity.

The sought source, then, is that oscillating volume velocity associated with the interaction of a single line vortex with the downstream edge of the cavity. The phase of the effective volume source is estimated here by making use of experimental measurements of vortex-corner interaction. Tang and Rockwell (1983) made detailed measurements of the wall pressure distributions associated with the interaction of line vortices and a square corner for vortices shed from an upstream edge at a constant frequency. The measurements were performed for a number of vortex paths, including paths for which the vortices impinged directly on the corner and paths for which the vortices were swept past the corner. In all cases, the time history of the pressure at any point on the wall was nearly sinusoidal.

They found that near the corner, the pressure distribution was dipole-like, being 180° out of phase between the top and front sides of the corner at all times during an oscillation cycle. This is to be expected, as vorticity inhomogeneities near solid boundaries act as dipole sound sources (Curle, 1955). Tang and Rockwell also observed that the phase of the dipole moment at the downstream edge depended on the path of the vortices path the corner. In the present discussion, it is assumed that the vortex is convected past the corner, incurring only slight distortion, since this path corresponds to that seen in the flow-visualization studies of Nelson *et al.* (1981) and Bruggeman (1987). In this case, the dipole moment at the corner passed through a zero (decreasing) at the instant when the leading edge of the vortex was approximately flush with the corner.

The dipole moment is essentially a local value of the pressure gradient existing at the corner, and will locally induce a flow that can be determined from the constant-frequency version of Euler's equation:

$$v=(-i/\omega\rho)\nabla p. \tag{16}$$

Since the time dependence of each side is $e^{-i\omega t}$, the velocity disturbance leads the pressure gradient (dipole moment) by $\pi/2$. Thus when the dipole moment at the corner is zero and decreasing, velocity induced by the dipole moment will be at a minimum (that is, at its peak and pointing out of the cavity). This is the required information for specification of the phase of the source associated with the flow disturbance.

The source associated with the flow disturbance is taken to be located at the downstream edge of the cavity. Then the travel time from upstream edge to source position is $2d/U$, and the resultant phase lag will be $2\omega d/U$. At this instant, the flow source associated with the vortex shedding has a phase that is $3\pi/2$ greater than the phase of the resonator flow at the moment the vortex was shed. The flow source then has a phase of $3\pi/2-2\omega d/U$ relative to the resonator flow.

The flow source thus has the form

$$q_o=\kappa U^2 e^{-i(3\pi/2-2\omega d/U)}(q_r/|q_r|), \tag{17}$$

where κ is a quantity that is real, positive, and constant for a given frequency ω, having dimensions of length×time. Here ω is an unknown which is be solved for once all necessary describing functions are specified.

Equation (17) can be expressed in a form without dimensional constants when the pressure fluctuation associated with the flow disturbance is constant, as expected for the range of parameters studied here. This pressure fluctuation, p_{above}, relates to the flow disturbance q_o as

$$q_o=p_{\text{above}}/(-i\omega M), \tag{18}$$

so that the amplitude of q_o can be expressed as

$$|q_o|\propto\beta A^2\rho U^2/(-i\omega M) \tag{19}$$

for a pressure disturbance $p_{\text{above}}=\beta\rho U^2$.

For the purpose of calculating the forward gain function, the average "resonator" flow is taken as given. One can now specify the forward gain function. The resonator volume flow q_r is given, and the orifice volume flow is given by Eq. (17), so that the forward gain function is

$$\left(\frac{q_o}{q_r}\right)_{\text{forward}}=\frac{\beta A^2\rho U^2}{\omega M|q_r|}e^{-i(3\pi/2-2\omega d/U)}. \tag{20}$$

The absolute value $|q_r|$ appears because the phase of q_o was defined relative to q_r; the difference is phase is simply $3\pi/2-\omega T$, which in Eq. (20) appears entirely in the exponential term.

Equation (20) has several features worthy of note. It contains the essential nonlinearity of the system in that q_o depends nonlinearly on q_r. Since the flow source is saturated in amplitude while the resonator is sounding, the amplitude of q_o does not depend on that of q_r, but the phase of q_o is determined by the phase of q_r. It is this phase relationship which primarily governs the frequency of oscillation of the flow-resonator system.

C. Backward gain function

The backward gain function corresponds to the resonator flow caused by a specified orifice flow. It is assumed that the resonator response can be described by a lumped-element model. In such a model, the resonator is represented by an impedance composed of a mass, a stiffness, and a resistance. This represents a single-degree-of-freedom oscillator which responds to any disturbances to the flow outside the opening of the resonator. Here, these disturbances are represented by the "hydrodynamic" pressure disturbance p_{above} associated with the orifice flow q_o. The use of p_{above} rather than q_o as the driving quantity is simply a matter of convenience; since the two quantities are related by Eq. (18), specifying one automatically specifies the other.

The governing equation for this system is, for constant-frequency oscillations (as are assumed when a limit cycle is sought), the constant-frequency form of Eq. (10) for an $e^{-i\omega t}$ time dependence:

$$(-\omega^2 M-i\omega R+K)x=Ap_{\text{above}}(x,t). \tag{21}$$

Here x is the average displacement of the fluid in the cavity opening. This includes "orifice" and "resonator" displacements. Equation (21) applies for simple harmonic motion, which is assumed in this case since only the lowest-order Fourier component of the oscillatory flow is considered. Now, since for simple harmonic motion $q_{\text{total}}=-i\omega Ax$,

$$q_{\text{total}}=\frac{-i\omega p_{\text{above}}A^2}{-\omega^2M-i\omega R+K}. \tag{22}$$

The orifice flow q_o is specified by Eq. (18).

Thus, the resonator flow q_r is [as defined by Eq. (11)] the remaining part of the total volume velocity:

$$q_r=q_{\text{total}}-q_o=\frac{(K-i\omega R)A^2p_{\text{above}}}{i\omega M(K-M\omega^2-i\omega R)}. \tag{23}$$

Now the backward gain function can be stated as

$$\left(\frac{q_r}{q_o}\right)_{\text{backward}}=\frac{1-i\omega R/K}{(\omega/\omega_0^2)-(1-i\omega R/K)}. \tag{24}$$

Here $\omega_0=\sqrt{K/M}$ is the natural frequency of the resonator.

In terms of a nondimensional frequency parameter $\Omega=\omega/\omega_0$ and the quality factor $Q=\sqrt{KM}/R$, the backward gain function is

$$\left(\frac{q_r}{q_o}\right)_{\text{backward}}=\frac{1-i\Omega/Q}{\Omega^2-(1-i\Omega/Q)}. \tag{25}$$

D. Loop gain criterion

The criterion for stable self-excited oscillations is that the product of the two gain functions is identically one. If one thinks in terms of a feedback loop, this criterion implies that the total loop gain is equal to one. The criterion is given by an equation analogous to Eq. (12):

$$\left(\frac{q_o}{q_r}\right)_{\text{forward}}\left(\frac{q_r}{q_o}\right)_{\text{backward}}=1. \tag{26}$$

For given mean flow velocity and resonator dimensions, the real and imaginary parts of this equation comprise two equations in two variables—in dimensionless form, the two variables are the Strouhal number $\omega d/U$ and the velocity ratio $|u_r|/U$. These two equations can be solved numerically to yield possible solutions for stable oscillations.

This criterion can be written as

$$\frac{\beta A^2\rho U^2}{\omega M|q_r|}e^{-i(3\pi/2-2\omega d/U)}\frac{1-i\omega R/K}{(\omega/\omega_0^2)-(1-i\omega R/K)}=1. \tag{27}$$

In terms of nondimensional variables $\Omega=\omega/\omega_0$, $Q=\sqrt{KM}/R$, and $S=\omega d/U$, the criterion may be written as

$$\frac{\beta A^2\rho U^2}{\omega M|q_r|}e^{-i(3\pi/2-2S)}\frac{1-i\Omega/Q}{\Omega^2-(1-i\Omega/Q)}=1. \tag{28}$$

To seek a solution of this equation, one can specify a value of U. Solutions then occur when, for a particular amplitude and frequency,

$$\left(\frac{q_r}{q_o}\right)_{\text{backward}}=1\Bigg/\left(\frac{q_o}{q_r}\right)_{\text{forward}}. \tag{29}$$

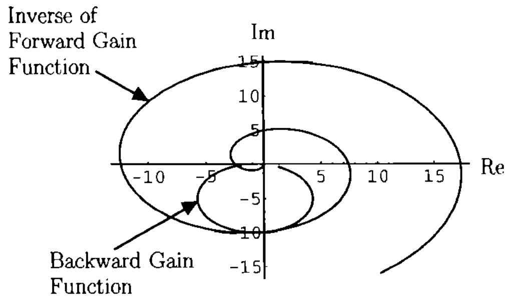

FIG. 3. The two gain functions plotted in the complex plane. The vertical axis is the imaginary part of the horizontal axis is the real part of the respective function.

The forward gain function and the inverse of the backward gain function can be plotted in the complex plane; neutrally stable oscillations occur when the two curves intersect at an equal frequency. Such a plot is shown in Fig. 3, for the resonator-flow system analyzed by Nelson *et al.* (1981). In this case, a solution exists near the negative imaginary axis, where $U=12$ m/s, and $f=609$ Hz (close to the natural frequency of the resonator). At this point, the Strouhal number is $\omega d/U=3.19$, close to π, meaning that the distance between successively shed vortices is close to the streamwise opening thickness d. The other intersection of the two curves (near the origin) is not a solution, since the two intersecting points correspond to different frequencies.

The solution occuring near the negative imaginary axis is the solution (for high-Q resonators) that corresponds to oscillations near the natural frequency of the resonator, as was previously observed by Elder (1978). It can be seen from Eq. (25) that near the resonator's natural frequency (where one expects the largest response), the backward gain function will be nearly purely negative and imaginary. When $\omega=\omega_0$, $\Omega=1$ and Eq. (25) becomes

$$\left(\frac{q_r}{q_o}\right)_{\text{backward}}=\frac{1-i/Q}{i/Q}\approx-iQ. \tag{30}$$

The phase of the forward gain function, as seen from Eq. (20), depends only on the nondimensional frequency $\omega d/U$. The inverse of the forward gain function crosses the negative imaginary axis at frequencies such that

$$\frac{\omega d}{U}=n\pi,\quad n=1,2,3,\dots . \tag{31}$$

It may be noted that Eq. (31) does not imply that high-amplitude oscillations take place near frequencies $2\omega_0$, $3\omega_0$, etc. Although the forward gain function can cross the negative imaginary axis near these frequencies, the backward gain function crosses this axis only at a frequency near ω_0, so that the loop gain criterion predicts high-amplitude oscillation only at frequencies near ω_0.

The phase criterion of Eq. (31) is equivalent to a criterion for the flow disturbance wavelength λ,

$$\lambda=d/n,\quad n=1,2,3,\dots, \tag{32}$$

since $\lambda=2fd/U$ for a convection velocity of $U/2$.

One can see from Eq. (20) that for fixed d and U, the forward gain function is inversely proportional to the radial frequency ω and to the resonator flow q_r. Thus solutions for higher n correspond to lower oscillation amplitudes. The highest-amplitude oscillations occur for $n=1$. Response for the $n=2$ flow-disturbance mode will be approximately half that of the $n=1$ mode, and responses associated with higher modes will be correspondingly lower in amplitude.

Thus, for resonators of moderate to high Q, the highest-amplitude oscillations occur in the vicinity of $\omega \approx \omega_0$ and $\omega d/U \approx \pi$. When $\omega d/U=\pi$, the wavelength of the flow disturbance is equal to the orifice diameter d, so that for the case calculated above, the distance between shed vortices is close to the opening's diameter.

E. Procedure for solution

The criterion for closure of the feedback loop [Eq. (28)] can be satisfied by numerically solving two equations [the real and imaginary parts of Eq. (28)] in two variables, the amplitude and frequency of oscillation. However, it is more efficient first to solve for the frequency of oscillation using the requirement that the phase of the loop gain be zero, and then to solve for the amplitude of oscillation.

Equation (28) can only be satisfied if the imaginary part of the loop gain is precisely zero, that is, if

$$\mathrm{Im}\left(e^{-i(3\pi/2-2S)} \frac{1-i\Omega/Q}{\Omega^2-(1-i\Omega/Q)} \right)=0. \tag{33}$$

Rationalizing the denominator of Eq. (33) yields

$$\mathrm{Im}\left[e^{-i(3\pi/2-2S)} \left(\Omega^2-\frac{\Omega^2}{Q^2}-1-\frac{i\Omega^3}{Q} \right) \right]=0. \tag{34}$$

Expanding the exponential and taking the imaginary part of the resulting expression gives

$$\left[\Omega^2\left(1-\frac{1}{Q^2}\right)-1 \right]\cos(2S)+\frac{\Omega^3}{Q}\sin(2S)=0, \tag{35}$$

so that one can write down the dispersion relation

$$\tan(2S)=\frac{Q^2-\Omega^2(Q^2-1)}{\Omega^3 Q}, \tag{36}$$

or

$$2S=\tan^{-1}\left(\frac{Q^2-\Omega^2(Q^2-1)}{\Omega^3 Q} \right)+2n\pi, \tag{37}$$

where n is an integer. The inverse tangent function has the range $-\pi/2<\tan^{-1}(x)<\pi/2$, and has a zero where the argument x is zero. In Eq. (37), it can be seen that near the natural frequency of the resonator ($\Omega=1$) the argument of the inverse tangent function will be small and near a zero crossing, so that the inverse tangent function will also be small.

For the case $n=1$ the resulting dispersion relation is

$$S=\frac{\omega d}{U}=\tan^{-1}\left(\frac{Q^2-\Omega^2(Q^2-1)}{\Omega^3 Q} \right) \Big/ 2+\pi. \tag{38}$$

The $n=1$ case corresponds to the experimental observations of Nelson *et al.* (1981). As shown in Sec. II D, it also corresponds to the highest-amplitude oscillations of a flow-excited resonator. Equation (38) yields values of S such that $3\pi/4 < S < 5\pi/4$ for $0<\Omega<\infty$. Equations of the form of Eq. (38), with higher values of n, yield values of S close to $n\pi$.

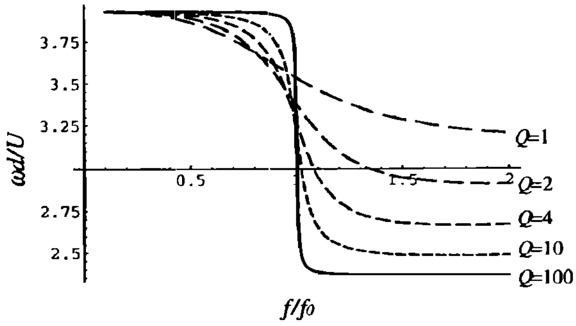

FIG. 4. Plot of nondimensional "Strouhal" frequency $\omega d/U$ versus sounding frequency f/f_0 for flow-disturbance mode $n=1$ ($\lambda \approx d$).

Solutions of Eq. (38) for a range of values of Q are shown in Fig. 4. In this figure, U can be thought of as a parameter increasing from left to right and from top to bottom along each of the curves shown. The figure shows that as Q increases, the resonator exerts more and more frequency control in the vicinity of resonance.

When a particular value of U is specified, Eq. (37) can easily be solved numerically for ω, yielding the frequency of oscillation. When the resulting value of ω is substituted into Eq. (28), one obtains a purely real equation for the "resonator" flow q_r. The total velocity response in the neck of the resonator can then be calculated as

$$|q_{\text{total}}|=|q_r||1+(q_r/q_o)^{-1}_{\text{backward}}|. \tag{39}$$

Of course, the forward gain function could be used in place of the inverse of the backward gain function here, since the product of the two gain functions is identically one when Eq. (28) is satisfied.

III. COMPARISON WITH EXPERIMENT

To show that the theory of presented here is credible, it is necessary to compare its results to experiment. Nelson *et al.* (1981) performed a detailed experimental investigation of flow excitation of a Helmholtz resonator. Their data provide a good check for the theory developed above, since they explicitly measured the parameters K, M, and R used in the theory presented here.

To perform such a comparison, one can find numerical solutions of Eq. (27), inserting the parameters describing the resonator of Nelson *et al.* (1981). These are (with the impedance terms stated in mechanical units, for consistency with the derivation in this paper):

$$M=2.2\times10^{-5}\ \text{kg}, \quad K=320\ \text{kg/s}^2, \\ R=8.4\times10^{-3}\ \text{kg/s}, \quad d=10\ \text{mm}. \tag{40}$$

This value of R is a radiation resistance, and was measured at the natural frequency of the resonator. For other frequencies, R will vary as $(ka)^2$, as for a piston in an infinite baffle. The above stiffness and mass yield a resonance frequency

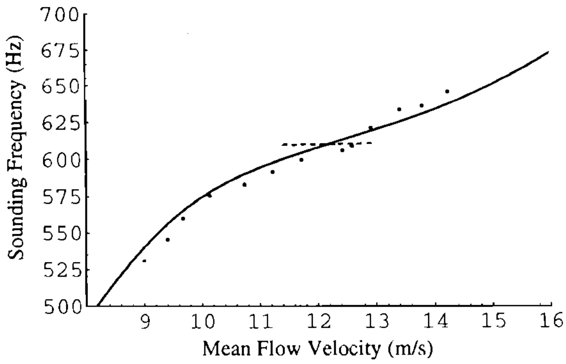

FIG. 5. Frequency of flow-excited oscillation (Hz) versus free-stream velocity (m/s). Line: theory; dots: experimental data from Nelson *et al.* (1981). The dashed line indicates the nominal natural frequency of the resonator, 609 Hz.

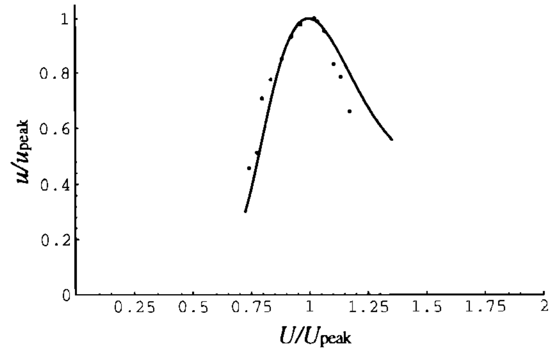

FIG. 6. Normalized oscillation velocity amplitude versus normalized free-stream velocity. Line: theory; dots: experimental data from Nelson *et al.* (1981).

$$f_0=\frac{1}{2\pi}\sqrt{\frac{K}{M}}=609\ \text{Hz}. \tag{41}$$

The resonator's quality factor is

$$Q=\sqrt{KM}/R=10. \tag{42}$$

Using the procedure outlined in Sec. II E, one can find numerical solutions for the frequency and amplitude of oscillation of a flow-excited resonator having parameters identical to the resonator of Nelson *et al.* (1981). Figure 5 shows the calculated (using the present model) and measured (by Nelson *et al.*) oscillation frequencies versus the free-stream velocity U (directly above the shear layer). Nelson *et al.* found that the free-stream velocity was $U=12$ m/s directly above the shear layer for a maximum velocity (far from the shear layer) of $U_\infty=22$ m/s. The measured convection speed for his vortices was 6 m/s, which corresponds to $U/2$; hence U and not U_∞ is used for comparison to the present theory (in which the vortex convection speed was assumed to be half the free-stream velocity). U is assumed here to be proportional to U_∞ by the constant 12/22, as it is for $U_\infty=22$ m/s. Figure 5 shows that the model predicts the frequency of oscillation quite well in the vicinity of resonance.

The relative amplitude of oscillation is predicted somewhat less well. The model predicts peak excitation amplitude for $U=13$ m/s, while Nelson *et al.* (1981) observed peak excitation for $U=12$ m/s. The general shape of the response curve (versus U), however, agrees approximately, as shown in Fig. 6. For the purpose of comparison, the mean-flow velocities were normalized to U_{peak}, the velocity at which the greatest response occurred. It is thus seen that the model yields a reasonable prediction of the range of flow velocities over which the resonator-flow system will oscillate with high amplitude.

It should be emphasized that the agreement with experiment seen in Figs. 5 and 6 was obtained without any adjustment of unknown parameters. Every parameter needed by the present theory was directly measured by Nelson *et al.* (1981). The present authors are not aware of any published study other than that of Nelson *et al.* in which detailed frequency results were presented along with measurements of all the parameters required by the present theory. For this reason, detailed quantitative comparisons of the kind shown in Figs. 5 and 6 could not be carried out using data from other studies.

The quantitative predictions of the present theory for the characteristics of peak-amplitude oscillation are in good agreement with the observations of many investigators. This theory's predictions are in agreement with the common observation that the peak response of a flow-excited resonator occurs near its natural frequency (Elder, 1973, 1978; Panton and Miller, 1975; Nelson *et al.*, 1981). The prediction that this occurs when the vortex wavelength matches the opening diameter agrees with the observations of Nelson *et al.* (1981), as well as those of Bruggeman (1987), Bruggeman *et al.* (1991), Panton (1990), and Blokhintsev (1945).

Likewise, Bruggeman (1987) found that for his flow-excited pipe resonator, strong excitation occurred for flow disturbance wavelengths that were an integral fraction of the streamwise opening diameter. He observed flow disturbance modes for which λ/d was equal to 1, 2, and 3; for each mode, the flow disturbance appeared as a sequence of discrete vortices. Analogous results for the oscillation characteristics of flow-excited depth-mode resonators have been obtained by Blokhintsev (1945), Ingard and Dean (1958), Hankey and Shang (1980), and Elder *et al.* (1982).

It should be noted that other investigators have observed different frequency relations for flow-excited resonator oscillations (Blake, 1986; Shakkotai *et al.*, 1987). The most commonly observed relation other than that stated above is that the flow disturbance wavelength is proportional to $d(n-1/4)$ (Heller, and Blogg, 1975; Elder, 1978; Blake, 1986). The explanation for the apparent discrepancy between these observations and the present theory is that the phase relationship between the resonator flow and the flow-disturbance/downstream edge interaction depends strongly on factors such as the flow geometry (Tang and Rockwell, 1983; Panton, 1990) and the mean-flow characteristics (Elder *et al.*, 1982). The theory presented in this paper assumed a phase relationship consistent with the flow visualizations of Nelson *et al.* (1981) and Bruggeman (1987); however, analogous reasoning incorporating a different phase relationship could be used to explain other experimental observations.

In Fig. 5, one sees the "locking-in" to the resonance frequency of the resonator near $f=f_0$. The slope of the frequency-velocity curve is considerably smaller in this region than in the off-resonant regions. However, the frequency of oscillation still increases with increasing flow velocity near resonance. The slope in this part of the curve is smaller for larger resonator Q; the higher the Q, the more control the resonator exerts over the flow.

Similar "lock-in" behavior is seen in many other studies of flow excitation of Helmholtz and standing-wave resonators (Ingard and Dean, 1958; Shakkotai *et al.*, 1987; Panton, 1990; Khosropour and Millet, 1990). These studies, however, are not amenable to detailed comparison with the present model—Ingard and Dean (1958), Parthasarathy *et al.* (1985), and Shakkotai *et al.* (1987) examined standing-wave resonators, Panton (1990) did not report the damping (or equivalently, the Q) of his Helmholtz resonators, and Khosropour and Millet (1990) examined the excitation of a Helmholtz resonator by an air jet, for which a more complicated dispersion relation applies (Fletcher and Thwaites, 1979). However, in all these cases the same qualitative behavior was seen. That is, resonators showed their peak response near their natural frequencies, the frequency of oscillation increased monotonically with the mean flow velocity, and the tightness of frequency control exerted by the resonator increased with increasing Q.

IV. CONCLUSION

A theory for flow-excited oscillations of lumped-element resonators has been presented. This theory is based on a simple model for the nonlinear interaction between a resonator's oscillations and disturbances which arise in an unstable mean flow; this nonlinear interaction is analyzed by the method of describing-function analysis. The results of the model are in good agreement with experimental observation.

The model's results indicate that there are two relevant feedback mechanisms in a flow-excited resonator. Each can be characterized by the corresponding induced volume velocity in the resonator's opening. One of these corresponds to a fluctuating volume velocity caused by the interaction of the unstable mean flow with the downstream edge of the opening; the other corresponds to the volume velocity associated with the resonant behavior of the system.

The flow-edge interaction mechanism is most important for frequencies not close to the resonance frequency of the resonator. In these regions, one sees a nearly linear dependence of the sounding frequency on the mean-flow velocity. This is the type of feedback that drives the edgetone, in which no resonator at all is present. The feedback associated with the resonator is most important near its resonance frequency, where one sees the frequency-velocity dependence of the system flatten out somewhat. However, since in this region one still sees a weak monotonic dependence of the sounding frequency on the mean flow velocity, one can conclude that even in this region the resonator does not act as the sole source of feedback. Instead, both "resonator" feedback and "edge" feedback are important for the entire range of mean-flow velocities in which limit cycles of the flow-resonator system occur.

Last, it should be noted that there is not and cannot be any universal formula for the oscillation frequencies of flow-excited resonators. For a given resonator-flow system, the details of the geometry and the characteristics of the mean flow result in a particular phase relationship between the resonator's oscillations and the flow disturbance/edge interaction. When this phase relationship is known, the present theory may be adapted to model a wide variety of flow-resonator systems.

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