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The structure of ideas in *The Port Royal Logic* ☆

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ABSTRACT

This paper addresses the degree to which *The Port Royal Logic* anticipates Boolean Algebra. According to Marc Dominicy the best reconstruction is a Boolean Algebra of Carnapian properties, functions from possible worlds to extensions. Sylvain Auroux's reconstruction approximates a non-complemented bounded lattice. This paper argues that it is anachronistic to read lattice algebra into the *Port Royal Logic*. It is true that the *Logic* treats extensions like sets, orders ideas under a containment relation, and posits mental operations of abstraction and restriction. It also orders species in a version of the tree of Porphyry, and allows that genera may be divided into species by privative negation. There is, however, no maximal or minimal idea. Abstraction is not binary. Neither abstraction nor restriction is closed. Ideas under containment, therefore, do not form a lattice. Nor are the relevant formal properties of lattices discussed. Term negation is privative, not a complementation operation. The technical ideas relevant to the discussion are defined. The *Logic*'s purpose in describing structure was not to develop algebra in the modern sense but rather to provide a new basis for the semantics of mental language consistent with Cartesian metaphysics. The account was not algebraic, but metaphysical and psychological, based on the concept of *comprehension*, a Cartesian version of medieval objective being.

0. Introduction

This paper investigates the structure of ideas in *The Port Royal Logic*. It has two immediate goals. The first is to explain what this structure is and why. The second is theoretical, to assess the degree to which the concepts of structure employed approximate those of modern algebra. Doing so will amount to a

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reexamination of the interpretations of the *Logic* by Sylvain Auroux and Marc Dominicy,[1] who argue that the structure of ideas in the *Logic* approximates that of Boolean algebra. In my opinion that claim is exaggerated. I hope to show that although there are novelties in the approach to structure, these are not algebraic. Rather, innovations derive not from mathematical developments, but rather from the need to rework various foundational concepts of medieval logic necessitated by Cartesian metaphysics and epistemology. To preserve the wider corpus of logical doctrines that had been accepted since the Middle Ages, the *Logic*'s authors found it necessary to redefine basic notions like signification. The rebuilding did have novel implications for the structure of ideas, but, as I hope to show, these were extensions of and changes in earlier doctrines. They were not couched in the language of mathematics but rather in that of medieval psychology and metaphysics.

In the discussion I will make use of concepts from modern logic, but today's logic can be applied to history is different ways. In rare cases a modern concept fits an earlier one exactly. An example is Bocheński's observation that Philo's 4th century B.C. implication is the same as the material conditional, which he then illustrated by a modern truth-table.[2] Let us call an interpretation that identifies an historical concept with a modern one a *paraphrase*. We shall see that Auroux and Dominicy paraphrase some of the *Logic*'s operations in terms of Boolean algebra.[3]

More commonly an earlier account is explained by extending it, reformulating both the original and the extension in modern terms. Bits and pieces of an earlier doctrine can be recast into modern vocabulary and elaborated in a way that results in a recognizable theory according to modern standards. Let us call such an interpretation an *extension*. Natural deduction completeness proofs of Aristotle's syllogistic are examples. Some of the points I will make below about the *Logic* will count as extensions in this sense.

A third type of reading uses modern logic to correct an earlier version. Typically these are extensions with revisions. The revision may clarify poorly defined terms, reorganize definitional order, or correct inconsistencies. Let us call a reading of this sort a *reconstruction*. Because Auroux and Dominic both find the treatment of term negation in the *Logic* to be flawed, the interpretations they give count as reconstructions in this sense.

The central topic of the paper is the degree to which the structure of ideas in the *Logic* approximates that of modern algebra. The key concepts from modern algebra to which this structure will be compared are *partial ordering, lattice* and *Boolean algebra*. Since the paper addresses historians as well as logicians, some introduction to the technical material is in order.

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[1] Auroux [7], Dominicy [14]. There have been other skeptics about how much the 17th century work anticipates modern logic. See, for example, Conimbricenses [12], pp. 318–320, and Pariente [21], p. 246, who writes:

> L'originalité du livre ne réside pas, il est vrai dans ses innovations formelles. Arnauld et Nicole ne sont pas des inventeurs sur le plan du calcul logique. Rien n'est plus éloigné de leur style de réflexion que les efforts diversifiés et inlassables d'un Leibniz pour mettre sur pied un formalisme efficace et rationnel.

More directly relevant to this paper is Russell Wahl's judgment, "It is a mistake, I believe, to read into the *Logic* a prelude to set theory". Wahl [23], p. 673.

[2] Bocheński [8]. §§ 20.07 and 20.071, p. 117.

[3] We shall be making use of standard concepts from set theory and abstract algebra. Those from algebra are defined as follows. $\langle B, \leq\rangle$ is a *partial ordering* iff $\leq$ is a reflexive, transitive and anti-symmetric binary relation on B. If $\langle B, \leq\rangle$ and $\langle B', \leq'\rangle$ are partial orderings, a function f from B into B', f is said to be *monotonic* iff, for any $x, y \in B$, if $x \leq y$ then $f(x) \leq' f(y)$; *antitonic* iff, for any $x, y \in B$, if $x \leq y$ then $f(x) \geq' f(y)$; and B is *dual to B' relative to f* iff f is onto and antitonic. $\langle B, \wedge\rangle/\langle B, \vee\rangle$ is a *meet/join semi-lattice* iff B is closed under a binary operation $\wedge/\vee$ that is associative, commutative, and idempotent. An ordering relation $\leq$ on a *meet/join* semi-lattice B is defined as follows: $x \leq y$ iff $x \wedge y = x/x \vee y = y$. It follows that if $\langle B, \wedge\rangle/\langle B, \vee\rangle$ is a *meet/join semi-lattice*, then $\langle B, \leq\rangle$ is a partial ordering. $\langle B, \wedge, \vee\rangle$ is a *lattice* iff $\langle B, \wedge\rangle$ and $\langle B, \vee\rangle$ are *meet* and *join semi-lattices*. In a partial ordering $\langle B, \leq\rangle$ *the greatest lower bound* of $\{xy\}$, briefly $glb\{x, y\}$ if it exists, is the $z \in B$ such that $z \leq x$, $z \leq y$, and for any w in B, if $w \leq x$ and $w \leq y$, then $w \leq z$; *the least upper bound* of $\{x, y\}$, briefly $lub\{x, y\}$, if it exists, is the $z \in B$ such that $x \leq z$, $y \leq z$, and for any w in B, if $x \leq w$ and $y \leq w$, then $z \leq w$. It follows that $\langle B, \wedge, \vee\rangle$ is a lattice iff $\langle B, \leq\rangle$ is a partial ordering closed under $lub = \wedge$ and $glb = \vee$. 0 is *the least element* of a lattice B iff, $0 \in B$ and for any x in B, $0 \leq x$, $0 \wedge x = 0$ and $0 \vee x = x$; 1 is *the greatest element* iff $1 \in B$ and for any x in B, $x \leq 1$, $1 \wedge x = x$ and $1 \vee x = 1$.

A partial order is a relation that organizes groups of "points" so that within the group some points are lined up as "prior to" others but in a way that allows all sorts of branching and coming together, both forward and backward. Formally a *partial ordering* is defined as a set in which its elements stand in a binary relation, which is symbolized by $\leq$, that is reflexive, transitive, and antisymmetric. Examples include the *less-than-or-equal-to* relation on numbers and the subset relation $\subseteq$ on sets. Many loosely organized sets count as partial orderings. These orderings can be illustrated by connecting points by vertical lines with the understanding that if point A is below point B then $A \leq B$. Examples include vertical lines, trees, diamonds, railroad systems, any combination of these, as well as the trivial cases of scattered unconnected individual points and even a single point. We will see that the *Logic* posits various "containment" relations that qualify as partial orderings.

A lattice has more structure than a partial ordering. It conforms to the additional constraint that any divergent route "upward" or "downward" always returns to a definite single point. Formally, a *lattice* is defined as a partial ordering that has the additional property that any two elements A and B possess a least upper bound and a greatest lower bound. The *least upper bound* of A and B is that element C such that $A \leq C$ and $B \leq C$ and there is no element D distinct from C such that $A \leq D \leq C$ and $B \leq D \leq C$. The *greatest lower bound* of A and B is that element C such that $C \leq A$ and $C \leq B$ and there is no element D distinct from C such that $A \leq D \leq C$ and $B \leq D \leq C$. It follows that in a lattice there cannot be two distinct points that are higher than all points below them, nor any two points that are lower than all those above them. In other words, a lattice has a unique maximal point, which is called its *highest element*, if it has one at all. Similarly, if it has a minimal element, it has only one, which called its *least element*, Trees, most rail systems, and scattered unconnected points, which are partial orderings, do not count as lattices. In a lattice it is possible to define an operation that pairs with any two points the next higher point, and another operation that maps them to the next lower point. Formally, the *meet* of A and B, symbolized $A \wedge B$, is the least upper bound of A and B, and the *join* of A and B, symbolized $A \vee B$, is their greatest lower bound. Both operations are well-defined for any pair from the set (i.e. for any A and B, $A \wedge B$ and $A \vee B$ exist and are unique), and both have the properties that they are associative and commutative. The union operation $\cup$ on sets is an example of a meet operation, and the intersection operation $\cap$ is a join operation. Auroux and Dominicy believe that in the *Logic* there is a greatest and a least idea, and that the mental operation of idea-restriction is a meet operation and the mental operation of abstraction is a join operation. That is, they interpret ideas as having, at a minimum, the structure of a lattice.

A Boolean algebra is a special sort of symmetrically balanced lattice. It has a maximal and a minimal element, and for any point there is another (its complement) that is its symmetric opposite in the sense that the join of a point and its complement is the structure's maximal element and their meet is the minimal element. Formally, a *Boolean algebra* is defined as a lattice such that (1) meet and join are distributive over one another, (2) there is a maximal element, which is symbolized by 1, and a minimal element, which is symbolized by 0, and (3) for any point A, there is another point, called its *complement* and symbolized by $-A$, such that $A \vee -A = 1$, and $A \wedge -A = 0$. It follows that in a Boolean algebra, *complementation* is well-defined: $-A$ exists and is unique. If complementation were well defined for ideas, then it would be a variety of term negation and a complemented term would be a kind of negative idea. Among the issues of this paper is whether the *Logic* envisages term negation and if so, whether it is Boolean complementation. Meet and join conform to various rules familiar from elementary logic including DeMorgan's laws and double negation. One illustration of the high degree of structure possessed by a Boolean algebra is the fact that if the ordered points above A are isolated, this substructure will be the exact mirror of the substructure consisting of all the points below $-A$. A diamond, a cube rising from one corner, and the family of the subsets of a set A ordered by $\subseteq$ (the so-called *power set algebra* of A) are examples of Boolean algebras. Auroux and Dominicy argue that ideas in the *Logic* fall short of Boolean algebra because, in their view, term negation on ideas is unwittingly defective.

The central question of this paper, then, is whether the *Logic*'s authors anticipate the modern notions of lattice, Boolean algebra, or any of its component concepts including partial order, maximal and minimal elements, and meet, join and complementation operations. Section 1 addresses the degree to which the *Logic*'s notion of idea-containment captures that of partial order. Section 2 investigates the special properties of species. Section 3 considers abstraction and whether it is a meet operation. Section 4 takes up restriction. Section 5 address negation.

1. Comprehension, containment and order

Ideas in the *Logic* are described as standing in a relation of "idea-containment." In this section we will consider how containment is defined and whether it qualifies as a partial ordering. Roughly, one idea is said to be contained in another if the first is part of the definition of the second. To a modern reader definitional containment immediately suggests an "analytical" definition, which in modern philosophy is usually understood to be a matter of convention. Although the *Logic*'s notion of definition contains innovations motivated by Descartes' metaphysics, it remains largely Aristotelian.

According to standard medieval theory a concept is a quality of the soul. In Cartesian vocabulary a concept is called an *idea* and a quality a *mode*. Ideas, however, are not only thoughts; they also constitute the terms of mental language. Nouns, both proper and common,[4] are provided by Providence with a kind of definition consisting of a series of modes. In the language of the *Logic* this series is called the idea's *comprehension*[5]:

> I call the *comprehension* of an idea the attributes that it contains in itself, and that cannot be removed without destroying the idea. For example, the comprehension of the idea of triangle contains extension, shape, three lines, and the equality of these three angles to two right angles, etc. .... none of these attributes can be removed without destroying the idea, as we have already said, whereas we can restrict its extension by applying it only to some of the subjects to which it conforms without thereby destroying it.[6]

A term's comprehension, in short, is a group of modes. As we shall see, these modes both constitute what we would call today the idea's intentional content and determine what the idea stands for in the world. They also have an important epistemological role. It is necessary, however, to say more about the background ontology.

According to a doctrine that was still current an idea has two kinds of being. Because it a mode of the soul, it is part of the soul's "form" and therefore has *formal being*. But an idea also "brings to mind" an entity, or, if the idea is abstract, a kind of entity, that possesses those modes contained in the idea's comprehension. Today we would say the idea is "about" this entity or group. Inasmuch as the idea brings to mind an entity or group possessing these modes, the idea has *objective being*.[7]

In medieval logic objective being was posited at various times as an explanatory entity. For example, at one point William of Ockham appealed to a *fictum*, which was his term for objective being, as the object of knowledge when we understand an abstract concept. Peter Aureol similarly proposed objective being as

[4] Proper nouns count as common nouns for logical reasons. According to the *Logic*'s treatment of the syllogistic, particular affirmative propositions are a special case of universal affirmatives (*LAP* II, 3 and III, 9), and thus by definition both have universal terms as subjects.

[5] I:6.

[6] *LAP* I:6, *KM* V 144, *B* 39.

[7] *VFI*:6, *KM* I:192, *G* 60. From *objectum*, past participle of *obicio*, to throw before, to present. *Objective* here does not have the modern meaning of an entity outside the mind.

the entity that is perceived in an illusion.[8] In Meditation III Descartes famously proves God's existence by appealing to the perfection possessed by the objective being of the idea of God.

In the *Logic*'s epistemology comprehension is used to explain certainty. In the normal case we know with certainty the proposition *every S is P* because we have a clear and distinct idea of S as P, and we do so only if the comprehension of P is included in that of S.[9]

In this paper, however, we are concerned with semantics, not epistemology. As the paper unfolds, it will become clear that virtually all the concepts important to the discussion are defined in terms of comprehension. At the root of the sematics is the definition of signification, which is the period's concept of reference. In the words of the *Logic*, an idea's comprehension determines what an idea *signifies, applies to*, or is *true of*. It *signifies* all those actual objects that instantiate all the modes in its comprehension.[10] Comprehension, thus, plays a role in the *Logic* similar to *Sinn* in Frege's semantics; it determines a term's reference. Moreover, the tie between an idea and its comprehension is so close that comprehensions provide the identity criterion for ideas: $A = B$ if, and only if, the comprehension of A is identical to that of B. It follows that there is no synonymy or ambiguity in mental language. If two ideas have the same comprehension, they are the same, and no idea can have more than one comprehension. Ideas and comprehensions, in short, are mapped one-to-one.

It is important for this paper that comprehension determines an ordering on ideas. Because comprehensions are collections of modes, one such collection can include or "contain" another. Because collections are essentially sets, the containment relation on comprehensions is what we would call today the subset relation on sets of modes. This formulation in modern terms is the first of several examples of "paraphrase" we shall meet. In this case, however, we cannot press the translation too far because the subset relation on sets possesses properties the *Logic*'s authors never addressed. We shall see examples shortly.

The authors did understand that what we call the subset ordering on comprehension-sets causes or "induces" a corresponding ordering of ideas, which the authors call the containment relation. Idea A *contains* idea B if, and only if, all the modes in the comprehension of B are also modes in the comprehension of A. In set theory, idea A *contains* idea B if, and only if, the comprehension of B is a subset of that of A. It follows that the authors envisioned two distinct but isomorphic orderings. The more basic is the set–inclusion relation on sets of modes. This relation in turn determines the containment relation on ideas. Although the authors do not say so in so many words, it is also not unreasonable to say that the authors also understood that these relations had the properties of a partial ordering. It is hard to imagine anybody understanding a containment relation without also understanding, at least at an intuitive level, that it has the properties reflexivity, transitivity and antisymmetry.

On the other hand, there are other properties of containment that the authors did not mention and probably did not imagine. For example, if a comprehension is a set of modes, then this set would be a member of the family of all subsets of modes, which is called today the *power set* of the set of modes. This power set is ordered by the subset relation, and forms a Boolean algebra, which is called the *power set algebra* of the set of modes. Moreover, if comprehensions stand 1–1 to ideas, this power set algebra on comprehension-sets (i.e. sets of modes) would generate an isomorphic Boolean algebra on ideas. Thus, it might be argued that ideas in the *Logic* form a Boolean algebra. Dominicy comes close to just this interpretation after he has corrected the *Logic*'s notion of negation by replacing it with Boolean complementation.[11]

There is no textual basis, however, for thinking that the authors believed that every imaginable set of modes corresponds to an idea. Questions of existence at the time, including the existence of ideas, were metaphysical, not mathematical. The *Logic*'s authors did not reason using existence axioms from set theory,

[8] See [22].

[9] IV:2 *KM* V:366–367, *B* 237–238; IV:6, *KM* V:378, *B* 247; IV:7, *KM* V:382, *B* 250. A special case of certain knowledge in which the predicate is not part of the comprehension of the subject is our clear and distinct idea of ourselves as existing.

[10] *Ibid.*

[11] Dominicy [14], p. 43.

like the power set axiom, which given the set of modes, would insure the existence of the family of every possible subset of modes. Rather, Cartesian metaphysics posited only actual substances and their modes. Arnauld himself scorned the notion of possible entity, and the *Logic*'s authors disavowed concern with vague and abstruse metaphysics.[12] On their understanding, *the idea of the sun* is a mode in my soul, and the modes that comprise its comprehension, like *extended* and *spherical*, exist in the actual sun. My idea of a *golden mountain* likewise is a mode in my soul, and the modes in its comprehension, like *gold* and *mountainous* exist, if at all, as instantiated in actual substances outside the mind. These modes are not mathematical entities the existence of which is assured by mathematical axioms. In short, if the authors had directly discussed the ontological status of possible comprehensions, which they did not, they would not have identified the set of comprehensions with the family of all possible subsets of modes. The conclusion, then, is that attributing to comprehensions or ideas parallel Boolean orderings goes beyond the text. At this point we are justified in attributing to both only the parallel partial orderings induced by the set–inclusion relation on those comprehension-sets that actually exist.

There is yet another ordering on ideas. This is the ordering of extensions. "Extension" has a technical meaning in the *Logic*. Unlike the modern notion, a term's extension is composed not of things outside the mind, but of ideas. A term's *extension* is defined as the collection of its "inferior" ideas. The inferiority relation here is key to the definition. It turns out that despite the fact that extension is defined as consisting of ideas, which are mental, an idea's extension tracks those things in the world that the idea signifies. It does so because inferiority is defined in terms of signification, which is a relation of ideas to things outside the mind. Idea A is *inferior* to idea B if, and only if, everything that A signifies B also signifies. It follows that a term's extension consists of all ideas that signify only things that the term itself signifies. This use of *extension* or *étendue* is semantic, and is new to the *Logic*.[13]

To get a clear notion of the order appropriate to extensions, it helps to paraphrase into modern vocabulary what we know of the relevant ordering relations. Let us call the set of entities a term signifies its *significance range*. (Significance range is very similar to the modern notion of extension.) In addition, let us recognize that a term's extension, as defined in the *Logic*, is essentially a set, the set of an idea's inferior ideas.

Let us extend the theory a bit to illustrate how an idea's extension tracks its significance range. Let us assume that everything any idea signifies has a proper name. It would then follow that an idea signifies something if, and only if, that entity is also uniquely signified by some proper name in the idea's extension. An idea's significance ranges and the set of proper names in its extension would then stand in a 1–1 correspondence. For everything an idea signifies, there would be an inferior idea signifying that thing, and conversely.

The *Logic* is silent, however, on whether everything signified has a proper name. There is, nevertheless, a 1–1 correspondence between significance ranges and extensions in the *Logic*'s sense. The correspondence holds because an idea's extension is always limited to those inferior ideas that signify only what is in the idea's significance range. If an idea signifies an entity, there is at least one idea in its extension that signifies that entity, even if it is only that idea itself. Conversely, if something is signified by any idea in its extension,

[12] See *LAP* I:7, *KM* V 146, *B* 41, and also the remarks on realism at *LAP*, *Discours* I, *KM* V 112–113, *B* 11–12, and the more general remarks at *LAP* IV:1, *KM* V 358, *B* 230.

[13] In the *Logic* an idea's extension is a set of its "inferior" ideas, and this set is defined in terms of signification, i.e. an idea's *extension* is the set of ideas that signify only things that the idea signifies:

> J'appelle *étendue* de l'idée, les sujets à qui cette idée convient, ce qu'on appelle aussi les inférieurs d'un terme général, qui à leur égard est appelé supérieur, comme l'idée du triangle en général s'étend à toutes les diverses espèces de triangles. (*LAP* I:6, *KM* V 145; *B* 39 145.)

It follows that two ideas signify the same objects "outside the mind" if, and only if, they have the same extensions. *Extension* also had a long history of use in physics, and its physical sense continued to be used by the Cartesians as the essence of material substance.

then it is signified by the idea itself. As we shall prove shortly, it follows that there is a 1–1 correspondence between idea extensions and significance ranges.

In preparation let us distinguish the four related partial orderings we have encountered. Because these can be confusing, it will helpful to lay them out side by side. First there is the family of comprehension-sets ordered by the set–inclusion relation, which is a partial ordering. These are sets of modes. This structure is isomorphic to a second, the set of ideas ordered by idea-containment. The mapping from these sets of modes to ideas is 1–1 because each comprehension determines a unique idea and every idea has a unique comprehension. The mapping is order preserving because one idea is defined as being contained in another if the comprehension of the first is included in that of the second. Hence the mapping from ideas to comprehension-sets is a 1–1 order preserving isomorphism.

The structure of comprehension-sets, and hence that of ideas, is mapped many-one onto a third structure, the family of significance ranges, which is a family of sets of entities outside the mind. This structure is ordered by the set–inclusion relation. This ordering too is a partial ordering. The mapping of comprehension-sets to significance ranges is many-one because the modes in two comprehension-sets may be contingently true of the same subjects. The mapping, moreover, is order-reversing (antitonic) because the comprehension of A contains that of B if, and only if, the set of entities signified by B is a subset of the set of entities signified by A. Because the partial ordering of ideas is isomorphic to that of comprehension-sets, it follows that ideas are mapped many-one and antitonically onto significance ranges.

Lastly, there is a fourth structure, the family of extensions. This is a family of sets of ideas. Because it is a family of sets, it is partially ordered by set–inclusion. Moreover, the partial ordering of extensions is isomorphic to that of significance ranges. The proof is straightforward. First, a significance range uniquely determines an extension. It does so because the significance range of A determines the set of all ideas which have significance ranges that are included in that of A. But the set of all ideas that have significance ranges that are included in that of A is another name for the extension of A. Conversely, the extension of A determines its significance range because that range is just the union of the significance ranges of the ideas in A's extension. Lastly, if the significance range of A is a subset of that of B, the set of all ideas that have significance ranges that are included in that of A is a subset of the set of all ideas that have significance ranges that are included in that of B. Hence there is an isomorphic mapping from significance ranges to extensions.

In sum, because ideas are isomorphic to comprehension-sets, comprehension-sets are mapped many-one and antitonically to significance ranges, and significance ranges are isomorphic to extensions, it follows that ideas are mapped many-one and antitonically to extensions. This mapping of ideas antitonically to extensions resolves an issue raised in the literature. It establishes that ideas are "dual" to extensions.

Whether ideas are dual to extensions has been an issue of dispute between Auroux and Dominicy.[14] According to the standard definition of duality in algebra, one partial order is defined as *dual* to another if there is a many-one antitonic mapping from the first to the second. This condition is met by the mapping of ideas to extensions. If A contains B, then the extension of B is a subset of that of A. Auroux and Dominicy, who do not define the terms at issue very clearly, may also have been concerned about the converse mapping from extensions to ideas. Because the mapping of ideas to extensions is not 1–1, the duality of ideas to extensions does not insure the converse, that extensions are dual to ideas. The extension of B may be a subset of that of A, yet the comprehension of A not include that of B. For example, *prudent* may signify *Peter* because the modes in its comprehension are true of Peter, and, therefore, the idea *Peter* is inferior to *prudent* and included in its extension. The idea *prudent*, however, is an accident and is not in the comprehension of *Peter*.

[14] Dominicy holds that they are, but Auroux maintains that they are not. See [14], pp. 40–41, and [7], pp. 80–85, 133–135.

Although thus far we have established a variety of structural parallels, ideas themselves, as so far examined, have rather minimal structure, that of a partial ordering. In the next section we consider species, a variety of ideas that possess considerably more structure.

2. Species and the tree of Porphyry

Among ideas species have special status. Within mental language, the *Logic* classifies species as common nouns:

> For when general ideas represent their objects as things and are indicated by terms called substantives or absolutes, they are called *genus* or *species* . . . .
>
> An idea is called genus when it is so common that it extends to other ideas that are also universal, as the quadrilateral is a genus with respect to the parallelogram and the trapezoid. . . .
>
> Common ideas that fall under a more common and general idea are called species, just as the parallelogram and the trapezoid are species of the quadrilateral, and body and mind are species of substance.[15]

As the passage explains, the *Logic* holds to the traditional view that species conform to the Tree of Porphyry[16]: Each species has a real definition (*definitio rei*) that specifies its comprehension. Grammatically, the definition is a convertible universal affirmative proposition that takes a species as its subject term and a genus restricted by a difference as its predicate term. Examples are *man is a rational animal* and *time is the measure of motion*.[17] Because the genus too is a common noun, it is assumed to have its own logically prior definition in terms of a yet higher genus and difference. Like the species, its genus and difference also possess comprehensions.[18] The comprehension of the species, then, is the sum of the comprehensions of its two defining terms. This "content" is also called the species' "essence." The definition

> explains the nature of a thing by its essential attributes, of which the common one is called the *genus*, and the proper one the *difference*.[19]
>
> These ideas are called differences when their object is an essential attribute distinguishing one species from another, such as extended, thinking, and rational.[20]

It is fair to paraphrase this doctrine in modern terms as stipulating that species form a partially ordered set in which the order has the properties of a rooted tree. Here let us imagine the tree as rising upward from its least node – its *root* – to terminate at its maximal elements – its *leaf* nodes. At each ascent in the tree, the node below is a genus and the nodes immediate connected to it above are its species. We shall say that the genus is *the parent of* its species. Formally, for the definitional regress from species to genus to be well-founded, the tree must have a root node. This root is traditionally called *the highest genus*. In the *Logic*, the species that serves this purpose is *being* or *substance*.[21] It was traditional to view the tree's

[15] *LAP* I:7, *KM* V 144–147, *B* 40–41.

[16] On the basis of similarity in language between the *Logic* and the commentary of Julius Pacius, Auroux argues that its authors were familiar with the *Isogoge*. See [7], p. 68, and [6].

[17] A real definition is fixed by nature. It is distinct from a *nominal definition* (*definitio nominale*), which is the conventional association of a spoken word with a term in mental language, i.e. an idea. See *LAP* I:12–13, II 16, IV 3–5.

[18] Strictly, as explained below, a difference is an adjective, which "signifies secondarily" a defining mode or modes that determine what it signifies in a primary sense. Secondary signification performs the function for an adjective that comprehension does for a noun.

[19] *LAP* II:16, *KM* V 243, *B* 126.

[20] *LAP* I:7, *KM* V 144–147, *B* 41.

[21] See *LAP* I:7, *KM* V 146, *B* 41:

leaves as individual substances, like Socrates and Plato. In the *Logic*, however, the tree's leaves are ideas that name individuals, i.e. they are proper names.[22]

The tree structure lays bare the definitional order of species. This structure turns on two assumptions. The first is that the same difference does not enter into the definition of more than one distinct species. Prima facie, the view that a difference can be used in only one definition seems false. Counter-examples are easy to find even among the geometrical species of extended substance that the *Logic* prefers to cite. A quadrangle and an equiangular triangle, for example, would seem to be distinct species of plane figure, but both are differentiated as having equal angles. This limitation of a difference to a single definition, however, dates to Aristotle. In *Parts of Animals*, for example, he maintains that *two-footed* must be considered ambiguous in the two species definitions of *two-footed human* and *two-footed bird*.[23]

The second assumption, which is also not obvious, is more basic. A species and its difference must signify the same objects outside the mind. Equivalently, they must have the same extension:

> ... it is clear that the *difference* constitutes the species and distinguishes it from other species, *it must have the same extension* as *the species*. Thus they must be able to be said reciprocally of each other, for example, that everything that thinks is mind, and everything that is mind thinks.[24]

We can reformulate this doctrine in the extended vocabulary of the previous section. A species and its difference must have the same significance range. To grasp the full import of this principle, it is necessary to say more about the semantics of adjectives.

Differentiae are ideas, but in mental grammar they are classified as adjectives. Although in Latin grammar there is not much difference between an adjective and a noun, the *Logic* follows the medieval tradition of classifying adjectives as "connotative" terms. As the name suggests, an adjective signifies in two ways. In the jargon of the *Logic*, an adjective *signifies primarily* all actual objects that instantiate that mode (or modes) that it *signifies secondarily*. In definitional order *secondary signification* is the more primitive. It is determined by an assignment, which is fixed by Providence, that pairs with each adjective a mode or series of modes. These modes function for an adjective in the way comprehension does for a noun. They determine what the adjective signifies in the primary sense. An adjective *signifies primarily* all those actual subjects that instantiate all the modes that it signifies secondarily.[25] The modes that define an adjective are either necessary or contingent. We have already met examples of necessary modes in the differentiae of species. Contingent modes are called *accidents*.

In short, secondary significata are to adjectives as comprehensions are to nouns. They are essentially sets of modes; they uniquely identify an adjective, and they determine what it (primarily) signifies in the world. Secondary significata stand in a one-to-one correspondence to adjectives, just as comprehensions do to nouns. Therefore, the set–inclusion relation on secondary significata, like that on the comprehensions of nouns, determines a partial order on adjectives. This ordering, then, is part of the broader partial ordering on ideas in general.

There is also a genus that is not a species, namely the highest of all genera. Whether the genus is being or substance is unimportant, and is more a question of metaphysics than logic.

[22] Proper nouns count as common nouns for logical reasons. According to the *Logic*'s treatment of the syllogistic, particular affirmative propositions are a special case of universal affirmatives (*LAP* II, 3 and III, 9), and thus by definition both have universal terms as subjects.

[23] For Aristotle's version see *Parts of Animals* 3, 642b20–643a20.

[24] *LAP* I:7, *KM* V 147–148, *B* 42.

[25] The use of the terms "primary" and "secondary" to describe the types of adjective signification should not obscure the fact that primary signification is defined in terms of secondary. The terminology derives from Aristotle's doctrine that substances, which are signified primarily, exist in the primary sense, and that modes, which exist only in substances and which make up comprehensions, exist in a secondary sense.

It is now clear how a species and its difference may be distinct but have the same extension. A species and its difference are distinct due to the identity criterion for ideas. The set of modes that the difference comprehends is a proper subset of those that the species comprehends. For example, *human* is defined as *rational animal*. The comprehension of rational is a proper subset of that of *human*: $\{\textit{rationality}\} \subset \{\textit{rationality}, \textit{animality}\}$. A species and its difference, on the other hand, have the same extension because they signify the same actual entities. The modes in the comprehension of the species are true of the same actual objects in the secondary signification of the difference. This coincidence happens in part because the difference cannot enter into the definition of a second species.

We may now draw out the implications of these principles for the structure of species. Like all ideas, species are ordered by the containment relation defined on their comprehension-sets. In the case of species, however, this ordering takes the form of a finite tree.[26] That it does so follows from the definition of species comprehension. This definition may be recast in modern terms in the form of an inductive definition. The differentia of *being* or *substance*, the highest genus, occupies the root node of the comprehension tree. The comprehension-sets of subsequent genera branch at each node into a finite number of immediate descendant species. At the division the comprehension of the genus is augmented for each species-comprehension by the addition of its difference. In practice the division is usually binary, and as we shall see in Section 5, the difference added to the comprehension of one species is in some cases the privative negation of the difference added to the other. The branching stops at the tree's leaf nodes, which are proper names.

The *Logic*'s account of species-comprehension may be straightforwardly recast as an inductive definition in the modern sense. The translation conforms so closely to the original – it is largely a matter of reorganizing various disparate remarks – that, to use the terms introduced earlier, it counts more as a paraphrase than an extension of the theory.

The formal definition presupposes a family $\{D_i\}_{i \leq n}$ of pairwise disjoint finite sets of modes (species differences) such that $D_1 = \{d\}$. The intended interpretation of D_1 is that it is the comprehension of the highest genus, *being* or *substance*, and of $\{D_i\}$ that it contains the differentiae of species of rank i in the tree. The inductive definition of the set C of *species-comprehensions* is:

$D_1 = \{d'\}$ is in C, and $\{d'\}$ is of rank 1.

For any comprehension c of rank m in C, if $m < n$, then for any $d_i \in D_{n+1}$, $c \cup \{d_i\}$ is in C and is of rank $m + 1$.

Nothing else is in C.

It is easy to see that the subset relation on C determines a tree. Trivially the relation is a partial ordering. Because the number of differentiae at a rank is finite, the inductive clause insures that the number of a node's descendants is finite. The assumption of distinct differentiae, one for each species, insures that no species has more than one parent. That the partition of differentiae is finite insures that each branch terminates in a finite number of steps. The structure, therefore, is a finitely branching finite tree. Because each comprehension uniquely determines an idea, the comprehension tree maps 1–1 and onto a corresponding tree of species ordered by idea-inclusion. This is the *Logic*'s version of the tree of Porphyry.

It should be mentioned that although the definition stipulates that D_1 is non-empty and the *Logic* names the highest genus (*being* or *substance*), it does not discuss its comprehension. To be formally adequate, one of two alternatives would suffice. Its comprehension could be a general mode true of existing substance only. Equally, it could be an "empty" comprehension. The authors do not discuss the issue. In general they avoid

[26] By a (*rooted*) *tree* Tr is meant a partially ordered structure $\langle T, \leq, \mathbf{0} \rangle$ such that for any $t \in T$, $\{x \mid x \leq t\}$ is well-ordered and $\mathbf{0}$ is the unique $\leq$-least element in T (the *root* of Tr). An element t of T is *a leaf* (*node*) of Tr iff for any element t' of T, if $t \leq t'$, then $t = t'$. For any non-leaf node of t of Tr, *the set of children of* t, briefly $c(t)$, is $\{t' \in T \mid t < t' \& \sim \exists t'' \in T(t < t'' < t')\}$. For any t' in $c(t)$, *the parent of* t' is t. The set of $\leq$-*descendants of* $t \in T$ is $\{t' \in T \mid t \leq t'\}$. Tr is *finite* if T is finite; Tr is *finitely branching* if for every t, $c(t)$ is finite. In a finite tree, $c(t)$ is well-defined and non-empty for any non-leaf node.

what they regarded as issues of arcane ontology.[27] We shall see below, however, that which alternative is correct has implications for idea structure.

It should also be said that although it was common in the Middle Ages for commentators on the *Isagoge* to remark that the structure of genera and species forms a tree, neither Porphyry nor the *Logic*'s authors explicitly make use of the tree metaphor. Rather, the authors' structural remarks about genera and species must be abstracted from their remarks on the nature of a real definition. It is true that this theory entails that species form a tree, but attributing to the order the formal properties of a tree in the modern sense is an extension of the original.

Before turning in the next section to the issue of whether the *Logic*'s authors envisaged lattice operations on ideas, let us summarize what we know so far about the structure of ideas, both in general and in the special case of species. Every idea has a comprehension or secondary signification, and a comprehension is essentially a set of modes. The family of these sets of modes is partially ordered by set–inclusion. This ordering induces an isomorphic "containment" relation on ideas-in-general because each comprehension-set determines a unique corresponding idea. Through signification the partial ordering of comprehensions is mapped many-one and antitonically onto significance ranges ordered by set–inclusion. Significance ranges ordered by set–inclusion are in turn isomorphic to extensions ordered by set–inclusion. It follows that ideas ordered by containment are dual to extentions ordered by inclusion. Moreover, within ideas there is a special substructure of species. The set–inclusion relation on species-comprehension-sets forms a finitely branching tree and induces an isomorphic structure on species themselves.

Thus, although species have a good deal of structure, it is tree- rather than lattice-like. Many ideas, moreover, are not species and therefore are not nodes in the tree of Porphyry. Non-species include impossible ideas like *golden mountain* and contingent adjectives like *prudent*. Differentiae themselves, although necessary, also do not form a node in the species tree. The partial ordering on ideas, therefore, is in general not a tree. In addition, there is no evidence so far that the *Logic*'s authors envisaged a least or a greatest idea, or that comprehensions includes every possible combination of modes. Nor is there evidence so far that the structure of comprehensions, and hence of ideas, constitutes a Boolean algebra. In Sections 3 and 4 we will consider how restriction and abstraction provide additional structure, and whether they amount to lattice or Boolean operations.

3. Abstraction

Both Auroux and Dominicy interpret abstraction as described in the *Logic* as a meet operation in the algebraic sense. This reading in my view goes beyond the text. In accordance with the standard medieval accounts, the *Logic* describes abstraction as an operation of the soul.[28] In the *Logic*'s non-Aristotelian account of the origin of ideas, ideas have three possible causes. Some are innate and placed in the soul by God. Some, which we may call *sensations*, are caused to be instantiated in the soul by God on the occasion of movements in the body's sensory organs. Some are caused by the operations of abstraction and restriction.[29] Abstraction consists of forming a new abstract idea from a prior more particular idea or sensation. An abstract idea is also called a general idea or common noun.

[27] See *LAP* I:7, *KM* V 146, *B* 41, and also the remarks on realism at *LAP*, *Discours* I, *KM* V 112–113, *B* 11–12, and the more general remarks at *LAP* IV:1, *KM* V 358, *B* 230.

[28] The *Logic*'s account of abstraction follows the logical tradition closely. The *locus classicus* of the medieval doctrine is Aquinas, *Summa Theologica* I.I,Q. 85. A full account of the doctrine is found in Buridan. See [10], AdA III, 8, 2, pp. 650–652. For an account roughly contemporary to that of the *Logic* and readily available at the time, see [13] [*reprint of 1607*], Liber I, Q.V, A. I, p. 143. The only remarkable way in which the *Logic* differs from earlier accounts is in its insistence that the "less" rather than the "more" general idea is abstracted first from sensation. Aquinas and Buridan hold the reverse, that the more general concept is abstracted first. The *Logic*'s version may be said to fit better its "algebra" of ideas in which simpler ideas are "constructed" up the quasi-lattice by progressive "cutting away." On abstraction see *VFI* 6, *KM* I 207–210, *G* 74–76; 11 *KM* 234–235, *G* 98–100; *LAP* I:5, *KM* V 142–143, *B* 37–38; *LAP* I:11, *KM* V 168–170, *B* 58–59.

[29] For a discussion of the origin of ideas in the *Logic* see Nadler [19].

The *Logic* goes beyond earlier accounts by defining abstraction in terms of comprehension. In its version, the mind generates a new idea by removing a mode or modes from the comprehension of a prior idea or sensation. The process is also called *prescinsion* ("cutting away"). It is helpful to quote one of the few passages in which the process is described:

> The third way of conceiving things by abstraction takes place when, in the case of a single thing having different attributes, we think of one attribute without the other even though they differ only by a distinction of reason. Here is how this happens. Suppose, for example, I reflect that I am thinking, and in consequence, that I am the I who thinks. In my idea of the I who thinks, I can consider a thinking thing without noticing that it is I, although in me the I and the one who thinks are one and the same thing. The idea I thereby conceive of a person who thinks can represent not only me but all other thinking persons. By the same token, if I draw an equilateral triangle on a piece of paper, and if I concentrate on examining *it on this paper along with all the accidental circumstances determining it, I* shall have an idea of only a single triangle. But if I ignore all the particular circumstances and focus on the thought that the triangle is a figure bounded by three equal lines ... [I am] able to represent all equilateral triangles. Suppose I go further and, ignoring the equality of lines, I consider it only as a figure bounded by three straight lines. I will then form an idea that can represent all kinds of triangles. If, subsequently, I do not attend to the number of lines, and I consider it only as a flat surface bounded by straight lines, the idea I form can represent all straight-lined figures. Thus I can rise by degrees to extension itself. Now in these abstractions it is clear that the lower degree includes the higher degree along with some particular determination, just as the I includes that which thinks, the equilateral triangle includes the triangle, and the triangle the straight-lined figure. But since the higher degree is less determinate, it can represent more things.[30]

[30] *LAP* I:5, *KM* V 143, *B* 38. In *On True and False Ideas* Arnauld develops an extended example, which, because texts on abstraction are rare, is worth quoting in full:

> The philosopher Thales, having to pay twenty workers one drachma each, counted twenty drachmas and paid each worker. He would not have been able to do this unless there were at least two perceptions in his mind: one of twenty men and one of twenty drachmas. And I remind you for the last time that *idea* and *perception* are the same thing in my dictionary, and thus that, when I make use of the expression 'idea' and 'idea of an object', I understand by this the *perception of an object*.
>
> Having some spare time he began to reflect, and thinking about what the two *perceptions* or *ideas* have in common, namely that there is 20 in both, he abstracts from what is particular in them the abstract idea of the number 20, which can subsequently be applied to twenty horses, twenty houses, twenty stadiums. This is a third idea or perception.
>
> He then takes it into his head to reflect on this abstract idea of the number 20, i.e. he considers it with greater attention with a reflective vision, which is one of the most admirable faculties of the mind. And the first thing he discovers is that it can be divided into equal halves, for he easily sees that if he puts 10 on one side and 10 on the other this makes 20. And he sees at the same time that if he adds 1 to 20, the number 21 cannot be divided into two equal parts, because the closest one can get to an equal division is to put 10 on one side and 11 on the other. This leads him to judge that it is well to distinguish by different words the numbers that can and cannot be divided into two equal halves [*qu'il est bon de distinguer, par des mots particuliers, les nombres qui se peuvent ou ne se peuvent pas partager; en deux moitiés égales*], calling these *even* and *odd* respectively. Then, still considering what is contained in (*encore enfermé*) this idea or perception of the number 20, he asks what its factors are, i.e. what numbers taken together make exactly 20.
>
> He begins with *unity*, and sees immediately that unity must be one of the factors, since 1 taken twenty times makes 20. From this it is easy to derive [*aisé de faire*] the general rule that 1 is a factor of all numbers, since it is its own factor, 1 being 1, and all the other numbers are only definite multitudes of 1s.
>
> Next he takes 2, and finds that 2 is a factor of 20, for in counting in 2s – 2, 4, 6, etc. – he arrives at exactly 20. He takes 3 and discovers that this is not a factor of 20, for counting in 3s – 3, 6, 9, 12, etc. – he finds that after having done this six times he arrives at 18, after which there is only 2 left before 20. He then takes 4 and finds that it is a factor of 20, because 4 taken five times is exactly 20. He finds the same for 5, for 5 taken four times is exactly 20. He next finds that neither 6, 7, 8, 9 can be factors of 20, for the same reason as he found 3 not to be. But 10 is a factor because 10 taken twice is 20. Neither 11, 12, 13, 14, 15, 16, 17, 18, nor 19 can make exactly 20, when taken any number of times, so they cannot be a factor of it. But 20 can be a factor because 20 taken once is 20. From this various reflections follow:
>
> First, because there can be numbers having no other factor than unity and themselves, it is a good idea to give them a name which distinguishes them from others and they can be called prime *numbers*.
>
> Second, all the even numbers, since they are divisible into two equal parts, have 2 as a factor.
>
> Third, 2 is the only even number which is *prime* because it alone of all the even numbers has only unity and itself as a factor.
>
> *VFI* 6, *KM* I, 207–208, *G* 74–75 (translation adapted by the author).

According to the passage, a new idea is distinguished by a new comprehension, and this comprehension is formed by removing modes from the comprehension of a prior idea. An immediate consequence is that some abstracted ideas do not fall on the tree of Porphyry. There are several cases.

One type that we have already met consists of accidents abstracted from sensation. For example, the idea *sinner* abstracted from my sensation *Peter sinning by denying Christ* has in its comprehension the mode *sinning*, which is an accident. There is no idea *sinner* in the tree of Porphyry.

A second case consists of abstractions from species. Although the idea *man*, which has the comprehension {*rational, animate, living, substance*}, falls on the tree, the abstract idea with the comprehension {*rational, corporeal*} is not a species. The theory entails that the two ideas are coextensive, however, because both have the extension of *rational*. On the other hand, the two comprehensions are distinct sets, and hence by the criterion of idea identity, the two are distinct ideas.

Both Auroux and Dominicy generalize from the text to classify abstraction as a meet operation relative to idea containment. This translation in modern terms is an extension of the *Logic*'s account in the sense previously distinguished. Their reasoning is as follows. Ideas are uniquely determined by their comprehensions, and thus any structure on comprehensions induces an isomorphic structure on ideas. Comprehensions, moreover, are essentially sets of modes and as such are ordered by set–inclusion. The text, therefore, seems to be clearly committed to the principle that idea A contains idea B (briefly $B \leq A$) if, and only if, the comprehension of B is a subset of that of A. Moreover in set theory $b \subseteq a$ iff $\exists c b \cup c = a$ iff $\exists d a \cap d = b$. Hence, given that ideas correspond 1–1 to sets of modes and that $\cap$ is well-defined on sets, a corresponding greatest lower bound operation $\wedge$ is well-defined relative to the order $\leq$ on ideas. In brief, Auroux and Dominicy seem to reason that because the *Logic* accepts the inference that when A is abstracted from B, the comprehension of A is contained in that of B, it is also committed, at least implicitly, to recognizing the existence of a meet operation $\wedge$ on ideas with its various properties. These include association, commutation, and idempotence. Moreover, since $\wedge$ is understood to be closed, the operation is interdefinable with $\leq$.

There is, however, no textual evidence that the *Logic*'s authors themselves drew these algebraic implications. They refer to abstraction only in the manner of medieval logicians as a mental operation that when applied to a single idea, produces a new idea. That is, abstraction for the Cartesians is like abstraction in earlier logic. It is monadic, not dyadic. Nor do the authors explicitly mention association, commutation, or idempotence, properties definitive of the modern operation. In short, abstraction as understood by the *Logic*'s authors is not described as a matrix operation in the modern sense. It is more accurate to say that abstraction is an example, one of several, in which the *Logic* redefines a concept from earlier psychology by appeal to comprehension. It is explained as a process that generates ideas in the soul by the removal of modes from comprehensions. These are explanatory terms from psychology, not algebra. One algebraic observation that does fit the *Logic*'s account, which is perhaps implicit in what the authors do say, is that A is an abstraction from B only if B is contained in A.[31]

A related issue is whether, if abstractions are included in the set of ideas, the set of ideas has a least element. A minimal element of this sort is posited in the extensions of Auroux and Dominicy.[32] As we have seen, the *Logic* is explicit that *being* or *substance* is the highest genus. It is the root of the tree of Porphyry. We have seen, however, that some abstractions are not nodes in the tree. Is the comprehension of *being* included in the comprehensions of all ideas, species and non-species? The issue turns on the comprehension of *being*, a topic the *Logic* does not address. If this comprehension is the empty set, then using modern reasoning, it would follow that its comprehension is included in that of every idea. On the other hand, if *being* contains some mode true of every existing thing, there is reason to think that this mode is not part of the comprehension of every abstract noun or noun phrase. A noun may be a complex false idea and

[31] The converse fails because A may be contained in B, yet B not be caused by abstraction. B may be a sensation, innate or caused by restriction.

[32] Auroux [7], p. 94 and Dominicy [14], p. 43 ff.

comprehend a set of modes jointly true of nothing.[33] The false idea of *golden mountain*, for example, may be abstracted from *golden mountain in El Dorado*. Other examples come from errors due to sensation. My idea of *pain caused by fire* may be abstracted from the ill-conceived idea I formed in childhood of *pain in my finger caused by fire*.

The issue has implications for the algebra of ideas. If there is no minimal idea in general – an issue the *Logic* does not discuss – and the number of ideas is finite, then a binary meet operation defined in terms of containment would not be closed, and ideas would not meet the defining conditions of a meet semi-lattice.

All in all, it is unlikely that the *Logic*'s authors would have held without comment the distinctly modern view that an empty set is contained in every set, a property that would be necessary if the empty root were to be part of the order of comprehension-sets. The *Logic*, moreover, is committed to the view that there are false ideas, which are defined as true of nothing that exists. It is a stretch, then, to suggest that the *Logic*'s authors posit a minimal idea or to view ideas as conforming to a meet semi-lattice.

4. Restriction

Auroux and Dominicy interpret restriction in the *Logic* as a join operation on ideas. Restriction had been a part of logical theory since the Middle Ages, and was a standard topic in the logic of the period.[34] The *Logic* follows this tradition by describing restriction as an operation in mental grammar by which, in modern grammatical terminology, a complex noun phrase is formed by conjoining an adjectival phrase or relative clause to a simpler noun phrase. Because terms in mental grammar are ideas, restriction is an operation on ideas. The *Logic* describes the process as follows:

> Occasionally we join a term to various other terms, composing in the mind a complete idea, of which one can often affirm or deny what could not be affirmed or denied of each of these terms taken separately. For example, these are complex terms: “a prudent person,” “a transparent body,” “Alexander son of Philip.”[35]

Definite descriptions are a special case in which a head noun is restricted by adjectives or relative clauses to the extent that the modes in their combined comprehensions (or secondary significata) are jointly true of only a single individual.[36] What is novel in the *Logic*'s account is the explanation of restriction by appeal to comprehension. The restricted noun phrase has as its comprehension the combination – essentially the set theoretic union – of the comprehensions of its component ideas.

The *Logic* goes on to provide a new analysis of the traditional distinction between the two types of restriction called explication and determination.[37] Again the explanation is in terms of comprehension. An *explication* is a restriction in which the comprehension of the modifying adjective adds nothing to that of

[33] On false ideas see *LAP* I, ii, *KM* V 136, *B* 32; *Discours* I, *KM* V, 110, *B* 9–10; I, ix. *KM* V, 157–178; *B* 49–50; I, xi. *KM* V, 168–170; *B* 58–60. See also the discussion in Martin [16].

[34] For a description of restriction in medieval logic see Buridan, *Treatise on Supposition* 4.1.46–47 and 4.4.63, and *Treatise on Consequence* 6.3.1 in [9] and [11], Book III, pp. 286, 648, and 835. For a roughly contemporary account see [15], Liber VIII, Caput 40, pp. 740–741. Restriction is also the operation by which, according to Descartes, the soul acquires factitious ideas: *Meditation* III, *B* IXa, 29, and Letter to Mersenne *AT* III 382–383.

It has become a common interpretation of the *Logic* that its authors distinguish a second operation of restriction, one that fashions a new idea by combining the comprehension of a well-defined idea with the modes of what is called “an indeterminate idea” ([20], pp. 237–238 and [7], p. 74). I have argued elsewhere that this multiplication of senses is unnecessary [17]. There are no texts that explain, as such, an “indeterminate idea” or a second sense of restriction. There are only three passages, all very short, that refer to restriction using the expression “indeterminate idea” (*LAP* I:7, *KM* V 147–148, *B* 41–42; *LAP* I:7, *KM* V 150, *B* 44; *LAP* II 3, *KM* V 199, *B* 83). Rather than referring to a second unexplained category of idea and a second unexplained sense of restriction, “indeterminate idea” in these passages is better understood as equivalent to “some idea.” It merely indicates a metalinguistic existential quantification over ideas-in-the-usual-sense subject to restriction-in-the-usual-sense.

[35] *LAP* I:8, *KM* V 151–152, *B* 44–45.

[36] See Pariente [21].

[37] For a statement of the distinction as drawn in earlier logic see [11], p. 286.

the original noun in the sense that it fails to narrow the set of entities signified. This occurs if the adjective's secondary signification contains modes that are accidentally true of the entities signified by the head noun. A *determination* is a restriction in which the modifying adjective narrows the significance range of the head noun by comprehending modes that are true of only a proper subset of the head noun's significance range. Determination functions, the authors say,

> by joining another distinct or determinate idea to it [the head noun], as when I join the idea of having a right angle to the general idea of a triangle. Then I narrow this to the single species of a triangle, namely to right triangle.[38]
>
> . . . [*D*]*etermination*] occurs when the addition to a general word restricts its signification and causes it no longer to be taken through its entire extension, but only for a part of it, as when I say, "transparent bodies," "knowledgeable people," "a rational animal." These additions are not simple explications but determinations, because they restrict the extension of the first term, causing the word "body" to signify no more than some bodies, the word "people" only some people, and the word "animal" only some animals.[39]

Restriction of species provides another example of ideas that are not species and hence not on the tree of Porphyry, for example, *prudent man* and *prodigal man*.

Unlike medieval abstraction, restriction is plausibly a two-place operation, one that takes two terms as arguments and yields a unique term as value. Semantically, the grammatical operation of restriction corresponds to the union operation on comprehensions. Since comprehension-sets map 1–1 in an order-preserving way (isotonically) onto ideas, a full Boolean operation of set union on comprehensions would determine a corresponding join operation on ideas. If, in addition, the structure of ideas were closed under restriction, restriction would determine a join semi-lattice of ideas ordered by the containment relation.

The issue of closure, however, presents an interpretive problem. There is no question that restriction is important in the formation of complex ideas. We have already discussed restriction's role in the definition of a species and in the distinction between explication and determination. It is also important in the *Logic*'s truth-conditions for categorical propositions. A universal affirmative, for example, is said to be true if the extension of the predicate restricted by the subject is identical to that of the subject. Restriction is also key to the generation of false and confused ideas.[40]

The authors of the *Logic* themselves, however, do not address whether ideas in general are closed under restriction. The extensions of Auroux and Dominicy do postulate a maximal restriction – a kind of ultimate contradictory idea, which Auroux calls *Monde*. It would have as its comprehension the set of all modes.[41] Postulating a maximal restriction would be an consistent extension of the existing text, but the authors themselves do not do so. The reason they do not, I suggest, is that they were not concerned with the issue of closure, which would be important if they were concerned to insure that ideas formed a semi-lattice. As in the case of abstraction, attributing lattice structure to the *Logic*'s ideas, would be an elaboration of the theory that the authors themselves did not envisage.

Before turning to "idea negation" in the next section, it will be helpful to summarize what we know so far about the "algebra" of ideas. On the one hand, ideas possess some positive structure. They are identified by their comprehensions, which are essentially sets of modes that are partially ordered by the inclusion relation. This order is isomorphic to the containment relation on ideas. There is a many-one antitonic mapping from comprehensions to significance ranges and a 1–1 isomorphic correspondence between significance ranges and

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[38] *LAP* I:7, *KM* V 147, *B* 40.
[39] *LAP* I:8, *KM* V 151–152, *B* 44–45.
[40] On the use of restriction in the statement of truth-conditions see the axioms and discussion in *LAP* II, 17–19, *B* 129–133, and [17].
[41] See Auroux, p. 94 and Dominicy p. 43 ff.

extensions. It follows that ideas are dual to extensions. Species and their comprehensions are structured according to a version of the tree of Porphyry, a rooted, finitely branching, finite tree. These are genuine examples of structure.

There are, however, structural properties that ideas lack. Many non-species do not occupy nodes on the tree of Porphyry. Neither abstraction nor restriction are lattice operations. Abstraction is not binary, and the *Logic*'s authors do not posit a "most abstract" idea. Although restriction is binary, it is only partially defined, and the *Logic*'s authors do not posit a maximally restrictive idea.

5. Negation

At this point enough evidence has been gathered to show that idea structure in the *Logic* is much more like that of traditional logic than that of modern algebra. There remains only "idea negation" to consider. Did the *Logic*'s authors conceive of a complementation operation on ideas in the modern sense? Both Auroux and Dominicy discuss the issue at length. They agree, for different reasons, that the *Logic*'s account of term negation is incoherent. Auroux's criticism is metatheoretic. He argues that a complementation operation in the modern sense would be inconsistent with the *Logic*'s broader metatheory. Dominicy's objections are textual. He argues that when the *Logic* makes use of term negation, its account is equivocal because it systematically confuses bivalent sentence negation with non-Boolean term negation. I have investigated the issue elsewhere and draw a conclusion more consistent with the results of the earlier sections of this paper.[42] I agree with Auroux that the *Logic*'s term negation is not a complementation operation, but argue that properly understood, it is perfectly consistent with the *Logic*'s wider theory. It is not a modern complementation operation but rather a variety of privative negation, an operation familiar to logic since Aristotle, with properties quite different from those of lattice complementation. I also argue that in the texts Dominicy cites it is quite clear that the authors do not equivocate. What is true is that in some places they use sentence negation in the form of negative categorical propositions and in others they use privative term negation. They do so quite intentionally to say different things. The role of privative negation is to generate negative species within the tree of Porphyry. Where is occurs, it imparts order to the species under a genus. This is not the place to rehearse in full the arguments for these interpretive points. In the context of this general review of structure in the *Logic*, however, it is appropriate to review an example of idea negation and sketch its role.

Texts in the *Logic* mentioning term negation are rare. The key texts on which Dominicy bases his interpretation concern the division of the genus *animal* into the two species *humans* and *beasts*. The species *humans* is defined by the difference *rational* and *beasts* by *non-rational*.

> Finally, we should note that it is not always necessary for the two differences dividing a genus both to be positive, but it is enough if one is, just as two people are distinguished from each other if one has a burden and the other lacks one, although the one who does not have the burden has nothing the other one does not have. This is how humans are distinguished from beasts in general, since a human is an animal with a mind, *animal mente praeditum*, and a beast is a pure animal, *animal merum*. For the idea of a beast in general includes nothing [63] positive which is not in a human, but is joined only to the negation of what is in a human, namely the mind. So the entire difference between the idea of an animal and the idea of a beast is that the comprehension of the idea of an animal neither includes nor excludes thought – the idea even includes it in its extension because it applies to an animal that thinks – whereas the idea of a beast excludes thought from its comprehension and thus cannot apply to any animal that thinks.[43]

[42] Ref. [18].

[43] *LAP* I:7, *KM* V 148–149, *B* 42–43. Dominicy [14], p. 45 ff. also discusses two similar sets of text and makes the same points. For the originals see Arnauld [2], Livre 5, Chap. 1, pp. 351–352, and Descartes' *Meditations*, which influenced the *Logic*: *Adam (1897–1909)* [1] IXa 42, 60–61, and *Adam (1897–1909)* [1] IXa, 62, 96–97.

Dominicy correctly reads the authors as employing negation in two senses. The genus *animal* is divided into two species. The first is defined by the difference *rational*, and the second by *non-rational*. Describing the division, the authors use negation in two different senses. They say,

> a human is an animal with a mind, *animal mente praeditum*, and a beast is a pure animal, *animal merum*,

What the authors mean may be stated in the technical vocabulary of the *Logic*. The idea *thought* is not part of the comprehension of the idea *animal*. This metalinguistic fact may be expressed in the object language by the negative categorical proposition *no animal thinks*. In modern terms the negation occurring here is essentially a bivalent sentence negation. The authors go on to say,

> ...the idea of a beast in general includes nothing positive which is not in a human, but is joined only to the negation of what is in a human, namely the mind... [B]east excludes thought from its comprehension...

They are making the point that the species *beast* has as part of its comprehension the privative negation *non-rational*. This metalinguistic fact is expressed by an object language universal affirmative *every beast is non-rational*. This proposition has as its predicate the privative negation of the term *rational*. In another text Arnauld says of this division the privative species *beast* is less "noble" than *man*. He says,

> the more noble [*man*] contains all that is in the less noble [*beast*], and such that they differ only in that the more noble has something that the other does not.

The use of privative negation in division has its origin in Aristotle. The Stagyrite introduces privative negation at *Categories* X and makes use of it throughout his work.[44] The negated term, he says, is predicated of a subject that lacks a property it would normally or naturally possess,[45] and that this lack is an "imperfection".[46] In the *Topics* he remarks that one of the ways to define species is to divide a genus into two by choosing as the differentia of the second the privative negation of the first.[47] The observation that a genus may be divided into two species by defining the second by the privative negation of the differentia of the first became a standard feature of medieval logic. John Buridan describes the process as follows:

> For if a term that is dividing and a finite term that is divided are related to each other by univocal predication, then one of them will be a genus and the other its species or difference, or one will be a species and the other its individual. The infinite term, however, will be taken for the other single species or the other several species, for the other difference or differences, or for the other individual or individuals.
>
> This happens sometimes because some species or difference does not have a positive name imposed on it, as when we say that of sounds some are utterances and others are non-utterances. And that a name is not imposed sometimes occurs on account of our not knowing the species or difference. It is because of this mode of division that sometimes a species is defined by means of its genus and the negations of another species, or several other species, or their differences, as when Porphyry says that an accident is a predicable that is neither a genus, nor a species, nor a difference, nor a property, or if we said that brute is a non-rational animal.[48]

[44] *Categories* 12a30, *Categories* 11b15, *Metaphysics* 1022b22.
[45] 1011b23, 11b15.
[46] 1022b29.
[47] 109a34.
[48] *Summulae* 8.1.8, [11], p. 628.

Buridan says that when the Latin negative *non* is applied to a species or difference, it is not an "infinite negation" (i.e. bivalent set complementation), but is rather privative negation. He remarks further that we employ privative negation when the language does not possess a lexicalized privative for the species or difference. This is exactly the role of the privative *non-rational* in the division of the genus *animal*. It distinguishes the species *beast* from *human* within the wider genus. Buridan makes clear, as Arnauld does, that this division is such that the first species is more "perfect" or "noble" than the second: "we say that [a human] is a more perfect, or more noble, animal than a horse."[49]

The implications of the discussion for the wider structure of ideas is clear. The authors make no mention of a generally defined complementation operation on ideas of the sort familiar from Boolean algebra. Rather, to the extent that they recognized a term negation at all, it is a version of traditional privative negation and is defined only for differences within the tree of Porphyry. It follows that, to this degree, there are negative ideas. These too would we classified as adjectives and count as a variety of non-species.

6. Conclusions

What has become clear is that Aristotelianism retains a strong grip on Cartesian logic. It is true that Descartes' metaphysics was revolutionary in important ways. Its divorce of spirit from matter, in particular, motivated the reconception of signification in terms of comprehension. The containment relation on comprehensions in turn introduced a new conception of order among ideas and through signification on extensions as well. Operations on ideas, however, remained largely traditional. Although now defined in terms of comprehension, they remained operations of the soul. There is no recognition that they possess the formal properties of modern matrix operations. The prominent place of genera and species is perhaps the most striking residue of Aristotelianism. Implicit in the theory of species definition in terms of comprehension is a detailed account of structure. Species form a finite tree in which some species are ordered under their genus by privative negation. Apart from the explanation in terms of comprehension, however, there is little here that is new. The parts that are new are interesting contributions to the logic of the times, but they share very little with modern algebra.

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