Exact multiplicity of positive solutions for concave-convex and convex-concave nonlinearities

Philip Korman
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025
and

Yi Li
Department of Mathematics and Statistics
Wright State University
Dayton Ohio 45435

Abstract

This note gives an unified treatment of the exact multiplicity results for both \(S\)-shaped and reversed \(S\)-shaped bifurcation for positive solutions of the two-point problem

\[ u'' + \lambda f(u) = 0, \quad \text{for} \ -1 < x < 1, \quad u(-1) = u(1) = 0, \]

depending on a positive parameter \(\lambda\), for both concave-convex and convex-concave nonlinearities \(f(u)\).

Key words: Exact multiplicity of positive solutions, \(S\)-shaped and reversed \(S\)-shaped bifurcation.

AMS subject classification: 34B18.

1 Introduction

In this note we give an unified treatment of the exact multiplicity results for both \(S\)-shaped and reversed \(S\)-shaped bifurcation for positive solutions of the two-point problem

\[(1.1) \quad u'' + \lambda f(u) = 0, \quad \text{for} \ -1 < x < 1, \quad u(-1) = u(1) = 0.\]
We show that in essentially the same manner one can derive the result of
the authors on $S$-shaped bifurcation [9], and the recent result of K.C. Hung

Recall that positive solutions of (1.1) are even functions, with $u'(x) < 0$
for $x > 0$, so that they take the global maximums at $x = 0$. Moreover, the
maximum value $\alpha = u(0)$ is a global parameter, i.e., it uniquely identifies
the solution pair $(\lambda, u(x))$, see e.g., [8]. It follows that the two-dimensional
graph of $(\lambda, \alpha)$ gives a faithful representation of the solution curves.

We define $f(u)$ to be convex-concave if there is an $\gamma \in (0, \infty)$, such that

$$f''(u) > 0 \text{ for } u \in (0, \gamma), \quad f''(u) < 0 \text{ for } u \in (\gamma, \infty).$$

Similarly, $f(u)$ is called concave-convex if there is an $\gamma \in (0, \infty)$, such that

$$f''(u) < 0 \text{ for } u \in (0, \gamma), \quad f''(u) > 0 \text{ for } u \in (\gamma, \infty).$$

The following result is known, and it has proved to be quite useful, see
P. Korman, Y. Li and T. Ouyang [10], or T. Ouyang and J. Shi [12] (it also
holds for balls in $\mathbb{R}^n$)

**Theorem 1.1**

(i) Assume that $f(u)$ is convex-concave, and $f(0) \leq 0$. Then
the global solution curve admits at most one turn in the $(\lambda, \alpha)$ plane. More-
over, the turn is to the right.

(ii) Assume that $f(u)$ is concave-convex, and $f(0) \geq 0$. Then the global
solution curve admits at most one turns in the $(\lambda, \alpha)$ plane. Moreover, the
turn is to the left.

In this note we shall consider the cases when $f(0)$ has a “wrong sign”,
i.e., either $f(u)$ is concave-convex and $f(0) < 0$, or convex-concave, and $f(0) > 0$. Following P. Korman and Y. Li [9], we give conditions under
which the solution curve makes exactly two turns, so that it is either $S$-
shaped, or reversed $S$-shaped. We show that the argument in P. Korman
and Y. Li [9] can cover both cases. We also simplify that argument in several
places. We show that the reversed $S$-shaped bifurcation can be seen as a
“dual version” of the $S$-shaped bifurcation. In particular, we easily recover
one of the main results of K.C. Hung [6], and suggest a generalization. We
also provide some extra information on the reversed $S$-shaped curve. K.C.
Hung [6] also discusses the broken reversed $S$-shaped curves, in case $f(u)$
is concave-convex and has three roots (Theorems 1.2 and 2.2 in [6]). Our
results imply the Theorem 1.2 from that paper (which originated in [13]),
but not the stronger Theorem 2.2.
2 Solution curves with at most two turns

We shall need the linearized problem corresponding to (1.1)

\[ w'' + \lambda f'(u)w = 0, \quad -1 < x < 1, \quad w(-1) = w(1) = 0. \]  

(2.1)

Recall that \((\lambda, u(x))\) is called a critical point (or a singular solution) of (1.1), if the problem (2.1) admits non-trivial solutions. In such a case \(w(x)\) is an even function, and it does not change sign (thus for the rest of this paper we assume that \(w(x) > 0\) for all \(x\), see e.g., [8]. We assume that \(f(u) \in C^2[0, \infty)\), and define a function \(I(u) = f^2(u) - 2F(u)f'(u)\), where 

\[ F(u) = \int_0^u \int_t^1 f(t) dt. \]

The following lemma has originated from P. Korman, Y. Li and T. Ouyang [10], and P. Korman and Y. Li [9]. (Since both \(u(x)\) and \(w(x)\) are even functions, we may restrict the integrals below to the interval \((0, 1)\).)

**Lemma 2.1** (i) Assume that \(f(u)\) is convex-concave and there is a \(\beta > \gamma\), such that 

\[ I(\beta) = f^2(\beta) - 2F(\beta)f'(\beta) \geq 0. \]  

(2.2)

Let \((\lambda, u)\) be any critical point of (1.1), such that \(u(0) \geq \beta\), and let \(w(x)\) be any non-trivial solution of the linearized problem (2.1). Then 

\[ \int_0^1 f''(u(x))u'(x)w^2(x) dx > 0, \]  

(2.3)

\[ \int_0^1 f''(u(x))w^3(x) dx < 0, \]  

(2.4)

and the solution curve turns to the right.

(ii) Assume that \(f(u)\) is concave-convex and there is a \(\beta > \gamma\), such that 

\[ I(\beta) = f^2(\beta) - 2F(\beta)f'(\beta) \leq 0. \]  

(2.5)

Let \((\lambda, u)\) be any critical point of (1.1), such that \(u(0) \geq \beta\), and let \(w(x)\) be any non-trivial solution of the linearized problem (2.1). Then 

\[ \int_0^1 f''(u(x))u'(x)w^2(x) dx < 0, \]  

(2.6)

\[ \int_0^1 f''(u(x))w^3(x) dx > 0, \]  

(2.7)

and a turn to the left occurs.
Proof: We shall derive a convenient expression for the integral in (2.3). Differentiating (2.1), we get

\[ w'' + \lambda f'(u)w_x + \lambda f''(u)u'w = 0. \]  

(2.8)

Multiplying the equation (2.8) by \( w \), the equation (2.1) by \( w_x \), subtracting and integrating over \((0, 1)\), we express

\[ \lambda \int_0^1 f''(u)u'w^2 \, dx = w^2(1) - \lambda w^2(0) f'(u(0)). \]  

(2.9)

By differentiation, we verify that \( u''(x)w(x) - u'(x)w'(x) \) is equal to a constant for all \( x \), and hence

\[ u''(x)w(x) - u'(x)w'(x) = -\lambda w(0)f(u(0)), \quad \text{for all } x \in [0, 1]. \]

Evaluating this expression at \( x = 1 \), we obtain

\[ w'(1) = \frac{\lambda w(0)f(u(0))}{u'(1)}. \]  

(2.10)

Multiplying (1.1) by \( u' \), and integrating over \((0, 1)\), we have

\[ u'^2(1) = 2\lambda F(u(0)). \]  

(2.11)

Using (2.11) and (2.10) in (2.9), we finally express

\[ \int_0^1 f''(u)u'w^2 \, dx = \frac{w^2(0)}{2F'(u(0))} I(u(0)). \]  

(2.12)

Let us prove part (ii) of the lemma (part (i) is similar, and it was proved in P. Korman and Y. Li [9]). Since

\[ I'(u(0)) = -2F(u(0))f''(u(0)) < 0, \quad \text{for } u(0) > \beta, \]

we conclude that \( I(u(0)) < I(\beta) \leq 0 \), and, in view of (2.12), the inequality (2.6) follows. (Observe that \( F(u(0)) > 0 \) for any positive solution.)

Turning to the proof of (2.7), consider the function \( p(x) \equiv \frac{w(x)}{w'(x)} \). Since \( p(1) = 0 \), and since \( u''(x)w(x) - u'(x)w'(x) \) is equal to a constant for all \( x \), we have

\[ p'(x) = -\frac{u'(1)w'(1)}{u'(x)^2} < 0, \]
which implies that the function \( p(x) \) is positive and decreasing on \((0, 1)\). Let \( x_0 \) be the point where \( f''(u(x)) \) changes sign (i.e., \( u(x_0) = \gamma \), and \( f''(u(x)) > 0 \) on \((0, x_0)\), and \( f''(u(x)) < 0 \) on \((x_0, 1)\)). By scaling \( w(x) \), we may achieve that \( w(x_0) = -u'(x_0) \), or \( p(x_0) = 1 \). Then \( w(x) > -u'(x) \) on \((0, x_0)\), and \( w(x) < -u'(x) \) on \((x_0, 1)\), and using (2.6), we have

\[
\int_0^1 f''(u(x))w^3 \, dx > \int_0^1 f''(u(x))w^2(-u'(x)) \, dx > 0,
\]
concluding the proof.

We shall use the following lemma (see e.g., [8]).

**Lemma 2.2** Assume that \( f''(u) < 0 \) \((f''(u) > 0)\) for \( u \in (0, \delta) \), for some \( \delta > 0 \). Then only turns to the right (left) are possible on the curve of positive solutions of (1.1), while \( u(0) \in (0, \delta) \).

Recall that the maximum value of solution, \( \alpha = u(0) \), uniquely identifies the solution pair \((\lambda, u(x))\) of (1.1), and the solution set of (1.1) can be faithfully depicted by the planar curves in the \((\lambda, \alpha)\) plane. It is natural to ask: which way the solution curve travels through a given point \((\lambda, \alpha)\)? Define

\[
h(u) = 2F(u) - uf(u),
\]
where, as usual, \( F(u) = \int_0^u f(t) \, dt \). The following result is from P. Korman [7], see also the discussion in P. Korman [8] of the preceding results in [11], [2], and [14].

**Theorem 2.1** (i) Assume that

\[
(2.13) \quad h(\alpha) < h(u), \text{ for } 0 < u < \alpha.
\]

Then the positive solution of (1.1), with maximum value \( u(0) = \alpha \), travels to the left in the \((\lambda, \alpha)\) plane, i.e., \( \lambda'(\alpha) < 0 \). (This solution is unstable, see P. Korman [8] for the definition and details.)

(ii) Assume that

\[
(2.14) \quad h(\alpha) > h(u), \text{ for } 0 < u < \alpha.
\]

Then the positive solution of (1.1), with maximum value \( u(0) = \alpha \), travels to the right in the \((\lambda, \alpha)\) plane, i.e., \( \lambda'(\alpha) > 0 \). (This solution is stable.)

The following is the central result of this paper.
Theorem 2.2 (i) Assume that $f(u)$ is convex-concave, and $f(0) > 0$. Assume that $h(\gamma) \leq 0$, and $f(u) > 0$ for $u > \gamma$. Then the global solution curve admits at most two turns in the $(\lambda, \alpha)$ plane. Moreover, only turns to the right are possible if $u(0) > \gamma$.

(ii) Assume that $f(u)$ is concave-convex, and $f(0) < 0$. Assume that $h(\gamma) \geq 0$, and $f(u) > 0$ for $u > \gamma$. Then the global solution curve admits at most two turns in the $(\lambda, \alpha)$ plane. Moreover, only turns to the left are possible if $u(0) > \gamma$.

Proof: Let us prove the part (ii) first. We have $h(0) = 0$, $h'(u) = f(u) - uf'(u)$, $h''(u) = -uf''(u)$. Since $h''(u) > 0$ on $(0, \gamma)$, and $h(\gamma) > 0$, there exists $u_1 \in (0, \gamma)$ so that $h'(u) < 0$ on $(0, u_1)$ and $h'(u) > 0$ on $(u_1, \gamma)$, see Figure 1.

![Figure 1: The function $h(u)$ for part (ii)](image)

On $(\gamma, \infty)$ we have $h''(u) < 0$, so that either $h'(u) > 0$ on $(\gamma, \infty)$, or there is a point $u_2$ where $h'(u_2) = 0$. In the first case, the solution curve travels to the right for all $u(0) > \gamma$, in view of the Theorem 2.1. In that case the global solution curve has at most one turn, a turn to the right occurring where $u(0) < \gamma$ ($f(u)$ is concave in that range, see Lemma 2.2). Turning to the second case, we have by our assumptions $f(u_2) > 0$, and then $h'(u_2) = 0$ implies that $f'(u_2) > 0$. Since $h(u_2) > 0$, we have $f(u_2)u_2 < 2F(u_2)$ and then

$$I(u_2) = f^2(u_2) - 2F(u_2)f'(u_2) < f^2(u_2) - f(u_2)u_2f'(u_2) = 0.$$ 

By the second part of Lemma 2.1 (with $\beta = u_2$) it follows that only turns to the left are possible when $u(0) > u_2$. Since only turns to the right are possible when $u(0) < \gamma$, and the curve travels to the right when $u(0) \in (\gamma, u_2]$, the proof follows. (At most one turn, to the right, is possible for $u(0) \leq u_2$.)
The part (i) is proved similarly. The function $h(u)$ for this case is given in Figure 2. This time only turns to the left are possible when $u(0) < \gamma$, the solution curve travels to the left when $u(0) \in (\gamma, u_2]$, and only turns to the right are possible when $u(0) > u_2$, see [9] for more details.

![Figure 2: The function $h(u)$ for part (i)](image)

The following exact multiplicity result follows easily.

**Theorem 2.3** Assume that $f(u)$ is concave-convex, and $f(0) < 0$. Assume that $f(u)$ has exactly one root, i.e., $f(u) < 0$ on $[0, a)$, $f(u) > 0$ on $(a, \infty)$ for some $a > 0$, and

$$
\lim_{u \to \infty} \frac{f(u)}{u} = \infty.
$$

Assume also that $\gamma > a$, and we have $F(\gamma) > 0$ and $h(\gamma) \geq 0$. Define $\theta \in (0, \gamma)$ by $F(\theta) = 0$. Then all positive solutions of (1.1) lie on a unique solution curve, which is reversed $S$-shaped in the $(\lambda, u(0))$ plane. Namely, one end of this curve starts at $\lambda_1 = \frac{1}{2} \left( \int_0^\theta \frac{du}{\sqrt{-F(u)}} \right)^2$, $u(0) = \theta$ (and also $u'(\pm 1) = 0$). From the point $(\lambda_1, \theta)$, the curve travels to the left, it makes exactly two turns, and it tends to infinity as $\lambda \to 0$.

**Proof:** We begin with the positive solution of (1.1) satisfying $u'(\pm 1) = 0$. Since $\frac{1}{2}u'^2 + \lambda F(u) = \text{constant}$, for that solution we have

$$
\frac{1}{2}u'^2 + \lambda F(u) = 0,
$$

and, in particular, $u(0) = \theta$. (Existence of such solution follows by solving $u'' + f(u) = 0$, $u(0) = \theta$, $u'(0) = 1$, and then scaling, so that the first root
occurs at $x = 1$.) On $(0, 1)$ we have $\frac{du}{dx} = -\sqrt{2\lambda} \sqrt{-F(u)}$, and integrating over $(0, 1)$, we calculate $\lambda = \lambda_1 = \frac{1}{2} \left( \int_0^\theta \frac{du}{\sqrt{-F(u)}} \right)^2$. This solution, call it $u_1(x)$, is non-singular in the class of even functions. Indeed the function $u'_1(x)$ is an odd solution of the linearized problem (2.1) (computed at $u_1(x)$). Hence, (2.1) has no non-trivial even solutions. By the Implicit Function Theorem, we can continue the solution point $(\lambda_1, u_1(x))$ in $\lambda$ to obtain even solutions. It turns out that we get positive solutions for $\lambda < \lambda_1$ (and sign changing even solutions for $\lambda > \lambda_1$). Indeed, differentiating (1.1) in $\lambda$, we get

$$u'' + \lambda f'(u)u + f(u) = 0, \quad \text{for } -1 < x < 1, \quad u_1(-1) = u_1(1) = 0,$$

and then it is easy to verify that $u_1 = \frac{1}{2} x u'(x)$. (Both functions satisfy the same equation, and are zero at $x = 1$.) Since $u_1 < 0$ for $x \in (0, 1)$, we obtain positive solutions for $\lambda < \lambda_1$. We now continue this solution curve (which at first travels to the left). The graph of $h(u)$ is as in the Figure 1 (at least for $u \in (0, \gamma)$). By the Theorem 2.1, the solution curve travels to the right, by the time $u(0) = \gamma$. Hence the solution curve has made exactly one turn to the right before that (recall that $f''(u) < 0$ on $(0, \gamma)$, see Lemma 2.2). Since $f(u)$ is superlinear, the solution curve cannot travel to the right indefinitely, see e.g., [8]. By the Theorem 2.2, only turns to the left are possible for $u(0) > \gamma$, so that the solution curve will make exactly one turn to the left, and then tend to infinity. Using (2.15) again, we conclude that $\lambda \to 0$. ♦

We remark that the graph of $h(u)$ is exactly as in the Figure 1, i.e., there is $u_2$ such that $h'(u_2) = f(u_2) - u_2 f'(u_2) = 0$. Observe that $u_1$ and $u_2$ are the points where a straight line out of the origin is tangent to the graph of $f(u)$. Our conditions on $f(u)$ imply existence of two such points.

The case when $\gamma \leq a$ is covered by the following result.

**Theorem 2.4** Assume that $f(u)$ is concave-convex, and $f(0) < 0$. Assume that $f(u)$ has exactly one root, i.e., $f(u) < 0$ on $[0, a)$, $f(u) > 0$ on $(a, \infty)$ for some $a > 0$, and the condition (2.15) holds. Assume that $\gamma \leq a$. Then any positive solution of (1.1) is non-singular, i.e., the corresponding linearized problem (2.1) has only the trivial solution. Let $\theta > a$ be such that $F(\theta) = 0$. Then all positive solutions of (1.1) lie on a unique solution curve in the $(\lambda, u(0))$ plane. One end of this curve starts at $\lambda_1 = \frac{1}{2} \left( \int_0^\theta \frac{du}{\sqrt{-F(u)}} \right)^2$, $u(0) = \theta$ (and also $u'(\pm 1) = 0$). From $(\lambda_1, \theta)$ the curve travels to the left, it makes no turns, and it tends to infinity as $\lambda \to 0$. 

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Proof: With \( k(u) \equiv f(u) - uf'(u) \), we have \( k(a) = -af'(a) \leq 0 \), \( k'(u) = -uf''(u) < 0 \) for \( u > a \). It follows that \( k(u) < 0 \) for \( u > a \), i.e., \( f'(u) > \frac{f(u)}{u} \) for \( u > a \). By the Theorem 3.1 in [8], any positive solution of (1.1) is non-singular, and the proof follows.

We now recall the result of P. Korman and Y. Li [9], which follows from the Theorem 2.2 in the same way as the Theorem 2.3.

**Theorem 2.5** Assume that \( f(u) > 0 \) for all \( u \geq 0 \), \( f(u) \) is convex-concave and \( \lim_{u \to \infty} \frac{f(u)}{u} = 0 \). Assume also that \( h(\gamma) \leq 0 \). Then the solution curve of (1.1) is exactly S-shaped. Namely, it starts at \((\lambda = 0, u = 0)\), it makes exactly two turns, and then continues for all \( \lambda \) without any more turns.

We now present generalizations of the above results, allowing for multiple changes of sign for \( f''(u) \).

**Theorem 2.6** (i) In the conditions of the Theorem 2.3, let \( \mu > u_2 \) be such that \( h(\mu) < h(u) \) for all \( u \in [0, \mu) \), see Figure 1. Then the Theorem 2.3 remains true if for \( u > \mu \) we no longer require that \( f''(u) > 0 \), replacing this by a weaker condition that \( h'(u) = f(u) - uf'(u) < 0 \).

(ii) In the conditions of the Theorem 2.5, let \( \mu > u_2 \) be such that \( h(\mu) > h(u) \) for all \( u \in [0, \mu) \), see Figure 2. Then the Theorem 2.5 remains true if for \( u > \mu \) we no longer require that \( f''(u) < 0 \), replacing this by a weaker condition that \( h'(u) = f(u) - uf'(u) > 0 \).

**Proof:** Let us prove the case (i), and the other one is similar. We know from the proof of the Theorem 2.3 that the solution curve is exactly reversed \( S \)-shaped for \( u(0) \in [0, \mu) \). By the Theorem 2.1, the solution curve continues to travel to the left for all \( u(0) > \mu \). \( \diamond \)

**References**


