Abstract

We extend the classical Pohozaev’s identity to semilinear elliptic systems of Hamiltonian type, providing an alternative and simpler approach to the results of E. Mitidieri [8], R.C.A.M. Van der Vorst [15], and Y. Bozhkov and E. Mitidieri [2].

Key words: Pohozaev’s identity, Non-existence of solutions.

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1 Introduction

Any solution \( u(x) \) of semilinear Dirichlet problem on a bounded domain \( \Omega \subset \mathbb{R}^n \)

\[
\Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega
\]  

(1.1)

satisfies the well known Pohozaev’s identity

\[
\int_{\Omega} [2nF(u) + (2 - n)uf(u)] \, dx = \int_{\partial \Omega} (x \cdot \nu)|\nabla u|^2 \, dS.
\]  

(1.2)

Here \( F(u) = \int_0^u f(t) \, dt \), and \( \nu \) is the unit normal vector on \( \partial \Omega \), pointing outside. (From the equation (1.1), \( \int_{\Omega} uf(u) \, dx = \int_{\partial \Omega} |\nabla u|^2 \, dS \), which gives an alternative form of the Pohozaev’s identity.) A standard use of this identity is to conclude that if \( \Omega \) is a star-shaped domain with respect to the origin, i.e. \( x \cdot \nu \geq 0 \) for all \( x \in \partial \Omega \), and \( f(u) = u|u|^{p-1} \), for some constant \( p \), then
the problem (1.1) has no non-trivial solution in the super-critical case, when \( p > \frac{n+2}{n-2} \). In this note we present a proof of Pohozaev’s identity, which appears a little more straightforward than the usual one, see e.g. L. Evans [3], and then use a similar idea for systems, generalizing the well-known results of E. Mitidieri [8]. After completing this work, we found out that this result appeared previously in Y. Bozhkov and E. Mitidieri [2]. However, our proof is different, and it appears to be much simpler. Similarly, we derive Pohozhaev’s identity for a version of \( p \)-Laplace equation.

Let \( z = x \cdot \nabla u = \sum_{i=1}^{n} x_i u_{x_i} \). It is easy to verify that \( z \) satisfies
\[
\Delta z + f'(u)z = -2f(u). \tag{1.3}
\]

We multiply the equation (1.1) by \( z \), and subtract from that the equation (1.3) multiplied by \( u \), obtaining
\[
\sum_{i=1}^{n} \left[ (zu_{x_i} - uz_{x_i})_{x_i} + x_i \frac{\partial}{\partial x_i} (2F(u) - uf(u)) \right] = 2f(u)u. \tag{1.4}
\]

Clearly,
\[
\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} (2F - uf) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [x_i (2F - uf)] - n(2F - uf).
\]

We then rewrite (1.4)
\[
\sum_{i=1}^{n} \left[ (zu_{x_i} - uz_{x_i}) + x_i (2F(u) - uf(u)) \right]_{x_i} = 2nF(u) + (2 - n)uf(u). \tag{1.5}
\]

Integrating over \( \Omega \), we conclude the Pohozaev’s identity (1.2). (The only non-zero boundary term is \( \sum_{i=1}^{n} \int_{\partial \Omega} z_{u_{x_i}} u_i dS \). Since \( \partial \Omega \) is a level set of \( u \), \( \nu = \pm \frac{\nabla u}{|\nabla u|} \), i.e. \( u_{x_i} = \pm |\nabla u| u_i \). Then \( z = \pm (x \cdot \nu) |\nabla u| \), and \( \sum_{i=1}^{n} u_{x_i} u_i = \pm |\nabla u| \).

It appears natural to refer to (1.5) as a \textit{differential form} of Pohozaev’s identity. For radial solutions on a ball, the corresponding version of (1.5) played a crucial role in the study of exact multiplicity of solutions, see T. Ouyang and J. Shi [9], and also P. Korman [6].

2 Non-existence of solutions for a class of systems

The following class of systems has attracted considerable attention recently
\[
\begin{align*}
\Delta u + H_v(u, v) &= 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \\
\Delta v + H_u(u, v) &= 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega,
\end{align*}
\tag{2.1}
\]

where \( H(u, v) \) is a given differentiable function, see e.g. the following surveys: D.G. de Figueiredo [4], P. Quittner and P. Souplet [13], B. Ruf [14], see also P. Korman [5]. This system is of \textit{Hamiltonian} type, which implies that it has some of the properties of scalar equations.

More generally, we assume that \( H(u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m) \), with integer \( m \geq 1 \), and consider the Hamiltonian system of \( 2m \) equations
\[
\begin{align*}
\Delta u_k + H_{v_k} &= 0 \text{ in } \Omega, \quad u_k = 0 \text{ on } \partial \Omega, \quad k = 1, 2, \ldots, m, \\
\Delta v_k + H_{u_k} &= 0 \text{ in } \Omega, \quad v_k = 0 \text{ on } \partial \Omega, \quad k = 1, 2, \ldots, m.
\end{align*}
\tag{2.2}
\]
We call solution of (2.2) to be positive, if \( u_k(x) > 0 \) and \( v_k(x) > 0 \) for all \( x \in \Omega \), and all \( k \). We consider only the classical solutions, with \( u_k \) and \( v_k \) of class \( C^2(\Omega) \cap C^1(\Omega) \). We have the following generalization of Pohozaev’s identity, see also [2].

**Theorem 2.1** Assume that \( H(u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m) \in C^2(\mathbb{R}_+^m \times \mathbb{R}_+^m) \cap C(\overline{\mathbb{R}}_+^m \times \overline{\mathbb{R}}_+^m) \).

For any positive solution of (2.2), and any real numbers \( a_1, \ldots, a_m \), one has

\[
\int_{\Omega} \left[ 2nH(u, v) + (2 - n)\Sigma_{k=1}^m (a_k u_k H_{u_k} + (2 - a_k) v_k H_{v_k}) \right] \, dx = 2\Sigma_{k=1}^m \int_{\partial \Omega} (x \cdot \nu) |\nabla u_k| |\nabla v_k| \, dS. \tag{2.3}
\]

**Proof.** Define \( p_k = x \cdot \nabla u_k = \Sigma_{i=1}^n x_i u_{kx_i} \), and \( q_k = x \cdot \nabla v = \Sigma_{i=1}^n x_i v_{kx_i} \), \( k = 1, 2, \ldots, m \). These functions satisfy the system

\[
\begin{align*}
\Delta p_k + \Sigma_{j=1}^m H_{u_k u_j} p_j + \Sigma_{j=1}^m H_{u_k v_j} q_j &= -2 H_{u_k}, \quad k = 1, 2, \ldots, m \\
\Delta q_k + \Sigma_{j=1}^m H_{u_k u_j} q_j + \Sigma_{j=1}^m H_{u_k v_j} p_j &= -2 H_{v_k}, \quad k = 1, 2, \ldots, m. \tag{2.4}
\end{align*}
\]

We multiply the first equation in (2.2) by \( q_k \), and subtract from that the first equation in (2.4) multiplied by \( v_k \). The result can be written as

\[
\Sigma_{i=1}^n \left[ (u_{kx_i} q_k - p_{kx_i} v_k) x_i + (-u_{kx_i} q_k - v_{kx_i}) \right] + H_{v_k} q_k - \Sigma_{j=1}^m H_{u_k u_j} p_j v_k - \Sigma_{j=1}^m H_{u_k v_j} q_j v_k = 2 v_k H_{v_k}. \tag{2.5}
\]

Similarly, we multiply the second equation in (2.2) by \( p_k \), and subtract from that the second equation in (2.4) multiplied by \( u_k \), and write the result as

\[
\Sigma_{i=1}^n \left[ (v_{kx_i} p_k - q_{kx_i} u_k) x_i + (-v_{kx_i} p_k - u_{kx_i}) \right] + H_{u_k} p_k - \Sigma_{j=1}^m H_{u_k u_j} p_j u_k - \Sigma_{j=1}^m H_{u_k v_j} v_j u_k = 2 u_k H_{u_k}. \tag{2.6}
\]

Adding the equations (2.5) and (2.6), we get

\[
\Sigma_{i=1}^n \left[ (u_{kx_i} q_k - p_{kx_i}) x_i + v_{kx_i} p_k - q_{kx_i} u_k \right] + H_{u_k} p_k + H_{v_k} q_k - \Sigma_{j=1}^m H_{u_k u_j} p_j u_k - \Sigma_{j=1}^m H_{u_k v_j} q_j v_k = 2 u_k H_{u_k} + 2 v_k H_{v_k}. \]

We now sum in \( k \), then switch the orders of summation in \( i \) and \( k \) in the second group of terms on the left (the ones involving \( H \)), putting the result into the form

\[
\Sigma_{i=1}^n x_i \left( 2 H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k} \right) x_i = 2 \Sigma_{k=1}^m u_k H_{u_k} + 2 \Sigma_{k=1}^m v_k H_{v_k}. \]

Writing,

\[
\Sigma_{i=1}^n x_i \frac{\partial}{\partial x_i} (2 H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k}) = \Sigma_{i=1}^n x_i \left( 2 H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k} \right),
\]

we obtain the differential form of Pohozaev’s identity

\[
\Sigma_{k=1}^m \Sigma_{i=1}^n \left[ u_{kx_i} q_k - p_{kx_i}, v_k + v_{kx_i}, p_k - q_{kx_i}, u_k + x_i (2 H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k}) \right] = 2n H + (2 - n) \left( \Sigma_{k=1}^m u_k H_{u_k} + \Sigma_{k=1}^m v_k H_{v_k} \right). \]
Integrating, we obtain as before
\[
\int_{\Omega} [2nH(u, v) + (2 - n)(\Sigma_{k=1}^{m} u_k H_{u_k} + \Sigma_{k=1}^{m} v_k H_{v_k})] \, dx \\
= 2\Sigma_{k=1}^{m} \int_{\partial\Omega} (x \cdot \nu) |\nabla u_k| |\nabla v_k| \, dS.
\] (2.7)

(\text{Since we consider positive solutions, and } \partial\Omega \text{ is a level set for both } u_k \text{ and } v_k, \text{ we have } \nu = -\frac{\nabla u_k}{|\nabla u_k|} = -\frac{\nabla v_k}{|\nabla v_k|}; \text{ i.e., } u_{ki} = |\nabla u_k| \nu_i \text{ and } v_{ki} = |\nabla v_k| \nu_i \text{ on the boundary } \partial\Omega.)

From the first equation in (2.1), \( \int_{\Omega} v_k H_{v_k} \, dx = \int_{\Omega} \nabla u_k \cdot \nabla v_k \, dx \), while from the second equation \( \int_{\Omega} u_k H_{u_k} \, dx = \int_{\Omega} \nabla u_k \cdot \nabla v_k \, dx \), i.e., for each \( k \)
\[
\int_{\Omega} v_k H_{v_k} \, dx = \int_{\Omega} u_k H_{u_k} \, dx.
\]

Using this in (2.7), we conclude the proof.

\textbf{Remark} Here and later on, we consider only the classical solutions. Observe that by our conditions and elliptic regularity, classical solutions are in fact of class \( C^3(\Omega) \), so that all quantities in the above proof are well defined. Also, it suffices to assume that \( \Omega \) is star-shaped with respect to any one of its points (which we then take to be the origin).

As a consequence, we have the following non-existence result.

\textbf{Proposition 1} Assume that \( \Omega \) is a star-shaped domain with respect to the origin, and for some real constants \( \alpha_1, \ldots, \alpha_m \), and all \( u_k > 0, v_k > 0 \), we have
\[
nH + (2 - n)\Sigma_{k=1}^{m} (\alpha_k u_k H_{u_k} + (1 - \alpha_k) v_k H_{v_k}) < 0.
\] (2.8)

Then the problem (2.2) has no positive solutions.

\textbf{Proof.} We use the identity (2.3), with \( \alpha_k/2 = \alpha_k \). Then, assuming existence of positive solution, the left hand side of (2.3) is negative, while the right hand side is non-negative, a contradiction.

\textbf{Example} Assume that \( m = 2 \), and consider \( H(u_1, v_1, u_2, v_2) = \frac{1}{p} (u_1^p + u_2^p) + u_1^r u_2^s \), with \( p > \frac{2n}{n-2} \), and \( r + s > \frac{2n}{n-2} \). Then the inequality (2.8) holds, with \( \alpha_1 = \alpha_2 = \frac{2}{3} \). It follows that the system
\[
\Delta u_1 + u_1^{p-1} = 0 \quad \text{in } \Omega, \quad u_1 = 0 \quad \text{on } \partial\Omega,
\]
\[
\Delta v_1 + ru_1^{-1} u_2^s = 0 \quad \text{in } \Omega, \quad v_1 = 0 \quad \text{on } \partial\Omega,
\]
\[
\Delta u_2 + u_2^{p-1} = 0 \quad \text{in } \Omega, \quad u_2 = 0 \quad \text{on } \partial\Omega,
\]
\[
\Delta v_2 + su_1^{-1} u_2^{s-1} = 0 \quad \text{in } \Omega, \quad v_2 = 0 \quad \text{on } \partial\Omega
\]
has no positive solutions.

In case \( m = 1 \), we recover the following result of E. Mitidieri [8]. We provide some details, in order to point out that some restrictions in [8] can be relaxed.

\textbf{Proposition 2} Assume that \( \Omega \) is a star-shaped domain with respect to the origin, and for some real constant \( \alpha \), and all \( u > 0, v > 0 \) we have
\[
\alpha uH_u(u, v) + (1 - \alpha)vH_v(u, v) > \frac{n}{n-2} H(u, v).
\] (2.9)
Then the problem (2.1) has no positive solution.

Comparing this result to E. Mitidieri [8], observe that we do not require that \( H_u(0,0) = H_v(0,0) = 0 \).

An important subclass of (2.1) is

\[
\Delta u + f(v) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \\
\Delta v + g(u) = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega,
\]

which corresponds to \( H(u,v) = F(v) + G(u) \), where as before, \( F(v) = \int_0^v f(t) \, dt \), \( G(u) = \int_0^u g(t) \, dt \). Unlike [8], we do not require that \( f(0) = g(0) = 0 \). The Theorem 2.1 now reads as follows.

**Theorem 2.2** Let \( f, g \in C(\bar{\mathbb{R}}_+) \). For any positive solution of (2.10), and any real number \( a \), one has

\[
\int_{\Omega} [2n(F(v) + G(u)) + (2 - n)(avf(v) + (2 - a)ug(u))] \, dx \\
= 2\int_{\partial \Omega} (x \cdot \nu)[\nabla u][\nabla v] \, dS.
\]

(2.11)

We now consider a particular system

\[
\Delta u + v^p = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \\
\Delta v + g(u) = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega,
\]

(2.12)

with \( g(u) \in C(\bar{\mathbb{R}}_+) \), and a constant \( p > 0 \).

**Theorem 2.3** Assume that \( \Omega \) is a star-shaped domain with respect to the origin, and

\[
nG(u) + (2 - n) \left( 1 - \frac{n}{(n - 2)(p + 1)} \right) ug(u) < 0, \quad \text{for all } u > 0.
\]

(2.13)

Then the problem (2.12) has no positive solution.

**Proof.** We use Pohozaev’s identity (2.11), with \( f(v) = v^p \). We select the constant \( a \), so that

\[
2nF(v) + (2 - n)avf(v) = 0,
\]

i.e., \( a = \frac{2n}{(n - 2)(p + 1)} \). Then, assuming existence of a positive solution, the left hand side of (2.11) is negative, while the right hand side is non-negative, a contradiction. \( \Box \)

Observe that in case \( p = 1 \), the Theorem 2.3 provides a non-existence result for a biharmonic problem with Navier boundary conditions (in E. Mitidieri [8], a separate identity was used to cover the biharmonic case)

\[
\Delta^2 u = g(u) \quad \text{in } \Omega, \quad u = \Delta u = 0 \quad \text{on } \partial \Omega.
\]

(2.14)

**Proposition 3** Assume that \( \Omega \) is a star-shaped domain with respect to the origin, and the condition (2.13) holds. Then the problem (2.14) has no positive solution.
Finally, we consider the system
\[
\begin{align*}
\Delta u + v^p &= 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \\
\Delta v + u^q &= 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega.
\end{align*}
\] (2.15)

The curve \( \frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n} \) is called a critical hyperbola. We recover the following well known result of E. Mitidieri [8], see also R.C.A.M. Van der Vorst [15]. (Observe that we relax the restriction \( p, q > 1 \) from [8].)

**Proposition 4** Assume that \( p, q > 0 \), and
\[
\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2}{n}.
\] (2.16)

Then the problem (2.15) has no positive solution.

**Proof.** Condition (2.16) is equivalent to (2.13), and the Theorem 2.3 applies. ♦

In case \( p = 1 \), we have the following known result, see E. Mitidieri [8].

**Proposition 5** Assume that \( \Omega \) is a star-shaped domain with respect to the origin, and \( q > \frac{n+4}{n-4} \). Then the problem
\[
\Delta^2 u = u^q \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial \Omega
\]
has no positive solutions.

## 3 Pohozaev's identity for a version of \( p \)-Laplace equation

We consider the following version of \( p \)-Laplace equation
\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \varphi(u_{x_i}) + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\] (3.1)

Here \( \varphi(t) = t|t|^{p-2} \), with a constant \( p > 1 \). This is a variational equation for the functional
\[
\int_{\Omega} \left[ \frac{1}{p} (|u_{x_1}|^p + \ldots + |u_{x_n}|^p) - F(u) \right] \, dx.
\]
This equation is known to the experts, see P. Lindqvist [7], but it has not been studied much.

Observe that \( \varphi(at) = a^{p-1} \varphi(t) \), for any constant \( a > 0 \). Also, \( \varphi'(t) = (p-1)|t|^{p-2} \), i.e.,
\[
t \varphi'(t) = (p-1) \varphi(t).
\] (3.2)

Letting, as before, \( z = x \cdot \nabla u = \sum_{i=1}^{n} x_i u_{x_i} \), we see that \( z \) satisfies
\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [\varphi'(u_{x_i}) z_{x_i}] + f'(u) z = -pf(u).
\] (3.3)

To derive (3.3), we consider \( u^s(x) \equiv u(sx) \), which satisfies
\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \varphi(u^s_{x_i}) = -s^p f(u^s).
\] (3.4)
(To see that, it is convenient to write (3.1) as \( \sum_{i=1}^{n} \varphi'\left( \frac{\partial}{\partial x_i} u \right) \frac{\partial^2}{\partial x_i^2} u + f(u) = 0 \).) Then differentiating (3.4) with respect to \( s \), and setting \( s = 1 \), we obtain (3.3). (Alternatively, to derive (3.3), one could differentiate (3.1) in \( x_j \), then multiply by \( x_j \), and sum in \( j \).)

**Proposition 6** Any solution of (3.1) satisfies

\[
\int_{\Omega} \left[ pnF(u) + (p - n)uf(u) \right] \, dx = (p - 1) \int_{\partial\Omega} (x \cdot \nu) |\nabla u| \sum_{i=1}^{n} \varphi(|\nabla u|\nu_i) \nu_i \, dS, \tag{3.5}
\]

where \( \nu_i \) is the \( i \)-th component of \( \nu \), the unit normal vector on \( \partial\Omega \), pointing outside.

**Proof.** Multiply the equation (3.1) by \( z \), and write the result as

\[
(p - 1) \sum_{i=1}^{n} \left( [z\varphi(u_{x_i})] - (p - 1) \sum_{i=1}^{n} \varphi(u_{x_i})z_{x_i} + (p - 1)f(u)z \right) = 0. \tag{3.6}
\]

Multiply the equation (3.3) by \( u \), and write the result as

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( [u\varphi'(u_{x_i})z_{x_i}] - \sum_{i=1}^{n} u_{x_i}\varphi'(u_{x_i})z_{x_i} + f'(u)uz \right) = -pfuf(u). \tag{3.7}
\]

We now subtract (3.7) from (3.6). In view of (3.2), we have a cancellation, and so we obtain

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( [(p - 1)z\varphi(u_{x_i}) - u\varphi'(u_{x_i})z_{x_i}] + [(p - 1)f(u) - uf'(u)]z \right) = pfuf(u). \]

As before,

\[
[(p - 1)f(u) - uf'(u)]z = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} (pF(u) - uf(u)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [x_i (pF(u) - uf(u))] - n(pF(u) - uf(u)). \tag{3.8}
\]

This gives us a differential form of Pohozaev’s identity

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [(p - 1)z\varphi(u_{x_i}) - u\varphi'(u_{x_i})z_{x_i} + x_i (pF(u) - uf(u))] = pnF(u) + (p - n)uf(u). \tag{3.9}
\]

Integrating, and using the divergence theorem, we conclude the proof. \( \diamond \)

For star-shaped domains, the right hand side of (3.5) is non-negative, so if

\[
 pnF(u) + (p - n)uf(u) < 0 \text{ for all } u,
\]

then the problem (3.1) has no non-trivial solutions.

**Example** For star-shaped domains, the problem

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \varphi(u_{x_i}) + u|u|^{r-1} = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega
\]

has no non-trivial solutions, provided the constant \( r \) satisfies \( r > \frac{mp - n + p}{n - p} \).

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