A remark on Pinney’s equation

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Abstract
We show that Pinney’s equation [2] with a constant coefficient can be reduced to its linear part by a simple change of variables. Also, Pinney's original solution is simplified slightly.

Key words: Pinney’s equation, general solution.

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In 1950 Edmund Pinney published a very influential paper [2], which was less than half a page long. That paper provided a general solution of the nonlinear differential equation

\[ y'' + a(x)y + \frac{c}{y^3} = 0, \quad y(x_0) = q \neq 0, \ y'(x_0) = p, \]  

with a given function \( a(x) \) and a constant \( c \neq 0 \). Namely, the solution is

\[ y(x) = \pm \sqrt{u^2(x) - cv^2(x)}, \]

where \( u(x) \) and \( v(x) \) are the solutions of the linear equation

\[ y'' + a(x)y = 0, \]

for which \( u(x_0) = q, \ u'(x_0) = p, \) and \( v(x_0) = 0, \ v'(x_0) = \frac{1}{q}. \) One takes “plus” in (2) if \( q > 0 \), and “minus” if \( q < 0 \). Clearly, \( u(x) \) and \( v(x) \) form a
fundamental set of (3), and by Liouville’s formula their Wronskian at any \( x \) is the same as at \( x_0 \), i.e.,

\[
    u'(x)v(x) - u(x)v'(x) = 1, \quad \text{for all } x.
\]

A substitution of \( y = \sqrt{u^2(x) - cv^2(x)} \) into (1) gives

\[
y'' + a(x)y + \frac{c}{y^3} = -c \frac{[u'(x)v(x) - u(x)v'(x)]^2 - 1}{[u^2(x) - cv^2(x)]^{\frac{3}{2}}} = 0.
\]

If \( c < 0 \), the solution is valid for all \( x \), while for \( c > 0 \) some singular points are possible.

The nonlinear equation equation (1) possessing a general solution is very special, and it attracted a lot of attention (there are currently 92 MathSciNet and 543 Google Scholar citations). It turns out that this equation was considered back in 1880 by Ermakov [1].

Our remark is that in case of constant \( a(x) = a_0 \), Pinney’s equation becomes linear for \( z(x) = y^2(x) \). Indeed, we multiply the equation

(4) \[
y'' + a_0 y + \frac{c}{y^3} = 0
\]

by \( y' \), and integrate to get

(5) \[
y'^2 + a_0 y^2 - c y^{-2} = p^2 + a_0 q^2 - c \frac{1}{q^2}.
\]

Now multiply the same equation by \( y \):

(6) \[
yy'' + a_0 y^2 + cy^{-2} = 0,
\]

and set \( z = y^2 \). Since \( yy'' = \frac{1}{2} z'' - y'^2 \), by using (5), one transforms (6) to

\[
z'' + 4a_0 z = 2 \left( p^2 + a_0 q^2 - c \frac{1}{q^2} \right), \quad z(0) = q^2, \quad z'(0) = 2pq.
\]

References
