A simplified proof of a conjecture for the perturbed Gelfand equation from combustion theory

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Abstract

For the perturbed Gelfand’s equation on the unit ball in two dimensions, Y. Du and Y. Lou [5] proved that the curve of positive solutions is exactly $S$-shaped, for sufficiently small values of the secondary parameter. We present a simplified proof and some extensions. This problem is prominent in combustion theory, see e.g., the book of J. Bebernes and D. Eberly [1].

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1 Introduction

The following Dirichlet problem for the perturbed Gelfand’s equation is prominent in combustion theory

\[ \Delta u + \lambda e^{\frac{u}{1+\epsilon u}} = 0, \quad \text{for } |x| < 1, \quad u = 0 \quad \text{when } |x| = 1, \]

(1.1)

see e.g., J. Bebernes and D. Eberly [1]. Here \( \lambda \) and \( \epsilon \) are positive parameters, and we think of \( \lambda \) as the primary parameter, while \( \epsilon \) is the secondary, or “evolution parameter”. By the maximum principle, the solution of (1.1) is positive, and then by the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [6] it is radially symmetric, i.e., \( u = u(r) \), with \( r = |x| \), and it satisfies

\[ u'' + \frac{n-1}{r}u' + \lambda e^{\frac{u}{1+\epsilon u}} = 0, \quad 0 < r < 1, \quad u'(0) = u(1) = 0. \]

For the perturbed Gelfand’s equation on a unit ball in two dimensions

(1.2)

Y. Du and Y. Lou [5], building on the earlier results of P. Korman, Y. Li and T. Ouyang [12], [13], proved the following theorem, thus settling a long-standing conjecture of S.V. Parter [16].

**Theorem 1.1** Suppose that \( n = 2 \) and \( \epsilon > 0 \) is sufficiently small, then solution curve \( \{(\lambda, u)\} \) of (1.2) is exactly S-shaped. Moreover, the solutions lying on the upper branch and lower branch of the solution curve are asymptotically stable, while the solutions on the middle branch are unstable. (See Figure 1.)

Their proof was rather involved, and it was relying on some previous results of E.N. Dancer [4]. The purpose of this note is to give a simpler and self-contained proof of the S-shaped part. We also observe that some more general results can be obtained without too much extra effort. While simplifying the proof in [5], we retain several of the crucial steps from that paper: the change of variables, Lemma 4.1 and Theorem 3.1, although we generalize or simplify these results. Our new tool involves showing that the turning points are non-degenerate, so that they persist when the secondary parameter \( \epsilon \) is varied. We also observe that computer assisted validation of bifurcation diagrams is possible for \( \epsilon \) not being small. Next, we state the result in the form we prove it.
Theorem 1.2 For $\epsilon$ sufficiently small the solution curve of (1.2) is exactly S-shaped. Moreover, at any $\lambda$ where either two or three solutions occur, these solutions are strictly ordered. (See Figure 1.)

When $\epsilon \geq 0.25$, the solution curve of (1.2) is monotone. Indeed, in that case the function $f(u) = e^{u + \epsilon u}$ satisfies $uf''(u) < f(u)$ for all $u > 0$, except for $u = 4$ when $\epsilon = 0.25$, so that the corresponding linearized problem has only the trivial solution by the Sturm’s comparison theorem, and so the implicit function theorem applies. It is natural to conjecture that there is a critical $\epsilon_0 > 0$, so that for $\epsilon \geq \epsilon_0 > 0$ the solution curve is monotone, while for $\epsilon < \epsilon_0 > 0$ it is exactly S-shaped. For the one-dimensional case the same statement is known as the long-standing S.-H. Wang’s conjecture [18], for which we gave a computer assisted proof in [14], and which was proved very recently by S.-Y. Huang and S.-H. Wang [8]. Our numerical computations, for the two-dimensional case (1.2), show that remarkably $\epsilon_0$ is close to 0.25, see Figures 1 and 2 ($\epsilon_0 \approx 0.2385$). (We used the shoot-and-scale method, described in detail in [10], which we implemented using Mathematica.) Previous contributions to the $n = 1$ case included K.J. Brown et al [2], S.P. Hastings and J.B. McLeod [7] (who proved the above theorem in one dimension), S.-H. Wang [18], S.-H. Wang and F.P. Lee [19], P. Korman and Y. Li [11].

When $n > 2$, this theorem does not hold. Indeed, if $\epsilon = 0$ and $3 \leq n \leq 9$, by the classical result of D.D. Joseph and T.S. Lundgren [9] the solution curve of the problem (1.2) makes infinitely many turns. It is natural to
expect that these turns will persist for small $\epsilon > 0$, which is indeed the case, as shown by E.N. Dancer [4].

2 Non-degenerate critical points

We consider positive solutions of the Dirichlet problem

\begin{equation}
\Delta u + \lambda f(u, \epsilon) = 0 \quad \text{for } x \in D, \; u = 0 \text{ on } \partial D,
\end{equation}

depending on positive parameters $\lambda$ and $\epsilon$, on a general domain $D \subset \mathbb{R}^n$. We are interested in the critical points of (2.1), i.e., the solution triples $(\lambda, u, \epsilon)$, for which the corresponding linearized problem

\begin{equation}
\Delta w + \lambda f_u(u, \epsilon)w = 0 \quad \text{for } x \in D, \; w = 0 \text{ on } \partial D
\end{equation}

has a nontrivial solution $w(x)$. We wish to continue the critical points, when the secondary parameter $\epsilon$ is varied. The following lemma was first proved by E.N. Dancer [4] (see also J. Shi [17]). We present a simpler proof for completeness.

**Lemma 2.1** Let $(\lambda_0, u_0, \epsilon_0)$ be a critical point of (2.1). Assume that the null-space of (2.2) is one-dimensional, spanned by some $w_0(x) \in W^{2,2}(D) \cap W^{1,2}_0(D)$. Assume also that

\begin{equation}
\int_D f(u_0(x))w_0(x) \, dx \neq 0,
\end{equation}
(2.4) \[ \int_D f''(u_0(x))w_0^3(x) \, dx \neq 0. \]

Then there is a unique critical point \((\lambda(\epsilon), u(\epsilon), \epsilon)\) near \((\lambda_0, u_0, \epsilon_0)\). Moreover, these are the only critical points in some neighborhood of \((\lambda_0, u_0, \epsilon_0)\).

**Proof:** We can normalize the solution of (2.2), so that

(2.5) \[ \frac{1}{2} \int_D w^2(x) \, dx = 1. \]

The equations (2.1), (2.2) and (2.5) give us three equations to find \(u, w\) and \(\lambda\) as a function of \(\epsilon\). We show that the implicit function theorem applies. Indeed, we define a map

\[ H(u, w, \lambda, \epsilon) : (W^{2,2}(D) \cap W^{1,2}_0(D)) \times (W^{2,2}(D) \cap W^{1,2}_0(D)) \times \mathbb{R} \times \mathbb{R} \to L^2(D) \times L^2(D) \times \mathbb{R} \]

as a vector whose entries are the left hand sides of the above equations:

\[ H(u, w, \lambda, \epsilon) = \begin{bmatrix} \Delta u + \lambda f(u, \epsilon) \\ \Delta w + \lambda f_u(u, \epsilon)w \\ \frac{1}{2} \int_D w^2(x) \, dx \end{bmatrix}. \]

The linearized operator with respect to the first three variables at a point \((u_0, w_0, \lambda_0, \epsilon_0)\) is

\[ H'(u, w, \lambda)(u_0, w_0, \lambda_0, \epsilon_0) \begin{bmatrix} v \\ \theta \\ \tau \end{bmatrix} = \begin{bmatrix} \Delta v + \lambda_0 f_u(u_0, \epsilon_0)v + \tau f(u_0, \epsilon_0) \\ \Delta \theta + \lambda_0 f_u(u_0, \epsilon_0)\theta + \lambda_0 f_{uu}(u_0, \epsilon_0)w_0v + \tau f_u(u_0, \epsilon_0)w_0 \\ \frac{1}{2} \int_D w_0^2 \theta \, dx \end{bmatrix}. \]

We need to show that this operator is both injective and surjective. To see that it is injective, we need to show that the system

(2.6)
\[ \begin{align*}
\Delta v + \lambda_0 f_u(u_0, \epsilon_0)v + \tau f(u_0, \epsilon_0) &= 0 \\
\Delta \theta + \lambda_0 f_u(u_0, \epsilon_0)\theta + \lambda_0 f_{uu}(u_0, \epsilon_0)w_0v + \tau f_u(u_0, \epsilon_0)w_0 &= 0 \\
\frac{1}{2} \int_D w_0^2 \theta \, dx &= 0
\end{align*} \]

has only the trivial solution \((v, \theta, \tau) = (0, 0, 0)\). The first equation in (2.6) can be regarded as a linear equation for \(v\), with its kernel spanned by \(w_0\), and the right hand side equal to \(-\tau f(u_0, \epsilon_0)\). Since by (2.3), \(f(u_0, \epsilon_0)\) is not orthogonal to the kernel, it follows that the first equation is solvable only if \(\tau = 0\). We then have \(v = kw_0\), with a constant \(k\). We now regard the second
equation in (2.6) as a linear equation for $\theta$ with the same kernel spanned by $w_0$, and the right hand side equal to $-k\lambda_0 f_u(u_0, \epsilon_0)w_0^2$. By our condition (2.4), the second equation is solvable only if $k = 0$. We then have $\theta = lw_0$, with a constant $l$. From the third equation in (2.6) we conclude that $l = 0$, completing the proof of injectivity.

Turning to the surjectivity, we need to show that for any $L^2(D)$ functions $a(x)$ and $b(x)$, and for any constant $c$ the problem

\begin{align}
\Delta v + \lambda_0 f_u(u_0, \epsilon_0)v + \tau f(u_0, \epsilon_0) &= a(x) \\
\Delta \theta + \lambda_0 f_u(u_0, \epsilon_0)\theta + \lambda_0 f_{uu}(u_0, \epsilon_0)w_0v + \tau f_u(u_0, \epsilon_0)w_0 &= b(x) \\
\int_D w_0 \theta \, dx &= c
\end{align}

is solvable. Proceeding similarly to the above, we regard the first equation in (2.7) as a linear equation for $v$ with the right hand side equal to $a(x) - \tau f(u_0, \epsilon_0)$. By by (2.3), we can choose $\tau$, so that this function is orthogonal to $w_0$. Then the first equation in (2.7) has infinitely many solutions of the form $v = \bar{v} + kw_0$, where $\bar{v}$ is any fixed solution, and $k$ is any constant. We now turn to the second equation in (2.7), where $\tau$ has been just fixed above. In view of our condition (2.4), we can fix $k$ so that $b(x) - \lambda_0 f_{uu}(u_0, \epsilon_0)w_0v - \tau f_u(u_0, \epsilon_0)w_0$ is orthogonal to $u_0(x)$, and hence the second equation is solvable. We then have $\theta = \bar{\theta} + lw_0$, where $\bar{\theta}$ is a fixed solution and $l$ is any number. Finally, from the third equation in (2.7) we uniquely determine $l$.

This lemma shows that the critical points continue on a smooth curve, when the secondary parameter $\epsilon$ varies. We call such critical points non-degenerate.

3  Positivity for the linearized problem

Let $u(r)$ be a positive solution of the Dirichlet problem

\begin{align}
&u'' + \frac{n-1}{r}u' + f(u) = 0, \quad r > 0, \quad u'(0) = u(1) = 0.
\end{align}

We wish to show that any non-trivial solution of the linearized problem

\begin{align}
&L[w] \equiv w'' + \frac{n-1}{r}w' + f'(u)w = 0, \quad r > 0, \quad w'(0) = w(1) = 0
\end{align}

does not vanish on $(0, 1)$.
Lemma 3.1  Assume that there is a function \( z(r) \in C^2(0, 1) \) (a “test function”), such that for some \( \xi \in [0, 1] \)

\[
(3.3) \quad z > 0, \quad \text{and} \quad L[z] = z'' + \frac{n-1}{r}z' + f'(u)z < 0 \quad \text{on} \quad (0, \xi),
\]

\[
(3.4) \quad z < 0, \quad \text{and} \quad L[z] = z'' + \frac{n-1}{r}z' + f'(u)z > 0 \quad \text{on} \quad (\xi, 1).
\]

Then \( w(r) \) does not vanish on \([0, 1]\), i.e., we may assume that \( w(r) > 0 \).

Proof: Without loss of generality we may assume that \( w(0) > 0 \). We claim that \( w(r) \) cannot vanish on \((0, \xi]\). Assuming the contrary, we can find \( \xi_0 \in (0, \xi] \), so that \( w(r) > 0 \) on \((0, \xi_0]\) and \( w(\xi_0) = 0, w'(\xi_0) < 0 \). Combining the equations in (3.2) and (3.3), we get

\[
\left[ r^{n-1} (w'z - wz') \right]' > 0.
\]

Integration over \((0, \xi_0]\) gives

\[
\xi_0^{n-1} w'(\xi_0)z(\xi_0) > 0,
\]

but \( w'(\xi_0) < 0 \) and \( z(\xi_0) \geq 0 \), which is a contradiction.

We show similarly that \( w(r) \) cannot vanish on \([\xi, 1)\). ♦

In a nutshell, we showed that \( z(r) \) oscillates faster than \( w(r) \) on both of the intervals \((0, \xi]\) and \([\xi, 1)\), and hence \( w(r) \) cannot vanish on either of the intervals. Observe that both the cases \( \xi = 0 \) and \( \xi = 1 \) are allowed. Only rarely can one use this lemma directly, but rather the idea of its proof is used.

We now present a generalization of a result of Y. Du and Y. Lou [5].

Theorem 3.1  For the problem (3.1) assume that \( n = 2 \), the function \( f(u) \in C^2(\bar{R}_+) \) satisfies \( f(u) > 0, f'(u) > 0 \) for all \( u > 0 \), and it is log-concave:

\[
(3.5) \quad (\log f(u))'' < 0, \quad \text{for all} \quad u > 0,
\]

i.e., \( f''(u)f(u) - f'^2(u) < 0 \) for all \( u > 0 \). Let \( u(r) \) be a positive solution of (3.1). Then any non-trivial solution of the corresponding linearized problem (3.2) may be assumed to satisfy \( w(r) > 0 \) on \([0, 1]\).
The function $g$ can vanish at most once on $(0, 1)$. Compute

$$L[z] = -2f(u) + \alpha f'(u) = e^{h(u)} \left[ -2 + \alpha h'(u) \right].$$

The function $g(u) \equiv -2 + \alpha h'(u)$ satisfies

$$\frac{d}{dr}g(u) = \alpha h''(u)u'(r) > 0,$$

and hence $L[z]$ can change sign at most once on $(0, 1)$ (from negative to positive).

Without loss of generality we may assume that $w(0) > 0$, and let us suppose that $w(\xi_0) = 0$ at some $\xi_0 \in (0, 1)$. Define $\alpha_1 > 0$ so that $z(r) = ru'(r) + \alpha_1$ also vanishes at $\xi_0$, and $z(r) > 0$ on $(0, \xi_0)$. For $\alpha > 0$ small, the function $g(u(r)) = -2 + \alpha h'(u(r))$ is negative, while for larger $\alpha$’s it changes sign. Let $\alpha_2$ be the supremum of $\alpha$’s for which $g(u(r)) < 0$ for all $r \in (0, 1)$, so that for $\alpha > \alpha_2$, $g(u(r))$ changes sign (exactly once) on $(0, 1)$.

Case 1. $\alpha_2 \geq \alpha_1$. Fix $z(r) = ru'(r) + \alpha_1$. Then on $(0, \xi_0)$, $z > 0$ and $L[z] < 0$, and hence $w(r)$ cannot vanish on $(0, \xi_0]$, a contradiction.

Case 2. $\alpha_2 < \alpha_1$. Then there is a point $\xi_1 \in (0, 1)$, such that $g(u(r)) = -2 + \alpha_1 h'(u(r)) < 0$ on $(0, \xi_1)$ and $g(u(r)) = -2 + \alpha_1 h'(u(r)) > 0$ on $(\xi_1, 1)$.

Sub-case (a). $\xi_0 \leq \xi_1$. Again, fix $z(r) = ru'(r) + \alpha_1$. Then on $(0, \xi_0)$, $z > 0$ and $L[z] < 0$, and hence $w(r)$ cannot vanish on $(0, \xi_0]$, a contradiction.

Sub-case (b). $\xi_1 < \xi_0$. Consider $z(r) = ru'(r) + \alpha$, with $\alpha < \alpha_1$. As we decrease $\alpha$ from $\alpha_1$, the root of the decreasing function $z(r)$ moves to the left (and the root is at $r = 0$, when $\alpha = 0$), while the root of the increasing function $g(u(r)) = -2 + \alpha h'(u(r))$ moves to the right. At some $\bar{\alpha} > 0$, these roots intersect at some $\bar{\xi} \in (\xi_1, \xi_0)$. Fix $z(r) = ru'(r) + \bar{\alpha}$. Then on $(0, \bar{\xi})$, we have $z > 0$ and $L[z] < 0$, and we have $z < 0$ and $L[z] > 0$ on $(\bar{\xi}, 1)$. By Lemma 3.1, $w(r)$ cannot vanish (or a contradiction is achieved on $(\xi, 1)$). □
Example. $f(u) = u^p + u^q$, with positive constants $p$ and $q$. A direct computation shows that $f(u)$ is log-concave for all $u > 0$, and the theorem applies, if and only if

$$(p-q)^2 - p - q \leq 0, \text{ or } (p-q)^2 - 2p - 2q + 1 < 0.$$ 

Example. $f(u) = e^{\frac{-u}{1+u}}$, with a constant $a \geq 0$. This function is log-concave for all $u > 0$, and the theorem applies.

This result does not hold in case $n > 2$.

**Theorem 3.2** In addition to the conditions of Theorem 3.1, assume that

$$uf'(u) > f(u), \text{ for all } u > 0.$$ 

Then any positive solution of (3.1) is non-singular, i.e., (3.2) has only the trivial solution.

**Proof:** By the Sturm comparison theorem, $w(r)$ must vanish on $(0,1)$, in contradiction with the Theorem 3.1. 

\[\diamondsuit\]

### 4 The limiting problem

We consider now the solutions of the perturbed Gelfand’s problem

\begin{equation}
(4.1) \quad u'' + \frac{1}{r}u' + \lambda e^{\frac{u}{1+u}} = 0, \quad 0 < r < 1, \quad u'(0) = u(1) = 0,
\end{equation}

on two-dimensional unit ball, with positive parameters $\lambda$ and $\epsilon$.

Following Y. Du and Y. Lou [5], we set $w(r) = e^2 u(r)$, and $\mu = \lambda e^2 e^{1/\epsilon}$, converting this problem into

\begin{equation}
(4.2) \quad w'' + \frac{1}{r}w' + \mu e^{-\frac{1}{1+u}} = 0, \quad 0 < r < 1, \quad w'(0) = w(1) = 0.
\end{equation}

The limiting problem at $\epsilon = 0$ is

\begin{equation}
(4.3) \quad v'' + \frac{1}{r}v' + \eta e^{-\frac{1}{2}} = 0, \quad 0 < r < 1, \quad v'(0) = v(1) = 0,
\end{equation}

where we changed the names of the variables for future reference. The exact multiplicity result for (4.3) will follow from the following theorem, which is not much harder to prove than the special case of the problem (4.3). Except for the last statement, it corresponds to Theorem 2 in Y. Du and Y. Lou [5].
Theorem 4.1 Consider the problem

\[(4.4) \quad v'' + \frac{1}{r}v' + \eta f(v) = 0, \quad r > 0, \quad v'(0) = v(1) = 0.\]

Assume that the function \(f(v) \in C^2(\mathbb{R}_+)\) satisfies \(f(0) = f'(0) = 0, f(v) > 0 \text{ for } v > 0\). Assume also that \(f(v)\) is log-concave and convex-concave for all \(v > 0\), so that (3.5) holds, and \(f''(v) > 0\) on \((0, \beta)\) and \(f''(v) < 0\) on \((\beta, \infty)\), for some \(\beta > 0\). Assume that \(\lim_{v \to \infty} f(v) = f_0 > 0\). Assume finally that \(vf'(v) > f(v)\) on \((0, \beta)\).

Then there is a critical \(\eta_0\), such that for \(\eta < \eta_0\) the problem (4.4) has no positive solutions, it has exactly one positive solution at \(\eta = \eta_0\), and there are exactly two positive solutions for \(\eta > \eta_0\). Moreover, all solutions lie on a single smooth solution curve, which for \(\eta > \eta_0\) has two branches, denoted by \(v^-(r, \eta) < v^+(r, \eta)\), with \(v^+(r, \eta)\) strictly monotone increasing in \(\eta\), and \(\lim_{\eta \to \infty} v^+(r, \eta) = \infty\) for all \(r \in [0, 1)\). For the lower branch, \(\lim_{\eta \to \infty} v^-(0, \eta) = 0\). Denote \(v_0 = v^-(r, \eta_0) = v^+(r, \eta_0)\). The turning point \((\eta_0, v_0)\) is the only critical point on the solution curve, and moreover \((\eta_0, v_0)\) is a non-degenerate critical point.

**Proof:** We follow [5] to prove that the problem (4.4) has a positive solution for \(\eta\) large. Let \(\phi \in C_0^\infty(B), \phi \geq 0, \max_B \phi > 0\) (\(B\) is the unit ball in \(\mathbb{R}^n\)). Let \(v > 0\) be the unique solution of \(\Delta v + \phi = 0\) in \(B, v = 0\) on \(\partial B\), and denote by \(\overline{v}\) be the unique solution of \(\Delta v + \eta f_0 = 0\) in \(B, v = 0\) on \(\partial B\). Then \(\underline{v} < \overline{v}\), for \(\eta\) large, and they form a lower-upper solution pair.

We now continue this solution for decreasing \(\eta\). Since \(f'(0) = 0\), the solution curve cannot enter the point \((0, 0)\) in the \((\lambda, u(0))\) plane, nor can a bifurcation from zero occur at some \(\lambda > 0\) (just multiply the PDE version of (4.4) by \(v\) and integrate). Hence, the solution curve must turn to the right. By the condition (4.5), and Theorem 3.2, any turns must occur in the region where \(v(0) > \beta\). By a result of P. Korman, Y. Li and T. Ouyang [12], only turns to the right are possible in that region. Hence, there is only one turn to the right. In particular, the proof in [12] showed that the condition (2.4) holds, and so the turning point is non-degenerate. Monotonicity of the upper branch, \(v^+(r, \eta)\), is proved as in [12]. (Near the turning point the monotonicity follows by the Crandall-Rabinowitz theorem. Then one uses the maximum principle to show that \(v^+(r, \eta) > 0, \text{ for all } \eta > \eta_0\)\). 

In particular, this theorem applies to the problem (4.3) (we define \(e^{-\frac{1}{r}}\) to be zero at \(v = 0\)).
The following important lemma was proved first in Y. Du and Y. Lou [5]. We present its proof for completeness.

**Lemma 4.1** Fix $\epsilon < v_0(0)$. Consider the solutions of (4.2), with $w(0) > v_0(0) - \epsilon$. Then $\mu$ is an increasing function of $w(0)$, i.e., the solution curve of (4.2) travels to the right (northeast) in the $(\mu, w(0))$ plane.

**Proof:** Denote by $v(r, \alpha)$ the solution of (4.3) satisfying $v(0, \alpha) = \alpha$. Assume that $\beta > \alpha > v_0(0)$, and $v(r, \beta)$ is the solution of (4.3) with $v(0, \beta) = \beta$. Recall that the solutions of (4.3) are uniquely identified by a global parameter $v(0)$, so that we have solution pairs $(\eta(\alpha), v(r, \alpha))$ and $(\eta(\beta), v(r, \beta))$, with $\eta(\beta) > \eta(\alpha)$ and $v(r, \beta) > v(r, \alpha)$ for all $r \in [0, 1)$. Denote by $a = a(\alpha)$ the point where $v(a, \alpha) = \epsilon$, and set $w(r) = v(r, \alpha) - \epsilon$. Then $w(r) > 0$ on $[0, a)$, and it satisfies

$$w'' + \frac{1}{r}w' + \eta(\alpha)e^{-\frac{1}{r}} = 0, \quad 0 < r < a, \quad w'(0) = w(a) = 0.$$ 

Scaling $r = at$, we see that $w(t)$ satisfies

$$w'' + \frac{1}{t}w' + \eta(\alpha)a^2(\alpha)e^{-\frac{1}{at}} = 0, \quad 0 < r < 1, \quad w'(0) = w(1) = 0.$$ 

We see that $w(t)$ is the solution of (4.2) with the maximum value of $\alpha - \epsilon$ and the corresponding parameter value $\mu(\alpha - \epsilon) = \eta(\alpha)a^2(\alpha)$. Similarly, we identify $w(r) = v(r, \beta) - \epsilon$ as the solution of (4.2) with the maximum value of $\beta - \epsilon$ and the corresponding parameter value $\mu(\beta - \epsilon) = \eta(\beta)a^2(\beta)$. Observe that $a(\beta) > a(\alpha)$, because $v(r, \beta) > v(r, \alpha)$, and then

$$\mu(\beta - \epsilon) = \eta(\beta)a^2(\beta) > \eta(\alpha)a^2(\alpha) = \mu(\alpha - \epsilon),$$

and the proof follows. $\diamond$

5 Proof of the Theorem 1.2

When $\epsilon > 0$, all solutions of the problem (4.2) lie on a unique solution curve joining $(0, 0)$ to $(\infty, \infty)$ in $(\mu, w(0))$ plane, see e.g., P. Korman [10]. When $\epsilon \geq 0.25$, the solution curve is monotone. When $\epsilon$ is small, the solution curve must make at least two turns. Indeed, this curve begins at $(0, 0)$, and it is close to the lower branch of (4.3) (solutions on the lower branch of (4.3) are non-singular, hence persist for small $\epsilon > 0$), and therefore a turn to the left occurs. After the turn, the solution curve has no place to go for decreasing $\lambda$, 11
so that eventually it must travel to the right, providing us with at least the second turn. Denote by \( \beta = \beta(\epsilon) \) the point where \( e^{-\frac{1}{c+w}} \) changes convexity. When \( \epsilon \) is small, the solution curve makes exactly one turn to the left in the region where \( w(0) \in (0, \beta) \), and it can make a number of turns when \( \beta < w(0) < v_0(0) - \epsilon \) (and no turns are possible when \( w(0) > v_0(0) - \epsilon \)).

If the theorem was false, then as \( \epsilon \to 0 \) there would be at least four turning points on every curve. Let \( (\mu_1(\epsilon), w_1(r, \epsilon)) \) and \( (\mu_2(\epsilon), w_2(r, \epsilon)) \) denote the second and the third turning points respectively. By Lemma 4.1, both \( w_1(r, \epsilon) \) and \( w_2(r, \epsilon) \) are bounded from above by \( v_0(0) - \epsilon \) for all \( r \). The quantity \( w_1(0, \epsilon) \) (and hence \( w_2(0, \epsilon) \)) is also bounded from below by, say \( \frac{1}{4} \), for sufficiently small \( \epsilon \). Indeed,

\[
\left( e^{-\frac{1}{c+w}} \right)'' = \frac{e^{-\frac{1}{c+w}}}{(w + \epsilon)^2} [1 - 2(w + \epsilon)] > 0,
\]

for \( w < \frac{1}{4} \) and \( \epsilon \) small, and hence only turns to the left are possible when \( w(0, \epsilon) < \frac{1}{4} \). Take a sequence \( \epsilon_k \to 0 \). Using elliptic estimates, along a subsequence, \( (\mu_1(\epsilon), w_1(r, \epsilon)) \) tends to a solution \( (\eta, v(r)) \) of (4.3). (By the above estimates, \( w_1(r, \epsilon) \) cannot tend to either infinity or zero.) This solution \( (\eta, v(r)) \) has to be singular, since in any neighborhood of it there are two solutions, that the turning point \( (\mu_1(\epsilon), w_1(r, \epsilon)) \) brings. (If solution of (4.3) is non-singular, then by the implicit function theorem there is a unique solution of (4.2) near it, for \( \epsilon \) small.)

Similarly, \( (\mu_2(\epsilon), w_2(r, \epsilon)) \) must converge along a subsequence to a singular solution of (4.3), as \( \epsilon \to 0 \). Both \( (\mu_1(\epsilon), w_1(r, \epsilon)) \) and \( (\mu_2(\epsilon), w_2(r, \epsilon)) \) must converge to the unique turning point of (4.3). By Theorem 4.1 this turning point is non-degenerate, which means that for \( \epsilon > 0 \) small there can be only one singular point of (4.2) in its neighborhood. But we have two singular points, \( (\mu_1(\epsilon), w_1(r, \epsilon)) \) and \( (\mu_2(\epsilon), w_2(r, \epsilon)) \), a contradiction.

Finally, the ordering property of solutions follows by using the positivity of solutions of the linearized problem, the bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [3], and the strong maximum principle, see e.g., [10] for similar arguments.

\[ \diamond \]

6 Extensions

Lemma 4.1 does not require \( \epsilon \) to be very small, only that \( \epsilon < v_0(0) \approx 1.53 \). This lemma shows that the set of \( w(0) \)'s, for which turns may occur for
(4.2), is bounded. It follows that computer generated figures of S-shaped bifurcation, like the one in Figure 1 at \( \epsilon = 0.22 \), can be validated, i.e., a computer assisted proof of their validity can be given.

Other results on perturbation of solution curves can be established similarly.

**Theorem 6.1** Consider the problem \((u = u(x), x \in \mathbb{R}^n)\)

\[
\Delta u + \lambda (u - \epsilon)(u - b)(c - u) = 0, \quad |x| < 1, \quad u = 0 \text{ when } |x| = 1,
\]

with constants \(0 < \epsilon < b < c\), such that \(c > 2b\). Then for \(\epsilon\) sufficiently small, the set of positive solutions of (6.1) consists of two curves. The lower one starts at \((0,0)\) in the \((\lambda, u(0))\) plane, and it is monotone, tending to \(\epsilon\) as \(\lambda \to \infty\). The upper curve is parabola-like, with a single turn to the right. Correspondingly, there is a \(\lambda_0 > 0\), so that the problem (6.1) has exactly one positive solution for \(\lambda \in (0, \lambda_0)\), exactly two strictly ordered positive solutions at \(\lambda = \lambda_0\), and exactly three strictly ordered positive solutions for \(\lambda > \lambda_0\).

**Proof:** By B. Gidas, W.-M. Ni and L. Nirenberg [6], positive solutions of (6.1) are radially symmetric. Define \(f(u) = u(u + \epsilon - b)(c - \epsilon - u)\). Then we can write (6.1) as

\[
\Delta u + \lambda f(u - \epsilon) = 0, \quad |x| < 1, \quad u = 0 \text{ when } |x| = 1.
\]

At \(\epsilon = 0\), all positive solution of (6.2) lie on a parabola-like curve, with a single turn to the right. Moreover, except for the turning point, all solutions are non-singular, and the turning point is non-degenerate, i.e., the conditions of Lemma 2.1 hold, and the upper branch is monotone increasing, see [13] and [15]. As in the proof of the Theorem 1.2, we show that this curve preserves its shape for \(\epsilon\) sufficiently small, while monotonicity of the lower solution curve is easy to prove \((f'(u) < 0, \text{ when } u < \epsilon)\).

This is the first exact multiplicity result for a cubic with three positive roots, in dimensions \(n > 1\). Similar result holds for cubic-like \(f(u)\), were considered in [12] and [15].

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References


