TREMAIN EQUIANGULAR TIGHT FRAMES

MATTHEW FICKUS, DUSTIN G. MIXON, JOHN JASPER, AND JESSE PETERSON

Abstract. We combine Steiner systems with Hadamard matrices to produce a new class of equiangular tight frames. This in turn leads to new constructions of strongly regular graphs and distance-regular antipodal covers of the complete graph.

1. Introduction

Consider the following problem: Given a finite-dimensional Hilbert space $\mathcal{H}$ and some $N \in \mathbb{N}$, find vectors $\varphi_1, \ldots, \varphi_N \in \mathcal{H}$ of unit length that minimize coherence:

$$\kappa := \max_{i,j \in \{1, \ldots, N\}, i \neq j} |\langle \varphi_i, \varphi_j \rangle|.$$ 

Geometrically, this problem amounts to packing lines through the origin. Given their resemblance to error-correcting codes, it comes as no surprise that ensembles of minimal coherence, called Grassmannian frames, lead to communication protocols with minimal cross-talk. Since their introduction by Strohmer and Heath in 2003 [11], Grassmannian frames have received considerable attention in the finite frame theory community.

There were several precursors to the modern study of Grassmannian frames. One notable example is a 1974 paper of Welch [15], which provides various lower bounds for coherence in terms of the dimension $M$ of $\mathcal{H}$. This first of these bounds is

$$\kappa \geq \sqrt{\frac{N - M}{M(N - 1)}},$$

cf. Rankin’s 1956 paper [10]. This has since been dubbed the Welch bound, and one can show that the coherence of an ensemble of unit-length vectors meets equality in the Welch bound precisely when the ensemble forms an equiangular tight frame [11], meaning $|\langle \varphi_i, \varphi_j \rangle|$ is some fixed constant for every choice of $i$ and $j$ with $i \neq j$ (i.e., the vectors are “equiangular”) and furthermore, the operator $S : \mathcal{H} \to \mathcal{H}$ defined by

$$Sx := \sum_{i=1}^{N} \langle x, \varphi_i \rangle \varphi_i \quad \forall x \in \mathcal{H}$$

Date: February 3, 2016.

2010 Mathematics Subject Classification. Primary: 42C15, Secondary: 51E10, 05B20, 05C12.

Key words and phrases. equiangular tight frame, Steiner system, Hadamard matrix, strongly regular graph, distance-regular antipodal cover of the complete graph.

This research was supported by an AFOSR Young Investigator Research Program award, NSF Grant No. DMS-1321779, and AFOSR Grant No. F4FGA05076J002. The views expressed in this article are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.
is some multiple of the identity operator (i.e., the ensemble forms a so-called “tight frame”). By exhibiting equality in the Welch bound, equiangular tight frames (ETFs) are necessarily Grassmannian.

The significance of equiangularity brings us to another precursor of sorts. In 1966, van Lint and Seidel [13] first introduced a useful identification between real equiangular ensembles and graphs, in which vertices correspond to vectors, and edges are drawn according to the sign of the corresponding inner product. Since the Gram matrix \( G = [\langle \varphi_i, \varphi_j \rangle]_{ij} \) can be expressed in terms of the adjacency matrix \( A \) of this graph, one may then identify spectral properties of \( A \) with those of \( G \). This is particularly important in the case of real ETFs, as the tightness criterion implies that \( G^2 \) is a multiple of \( G \), which in turn forces \( A \) to satisfy a related quadratic. Indeed, more recent treatments of this identification [14, 6] have established a one-to-one correspondence between real ETFs and a family of strongly regular graphs (SRGs), namely, graphs with the property that every vertex has the same number \( k \) of neighbors, that adjacent vertices have the same number \( \lambda \) of common neighbors, and that non-adjacent vertices also have the same number \( \mu \) of common neighbors. Given the maturity of the SRG literature, this identification has compelled finite frame theorists to direct their attention toward complex ETFs, where open problems abound.

However, recent developments in ETF research leave one questioning the insurmountable maturity of the SRG literature. In particular, a decades-old SRG construction by Goethals and Seidel [8] (which leverages a particular family of real equiangular ensembles) was recently generalized in [4] to construct new SRGs from known real ETFs exhibiting certain symmetries. The notion that ETF research could inform the design of graphs was further suggested by Coutinho, Godsil, Shirazi and Zhan [3], who established that complex ETFs with Gram matrices consisting of prime roots of unity can be used to construct distance-regular antipodal covers of the complete graph (referred to as DRACKNs).

The present paper produces a new infinite family of complex ETFs called Tremain ETFs, which the authors discovered by generalizing Example 7.10 in Tremain’s notes [12]. The construction can be written as a real ETF infinitely often, and in some cases, the corresponding SRGs are new (this is established with the help of Brouwer’s table of strongly regular graphs [1]). In addition, the complex cases of the construction lead to a new infinite family of DRACKNs.

(Throughout, we use \( \{\delta_s\}_{s \in S} \) to denote any fixed orthonormal basis of \( \mathbb{C}^S \). For any \( x \in \mathbb{C}^S \), we may refer to its \( s \)th entry with respect to this basis \( x(s) := \langle x, \delta_s \rangle \). Also, we identify a given vector \( x \in \mathcal{H} \) with the corresponding linear operator \( x: \mathbb{C} \rightarrow \mathcal{H} \) defined by \( x(a) := ax \) for every \( a \in \mathbb{C} \). In particular, this identification allows us to express the adjoint operator \( x^*: \mathcal{H} \rightarrow \mathbb{C} \) given by \( x^* := \langle \cdot, x \rangle \), along with the outer product operator \( xx^* \), etc.)

2. Preliminaries

In this paper, it is convenient to scale the vectors in an ETF so that the inner products between any two vectors lies in the complex unit circle \( \mathbb{T} \). This scaling then forces the squared norm of each vector to be the reciprocal of the Welch bound. For example, consider any \( N \times N \) complex Hadamard matrix, that is, an \( N \times N \) matrix with entries in \( \mathbb{T} \) such that \( HH^* = NI \). Then removing any row from this matrix will form an \((N-1) \times N\) matrix \( \Phi \) whose columns form a so-called unimodular simplex. The fact that such vectors form an ETF is easy to check: In this case, \( M = N - 1 \), and so the Welch bound is \( 1/(N - 1) \),
and indeed, each column has squared norm $N - 1$, whereas inner products between distinct columns lie in $\mathbb{T}$. If we let $a$ denote the row of $H$ that was removed to produce $\Phi$, then we say its entries $\{a_i\}_{i=1}^N$ form the **Naimark complement** of the columns $\{\varphi_i\}_{i=1}^N$ of $\Phi$. The Naimark complement is useful because sums of inner products satisfy

$$\langle \varphi_i, \varphi_j \rangle + a_i a_j = \begin{cases} N & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

by identifying the sum as an inner product between corresponding columns of $H$.

Another useful family of ETFs is the so-called **Steiner ETFs**, first constructed in [5] from Steiner systems. A $(t, K, V)$-**Steiner system** is a collection $V$ of $V$ points together with a collection $B$ of size-$K$ subsets of $V$ (called **blocks**) with the property that each $t$-element subset of $V$ is contained in exactly one block. In this paper, we introduce a certain family of operators that will simplify the construction of Steiner ETFs, as well as the construction of Tremain ETFs in the next section. In particular, for each $v \in V$, consider an **embedding operator** $E_v: \mathbb{C}^R \rightarrow \mathbb{C}^B$ such that for every $b \in B$ with $v \in b$, there exists $r \in [R]$ such that $E_v \delta_r = \delta_b$. These embedding operators satisfy a few useful properties:

**Lemma 2.1.** Given a $(2, K, V)$-Steiner system with embedding operators $\{E_v\}_{v \in V}$, we have

(a) $\sum_{v \in V} E_v E_v^* = K I_B$,

(b) $E_v^* E_v = I_R$, and

(c) for every $v, v' \in [R]$ with $v \neq v'$, there exist $i, j \in [R]$ such that $E_v^* E_{v'} = \delta_i \delta_j^*$.

**Proof.** For (a), we first note that, since $E_v$ is an isometry, $E_v E_v^*$ is an orthogonal projection onto its range. Since $\{\delta_b\}_{b \in B}, v \in B$ is an orthonormal basis for this range, we have

$$E_v E_v^* = \sum_{b \in B} \delta_b \delta_b^*.$$  

The desired sum can then be simplified using this identity:

$$\sum_{v \in V} E_v E_v^* = \sum_{v \in V} \sum_{b \in B} \delta_b \delta_b^* = \sum_{b \in B} \sum_{v \in V} \delta_b \delta_b^* = K \sum_{b \in B} \delta_b \delta_b^* = K I_B.$$  

For (b), the fact that $E_v^* E_v = I_R$ follows from $E_v$ being an isometry. For (c), suppose $v \neq v'$. Then

$$\langle E_v^* E_{v'} \delta_r, \delta_r \rangle = \langle E_v \delta_r, E_{v'} \delta_r \rangle = \langle \delta_{v(r)}, \delta_{v'(r)} \rangle \in \{0, 1\} \quad \forall r, r' \in [R].$$

Summing these quantities gives

$$\sum_{r=1}^R \sum_{r'=1}^R \langle E_v^* E_{v'} \delta_r, \delta_r \rangle = \sum_{r=1}^R \sum_{r'=1}^R \langle \delta_{v'(r')}, \delta_{v(r)} \rangle$$

$$= \sum_{b \in B} \sum_{b' \in B} \langle \delta_{v'}, \delta_b \rangle$$

$$= \sum_{b \in B} \sum_{b' \in B} \left\{ \begin{array}{ll} 1 & \text{if } b = b' \\ 0 & \text{else} \end{array} \right\} = \#\{b \in B : v, v' \in b\} = 1.$$

We therefore conclude that $\langle E_v^* E_{v'} \delta_r, \delta_r \rangle = 1$ for exactly one choice of $r$ and $r'$, and zero otherwise, thereby implying the result. 

$\square$
These properties are useful in the proof of the following construction:

**Proposition 2.2** (Steiner ETFs, see [5]). Given a unimodular simplex \( \{ \varphi_s \}_{s=1}^{R+1} \) in \( \mathbb{C}^R \) and a \((2, K, V)\)-Steiner system with embedding operators \( \{ E_v \}_{v \in V} \), the ensemble \( \{ E_v \varphi_s \}_{s=1}^{R+1}, v \in V \) forms an equiangular tight frame in \( \mathbb{C}^B \).

As an example, we use a \((2, 3, 7)\)-Steiner system known as the Fano plane to construct a Steiner ETF of 28 vectors in \( \mathbb{C}^7 \) (actually, \( \mathbb{R}^7 \)). The following matrix is organized into \( V = 7 \) blocks, each of size \( B \times (R+1) = 7 \times 4 \), and each obtained by applying a different embedding operator \( E_v \) to a unimodular simplex extracted from a \( 4 \times 4 \) Hadamard matrix:

\[
\begin{bmatrix}
+ & + & - & + & - & + & - \\
+ & + & - & - & + & + & - \\
+ & - & + & - & + & - & - \\
+ & - & + & - & - & + & + \\
+ & + & + & + & + & + & + \\
\end{bmatrix}
\]

Here, \(+\) denotes 1, \(-\) denotes \(-1\), and blank entries are zeros. It is clear from this example that the columns have squared norm 3 (the number of blocks incident to each point), columns from a common block have inner product \(-1\) (since they belong to a common embedded simplex), and the supports of columns from different blocks intersect at a single entry (since two points determine a block), leading to an inner product of \( \pm 1 \). Additionally, tightness follows from the fact that the rows of this matrix are orthogonal with equal norms.

**Proof of Proposition 2.2.** We first demonstrate tightness:

\[
\sum_{v \in V} \sum_{s=1}^{R+1} (E_v \varphi_s)(E_v \varphi_s)^* = \sum_{v \in V} E_v \left( \sum_{s=1}^{R+1} \varphi_s \varphi_s^* \right) E_v^* = (R+1) \sum_{v \in V} E_v E_v^* = K(R+1)I_B,
\]

where the last step follows from Lemma 2.1(a). Next, Lemma 2.1(b) gives

\[
\langle E_v \varphi_s, E_v \varphi_{s'} \rangle = \langle E_v^* E_v \varphi_s, \varphi_{s'} \rangle = \langle \varphi_s, \varphi_{s'} \rangle,
\]

which equals \( \| \varphi_s \|^2 = R \) when \( s = s' \), and otherwise resides in the complex unit circle \( \mathbb{T} \). Finally, for \( v \neq v' \), Lemma 2.1(c) gives

\[
\langle E_v \varphi_s, E_{v'} \varphi_{s'} \rangle = \langle E_v^* E_v \varphi_s, \varphi_{s'} \rangle = \langle \delta_i \delta_j^* \varphi_s, \varphi_{s'} \rangle = \varphi_s(j) \varphi_{s'}(i) \in \mathbb{T},
\]

thereby implying equiangularity.

\( \square \)

### 3. Tremain ETFs

What follows is our main result:

**Theorem 3.1** (Tremain ETFs). Given unimodular simplices \( \{ \varphi_s \}_{s=1}^{R+1} \) in \( \mathbb{C}^R \) and \( \{ \psi_t \}_{t=1}^{V+1} \) in \( \mathbb{C}^V \), corresponding Naimark complements \( \{ a_s \}_{s=1}^{R+1} \) and \( \{ b_t \}_{t=1}^{V+1} \) in \( \mathbb{C} \), and a \((2, 3, V)\)-Steiner system with embedding operators \( \{ E_v \}_{v \in V} \), the ensemble

\[
\left\{ E_v \varphi_s \oplus \sqrt{2} a_s \delta_v \oplus 0 \right\}_{s=1}^{R+1}, v \in V \cup \left\{ 0_B \oplus \sqrt{2} \psi_t \oplus \sqrt{2} b_t \right\}_{t=1}^{V+1}
\]

forms an equiangular tight frame in \( \mathbb{C}^B \oplus \mathbb{C}^V \oplus \mathbb{C} \).
For the sake of illustration, we turn to Example 7.10 in Tremain’s notes [12]. Consider the columns of the following matrix:

$$
\begin{bmatrix}
+ & + & + & + & + & - & + & + \\
+ & + & + & + & - & + & + & + \\
+ & + & + & + & - & + & + & + \\
+ & + & + & + & - & + & + & + \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
+ & + & + & + & - & + & + & + \\
\end{bmatrix}
$$

Here, $+$ and $-$ denote $\pm 1$ and blanks are zeros as before, and furthermore

$\bullet = \sqrt{2}, \quad \blacksquare = \sqrt{1/2}, \quad \square = -\sqrt{1/2}, \quad \diamond = \sqrt{3/2}.$

The first 28 columns come from the Steiner ETF of the previous section. Steiner ETF vectors in a common embedded simplex have inner product $-1$, and in the above matrix, these vectors receive an additional entry in the $\mathbb{C}^V$ component that changes their inner product to $+1$. Otherwise, the Steiner-based vectors in the Tremain ETF do not interact in the $\mathbb{C}^V$ or $\mathbb{C}$ components, thereby inheriting equiangularity from the Steiner ETF. The other vectors are supported solely in the $\mathbb{C}^V$ and $\mathbb{C}$ components: the portion in $\mathbb{C}^V$ forms a simplex extracted from an $8 \times 8$ Hadamard matrix whose inner products are $-1/2$, and the $\mathbb{C}$ component changes these inner products to $+1$. Also, the Steiner-based vectors have only one entry of support in common with these other vectors, allowing the inner products to have modulus $\sqrt{2} \cdot \sqrt{1/2} = 1$. Finally, the vectors all have squared norm 5, corresponding to the Welch bound for $M = 16$ and $N = 35$ (namely, $1/5$), and so this ensemble meets equality in the Welch bound, and equivalently forms an ETF.

**Proof of Theorem 3.1.** First, since $M = B + V + 1$ and $N = V(R + 1) + (V + 1)$, a persistent application of the identities $VR = BK = 3B$ and $V - 1 = R(K - 1) = 2R$ simplifies the Welch bound in this case:

$$
\sqrt{\frac{N - M}{M(N - 1)}} = \frac{1}{R + 2}.
$$

It suffices to demonstrate equality in the Welch bound, e.g., to show that each member of the purported ETF has squared norm $R + 2$, and that inner products between distinct members lie in $\mathbb{T}$. For convenience, we denote

$$
\tau_{v,s} := E_v \varphi_s \oplus \sqrt{2} a_s \delta_v \oplus 0 \quad \text{and} \quad \tau_t := 0_b \oplus \sqrt{1/2} \psi_t \oplus \sqrt{3/2} b_t.
$$

Checking the norms is straightforward:

$$
\|\tau_{v,s}\|^2 = \|E_v \varphi_s\|^2 + \|\sqrt{2} a_s \delta_v\|^2 = \|\varphi_s\|^2 + 2|a_s|^2 = R + 2,
$$

$$
\|\tau_t\|^2 = \left\| \sqrt{1/2} \psi_t \right\|^2 + \left\| \sqrt{3/2} b_t \right\|^2 = \frac{1}{2} \|\psi_t\|^2 + \frac{3}{2} \|b_t\|^2 = \frac{1}{2} V + \frac{3}{2} = R + 2.
$$

To demonstrate equiangularity, we proceed in cases:

Case I: Suppose $v = v'$ but $s \neq s'$. Then $\langle E_v \varphi_s, E_v \varphi_{s'} \rangle = \langle E_v^* E_v \varphi_s, \varphi_{s'} \rangle = \langle \varphi_s, \varphi_{s'} \rangle$ by Lemma 2.1(b), and so

$$
\langle \tau_{v,s}, \tau_{v',s'} \rangle = \langle E_v \varphi_s, E_v \varphi_{s'} \rangle + 2a_s a_{s'} = \frac{\langle \varphi_s, \varphi_{s'} \rangle}{5} + 2a_s a_{s'} = a_s a_{s'}.
$$
Case II: Suppose \( v \neq v' \). Then for every \( s, s' \in [R + 1] \), Lemma 2.1(c) gives
\[
\langle \tau_v, s, \tau_{v'}, s' \rangle = \langle E_v \varphi_s, \varphi_{s'} \rangle = \langle E_v^* \varphi_s, \varphi_{s'} \rangle = \langle \delta_i^* \varphi_s, \varphi_{s'} \rangle = \varphi_s(i) \varphi_{s'}(i).
\]

Case III: Suppose \( t \neq t' \). Then
\[
\langle \tau_t, \tau_{t'} \rangle = \frac{1}{2} \langle \psi_t, \psi_{t'} \rangle + \frac{3}{2} b_t \bar{b}_{t'} = \frac{1}{2} \left( \langle \psi_t, \psi_{t'} \rangle + b_t \bar{b}_{t'} \right) + b_t \bar{b}_{t'} = b_t \bar{b}_{t'}.
\]

Case IV: For every \( v \in V \), \( s \in [R + 1] \) and \( t \in [V + 1] \), we have
\[
\langle \tau_{v,s}, \tau_t \rangle = \left\langle \sqrt{2} a_s \delta_v, \sqrt{1/2} \psi_t \right\rangle = a_s \psi_t(v).
\]

All of these inner products lie in \( T \), and so we are done. \( \square \)

Since Steiner triple systems exist if and only if \( V \equiv 1 \) or 3 mod 6 with \( V \geq 3 \), and since complex unimodular simplices exist in every dimension, we have the following corollary:

**Corollary 3.2.** For every \( V \equiv 1 \) or 3 mod 6 with \( V \geq 3 \), there exists an equiangular tight frame of \( N \) vectors in \( \mathbb{C}^M \) with
\[
M = \frac{1}{6} (V + 2)(V + 3), \quad N = \frac{1}{2} (V + 1)(V + 2).
\]

As illustrated by Tremain’s original example above, this construction is real whenever the unimodular simplices come from real Hadamard matrices. This gives another corollary:

**Corollary 3.3.** If there exists an \( h \times h \) real Hadamard matrix with \( h \equiv 1 \) or 2 mod 3, then there exists an equiangular tight frame of \( N \) vectors in \( \mathbb{R}^M \) with
\[
M = \frac{1}{3} (h + 1)(2h + 1), \quad N = h(2h + 1).
\]

The following section demonstrates the novelty of these ETFs by using them to construct new strongly regular graphs and distance-regular antipodal covers of the complete graph.

4. New SRGs and DRACKNs

There are two ways that certain ETFs can be associated with strongly regular graphs (SRGs). First, there is a one-to-one correspondence between real ETFs (with \( N > M \)) and a certain class of SRGs, specifically SRGs with parameters satisfying \( 2\mu = k \):

**Proposition 4.1** (see [14, 6]). Suppose there exists an equiangular tight frame of \( N \) vectors in \( \mathbb{R}^M \) with \( N > M \), and set
\[
\beta = \sqrt{\frac{N - M}{M(N - 1)}}.
\]

Then there exists a strongly regular graph with parameters
\[
v = N - 1, \quad k = \frac{N}{2} - 1 + \frac{1}{2\beta} \left( \frac{N}{M} - 2 \right), \quad \lambda = \frac{3k - v - 1}{2}, \quad \mu = \frac{k}{2}.
\]

Combined with Corollary 3.3, this then implies the following:

**Corollary 4.2.** If there exists an \( h \times h \) real Hadamard matrix with \( h \equiv 1 \) or 2 mod 3, then there exists a strongly regular graph with parameters
\[
v = (2h - 1)(h + 1), \quad k = (h + 2)(h - 1), \quad \lambda = \frac{1}{2}(h^2 + 2h - 6), \quad \mu = \frac{1}{2}(h + 2)(h - 1).
\]
Examining small values of $h \equiv 1$ or $2 \mod 3$ for which there exists a real Hadamard, we list the real Tremain ETFs that we can construct as well as the parameters of the associated SRGs given by Corollary 4.2:

<table>
<thead>
<tr>
<th>$h$</th>
<th>$M$</th>
<th>$N$</th>
<th>$v$</th>
<th>$k$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>New?</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>10</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>No</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>36</td>
<td>35</td>
<td>18</td>
<td>9</td>
<td>9</td>
<td>No</td>
</tr>
<tr>
<td>8</td>
<td>51</td>
<td>136</td>
<td>135</td>
<td>70</td>
<td>37</td>
<td>35</td>
<td>No</td>
</tr>
<tr>
<td>16</td>
<td>187</td>
<td>528</td>
<td>527</td>
<td>270</td>
<td>141</td>
<td>135</td>
<td>No</td>
</tr>
<tr>
<td>20</td>
<td>287</td>
<td>820</td>
<td>819</td>
<td>418</td>
<td>217</td>
<td>209</td>
<td>Yes</td>
</tr>
<tr>
<td>28</td>
<td>551</td>
<td>1596</td>
<td>1595</td>
<td>810</td>
<td>417</td>
<td>405</td>
<td>?</td>
</tr>
</tbody>
</table>

Examining Brouwer’s table of strongly regular graphs [1], we see that the SRG with parameters $v = 819$, $k = 418$, $\lambda = 217$, and $\mu = 209$ appears to be new to the literature. The SRG with parameters $v = 1595$, $k = 810$, $\lambda = 417$, and $\mu = 405$ is too large to be reported on the table.

Of course, the Hadamard conjecture implies that there is an infinite family of real Tremain ETFs, but this is overkill. Indeed, Kronecker powers of the $2 \times 2$ Hadamard matrix easily produces an infinite family of admissible Hadamard matrices. In addition, it is not difficult to verify using Dirichlet’s theorem that Paley’s construction is admissible infinitely often. Moreover, admissible Hadamard matrices are closed under Kronecker products, so there is actually a large family of unconditional Hadamard matrices available to the real Tremain ETF construction.

While all real ETFs give rise to an SRG by way of Proposition 4.1, there is another method of constructing different SRGs from certain ETFs. A special case of this construction of SRGs was first observed by Goethals and Seidel [8].

**Proposition 4.3** (see [4]). Suppose there exists an equiangular tight frame $\{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^M$ with $N > M$, and set

$$\alpha = \frac{N}{M}, \quad \beta = \sqrt{\frac{N - M}{M(N - 1)}}.$$  

If there exists $x \in \mathbb{R}^M$ such that $\langle x, \varphi_i \rangle = 1$ for every $i \in [N]$, then there exists a strongly regular graph with parameters

$$v = N, \quad k = \frac{N - 1}{2} + \frac{\alpha - 1}{2\beta}, \quad \lambda = \frac{N}{4} - 1 + \frac{3\alpha - 4}{4\beta}, \quad \mu = \frac{N}{4} + \frac{\alpha}{4\beta}.$$  

Note the additional hypothesis in Proposition 4.3 asks for a vector $x \in \mathbb{R}^M$ such that $\langle x, \varphi_i \rangle = 1$ for every $i \in [N]$. We will show that real Tremain ETFs can be constructed to satisfy this property if the underlying Steiner triple system contains a parallel class, that is, a subset of the blocks that form a partition of the set of points. A parallel class from a $(2, K, V)$-Steiner system must contain exactly $V/K$ blocks, thus it is a necessary condition on the existence of a parallel class that $K$ divide $V$. Fortunately, for Steiner triple systems (meaning $K = 3$) it is known that there is a Steiner triple system with a parallel class if and only if 3 divides $V$. This leads to the following:
Corollary 4.4. If there exists an \( h \times h \) real Hadamard matrix with \( h \equiv 2 \mod 3 \), then there exists a strongly regular graph with parameters
\[
v = h(2h + 1), \quad k = \frac{(2h - 1)(h + 2)}{2}, \quad \lambda = \frac{(h - 1)(h + 4)}{2}, \quad \mu = \frac{h(h + 2)}{2}.
\]

Proof. Consider real Hadamard matrices \( H_1 \) and \( H_2 \) of sizes \( h \times h \) and \( 2h \times 2h \), respectively. Note that we may select \( H_1 \) and \( H_2 \) so that each has all 1s in the first row. Put \( R = h - 1 \) and \( V = 2h - 1 \), let \( \{ \varphi_s \}_{s=1}^{R+1} \) denote the columns of the matrix \( \Phi \) obtained by removing the last row \( a \) of \( H_1 \), and let \( \{ \psi_t \}_{t=1}^{V+1} \) denote the columns of the matrix \( \Psi \) obtained by removing the first row \( b \) of \( H_2 \). Since \( h \equiv 2 \mod 3 \), we have \( V \equiv 0 \mod 3 \), and so we may select a Steiner triple system with a parallel class \( P \subseteq B \). For each \( v \in V \), pick an embedding operator \( E_v \) such that \( E_v \delta_1 = \delta_b \) for the unique \( b \in P \) such that \( v \in b \); note that such a choice is available by the definition of \( P \). Combine these ingredients to produce a real Tremain ETF according to Theorem 3.1.

Now pick \( x \in \mathbb{C}^R \oplus \mathbb{C}^V \oplus \mathbb{C} \) of the form \( x = \chi \oplus 0 \oplus \sqrt{\frac{2}{3}} \), where \( \chi(b) = 1 \) for every \( b \in P \), and zero otherwise. Then for every \( v \in V \), our choice of \( E_v \) ensures that \( E_v^* \chi = \delta_1 \). Applying the notation in (3.1), we therefore have
\[
\langle x, \tau_v, s \rangle = \langle \chi, E_v \varphi_s \rangle = \langle E_v^* \chi, \varphi_s \rangle = \langle \delta_1, \varphi_s \rangle = \varphi_s(1) = 1.
\]
Also, \( \langle x, \tau_t \rangle = \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{2}{3}} = 1 \), and so the result then follows from Proposition 4.3. \( \square \)

For small values of \( h \), we have the following table of SRG parameters that arise via Corollary 4.4:

<table>
<thead>
<tr>
<th>( h )</th>
<th>( M )</th>
<th>( N )</th>
<th>( v )</th>
<th>( k )</th>
<th>( \lambda )</th>
<th>( \mu )</th>
<th>New?</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>No</td>
</tr>
<tr>
<td>8</td>
<td>77</td>
<td>136</td>
<td>136</td>
<td>75</td>
<td>42</td>
<td>40</td>
<td>No</td>
</tr>
<tr>
<td>20</td>
<td>431</td>
<td>820</td>
<td>820</td>
<td>429</td>
<td>228</td>
<td>220</td>
<td>Yes</td>
</tr>
<tr>
<td>32</td>
<td>1073</td>
<td>2080</td>
<td>2080</td>
<td>1071</td>
<td>558</td>
<td>544</td>
<td>?</td>
</tr>
<tr>
<td>44</td>
<td>2003</td>
<td>3916</td>
<td>3916</td>
<td>2001</td>
<td>1032</td>
<td>1012</td>
<td>?</td>
</tr>
<tr>
<td>56</td>
<td>3221</td>
<td>6328</td>
<td>6328</td>
<td>3219</td>
<td>1650</td>
<td>1624</td>
<td>?</td>
</tr>
</tbody>
</table>

Again, we consult Brouwer’s table of strongly regular graphs [1] to identify which SRGs are known. The SRG with parameters \( v = 820 \), \( k = 429 \), \( \lambda = 228 \), and \( \mu = 220 \) seems to be new, and all but the first two graphs from the table above are too large to be reported on the table.

Having constructed a couple of new families of SRGs, we turn to another interesting class of graphs. An \((n,r,c)\)-distance-regular antipodal cover of the complete graph (DRACKN) in a simple graph \( G \) on \( rn \) vertices along with a partition of the vertices into \( n \) sets (called fibres) of \( r \) vertices such that

(i) no two vertices in the same fibre are adjacent,

(ii) there is a perfect matching between any two fibres, and

(iii) nonadjacent vertices have \( c \) common neighbors.

DRACKNs have been investigated in [7, 9, 3]. Recently, Coutinho, Godsil, Shirazi and Zhan proved the following result that produces DRACKNs from certain ETFs:

Proposition 4.5 (Theorem 5.1 in [3]). Let \( p \) be prime. If there exists and equiangular tight frame \( \{ \varphi_i \}_{i=1}^N \) in \( \mathbb{C}^M \) such that \( \langle \varphi_i, \varphi_j \rangle \) is a \( p \)th root of unity whenever \( i \neq j \), then there
exists an \((N, p, c)\)-distance-regular antipodal cover of the complete graph, where

\[
c = \frac{1}{p} \left( N - 2 + \frac{2M - N}{\beta M} \right), \quad \beta = \sqrt{\frac{N - M}{M(N - 1)}}.
\]

In what follows, we show how a Tremain ETF can be constructed to satisfy the hypothesis of Proposition 4.5 and hence give DRACKNs. The crucial ingredient in the following theorem is the existence of so-called Butson-type Hadamard matrices, that is, complex Hadamard matrices whose entries are roots of unity. Notationally, an \(H(q, h)\) matrix is any \(h \times h\) Hadamard matrix whose entries are all \(q\)th roots of unity.

**Corollary 4.6.** Let \(p\) be prime. If there exist \(H(p, h)\) and \(H(p, 2h)\) matrices for some \(h \equiv 1\) or \(2\) mod 3, then there exists a \((h(2h + 1), p, 2h^2/p)\)-distance-regular antipodal cover of the complete graph.

**Proof.** Consider \(H(p, h)\) and \(H(p, 2h)\) matrices \(H_1\) and \(H_2\), respectively. Put \(R = h - 1\) and \(V = 2h - 1\), let \(\{\varphi_s\}_{s=1}^{R+1}\) denote the columns of the matrix \(\Phi\) obtained by removing the first row \(a\) of \(H_1\), and let \(\{\psi_t\}_{t=1}^{V+1}\) denote the columns of the matrix \(\Psi\) obtained by removing the first row \(b\) of \(H_2\). Since \(h \equiv 1\) or \(2\) mod 3, we have \(V \equiv 1\) or \(3\) mod 6, and so we may select a Steiner triple system with appropriate embedding operators, and combine these ingredients to produce a Tremain ETF according to Theorem 3.1.

By Proposition 4.5, it suffices to show that the inner products between all pairs of vectors in this ETF lie in the set \(T_p\) of \(p\)th roots of unity. These inner products are expressed in equations (3.2)–(3.5). Since \(a_s, b_t, \varphi_s(i), \psi_t(v) \in \mathbb{T}_p\) for every \(s \in [R + 1], t \in [V + 1], i \in [R]\) and \(v \in V\), we are done. \(\square\)

It is conjectured that there exists an \(H(p, \lambda p)\) matrix exists for every \(p\) and every positive integer \(\lambda\). One relevant result in this direction is the following, due to Butson himself:

**Proposition 4.7** (see [2]). If \(p\) is prime and \(k \geq m \geq 0\) are integers, then there exists an \(H(p, 2^mp^k)\) matrix.

As such, the construction in Corollary 4.6 gives an infinite family of DRACKNs.

**References**


DEPARTMENT OF MATHEMATICS AND STATISTICS, AIR FORCE INSTITUTE OF TECHNOLOGY, WRIGHT-PATTERSON AIR FORCE BASE, OH 45433, USA
E-mail address: Matthew.Fickus@gmail.com

DEPARTMENT OF MATHEMATICS AND STATISTICS, AIR FORCE INSTITUTE OF TECHNOLOGY, WRIGHT-PATTERSON AIR FORCE BASE, OH 45433, USA
E-mail address: Dustin.Mixon@afit.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF CINCINNATI, CINCINNATI, OH 45221, USA
E-mail address: jjasper@uoregon.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, AIR FORCE INSTITUTE OF TECHNOLOGY, WRIGHT-PATTERSON AIR FORCE BASE, OH 45433, USA
E-mail address: Jesse.Peterson@afit.edu