Orthogonal Projection is a Linear Transformation

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Orthogonal Projection onto a Vector Subspace $\mathbb{W}$

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an orthog basis for a vector subspace $\mathbb{W}$ of $\mathbb{R}^n$.

Theorem (Orthogonal Decomposition Theorem)

Each vector $\vec{x}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\vec{x} = \vec{p} + \vec{z} \quad \textit{where } \vec{p} \textit{ is in } \mathbb{W} \textit{ and } \vec{z} \textit{ is in } \mathbb{W}^{\perp}.$$

In fact,

$$\vec{p} = \sum_{i=1}^{k} \mathrm{Proj}_{\vec{b}_i}(\vec{x}) = \sum_{i=1}^{k} \frac{\vec{x} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i}\, \vec{b}_i \quad \textit{and } \vec{z} = \vec{x} - \vec{p}.$$

Definition

We call $\vec{p}$ the *orthogonal projection of $\vec{x}$ onto $\mathbb{W}$*, and write $\vec{p} = \mathrm{Proj}_{\mathbb{W}}(\vec{x})$.

Note that $\mathbb{R}^n \xrightarrow{\mathrm{Proj}_{\mathbb{W}}} \mathbb{R}^n$ is a linear transformation, so ....

Orthogonal Projection is a linear transformation

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an orthog basis for a vector subspace $\mathbb{W}$ of $\mathbb{R}^n$. Consider the LT $\mathbb{R}^n \xrightarrow{\mathrm{Proj}_{\mathbb{W}}} \mathbb{R}^n$ given by orthogonal projection onto $\mathbb{W}$, so

$$\mathrm{Proj}_{\mathbb{W}}(\vec{x}) = \sum_{i=1}^{k} \frac{\vec{x} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i}\, \vec{b}_i.$$

What are:

- the kernel and range of this LT?
- the standard matrix for this LT?
- the eigenvalues and eigenvectors for this LT?

It is not hard to check that $\mathcal{R}ng(\mathrm{Proj}_{\mathbb{W}}) = \mathbb{W}$, $\mathcal{K}er(\mathrm{Proj}_{\mathbb{W}}) = \mathbb{W}^{\perp}$, and for each $\vec{w}$ in $\mathbb{W}$, $\mathrm{Proj}_{\mathbb{W}}(\vec{w}) = \vec{w}$ (so 1 is an eigenvalue and $\mathbb{E}(1) = \mathbb{W}$), for each $\vec{z}$ in $\mathbb{W}^{\perp}$, $\mathrm{Proj}_{\mathbb{W}}(\vec{z}) = \vec{0}$ (so 0 is an eigenvalue and $\mathbb{E}(0) = \mathbb{W}^{\perp}$).

Finding the standard matrix for $\mathrm{Proj}_{\mathbb{W}}$ requires a little work, but this is a worthwhile exercise!

Matrix for Orthogonal Projection Onto a Vector

The *orthogonal projection* of $\vec{x}$ onto $\vec{u}$ is given by

$$\mathrm{Proj}_{\vec{u}}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \left(\vec{x} \cdot \vec{u}\right) \vec{u}$$

provided $\vec{u}$ is a *unit* vector.

Let's compute the standard matrix A for the LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ given by $T(\vec{x}) = \left(\vec{x} \cdot \vec{a}\right) \vec{b}$ where $\vec{a}, \vec{b}$ are fixed vectors in $\mathbb{R}^n$. Recall that $\mathrm{Col}_j(A) = T(\vec{e}_j) = \left(\vec{e}_j \cdot \vec{a}\right) \vec{b} = a_j \vec{b}$, where $a_1, a_2, \dots, a_n$ are the standard coords for $\vec{a}$.

Thus $A = [a_1 \vec{b}\ a_2 \vec{b} \cdots a_n \vec{b}] = \vec{b}[a_1\ a_2 \dots a_n] = \vec{b}\, \vec{a}^T \neq \vec{a}^T\, \vec{b}$.

Applying this to the LT $\vec{x} \mapsto \mathrm{Proj}_{\vec{u}}(\vec{x}) = \left(\vec{x} \cdot \vec{u}\right) \vec{u}$ we get a standard matrix $P = \vec{u}\, \vec{u}^T$. That is, $\boxed{\mathrm{Proj}_{\vec{u}}(\vec{x}) = P\vec{x}}$.

Don't forget, this requires that $\vec{u}$ be a *unit* vector!

Matrix for Orthogonal Projection Onto a Vector SubSpace

Let $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be *orthon* basis for a vector subspace $\mathbb{W}$ of $\mathbb{R}^n$.
The LT $\mathbb{R}^n \xrightarrow{\text{Proj}_{\mathbb{W}}} \mathbb{R}^n$ given by orthogonal projection onto $\mathbb{W}$,

$$\text{Proj}_{\mathbb{W}}(\vec{x}) = \sum_{i=1}^{k} (\vec{x} \cdot \vec{u}_i)\, \vec{u}_i = \sum_{i=1}^{k} \text{Proj}_{\vec{u}_i}(\vec{x})$$

has standard matrix

$$P = \sum_{i=1}^{k} \vec{u}_i\, \vec{u}_i^{\,T} = U\, U^T$$

where

$$U = \left[\vec{u}_1 \ \vec{u}_2 \cdots \vec{u}_k\right].$$

That is, $\boxed{\text{Proj}_{\mathbb{W}}(\vec{x}) = P\vec{x}}$. This requires that $\mathcal{U}$ be an *orthon* basis!