Pointwise Bounds for the 3-Dimensional Wave Propagator
(and spectral multipliers)

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AMS Southeastern Sectional Meeting
Charleston, SC
March 11, 2017

Support provided by Simons Foundation grant #281057.
Perturbed wave equation in $\mathbb{R}^3$:
\[
\begin{cases}
  u_{tt} - \Delta u + Vu = 0 \\
  u(0, x) = 0 \\
  u_t(0, x) = g(x)
\end{cases}
\]

Potential $V(x)$ has finite Kato norm
\[
\|V\|_K := \sup_y \int_{\mathbb{R}^3} \frac{|V(x)|}{|x - y|} dx
\]

and belongs to the norm-closure of $C_c(\mathbb{R}^3)$.

This has same scaling as $\frac{C}{|x|^2}$ or $L^{3/2}(\mathbb{R}^3)$, and is (just barely) sufficient to ensure that $V$ is compact relative to $-\Delta$. 
If $V(x) \equiv 0$, the fundamental solution of
\[
\begin{cases}
  u_{tt} - \Delta u = 0 \\
  u(0, x) = 0 \\
  u_t(0, x) = g(x)
\end{cases}
\]
is given by Kirchoff's formula:

\[
K_0(t, x, y) = \frac{\delta_0(t - |x - y|) \pm \delta_0(t + |x - y|)}{4\pi(t \text{ or } |x - y|)},
\]

which satisfies

\[
\int_{-\infty}^{\infty} |K_0(t, x, y)| \, dt = \frac{1}{2\pi|x - y|}.
\]

**Question:** Does the fundamental solution of (*) also satisfy

\[
\int_{-\infty}^{\infty} |K(t, x, y)| \, dt = \frac{C}{|x - y|}?
\]
Answer: Not always. If $V(x)$ has large negative part, then $H = -\Delta + V$ may have finitely many negative eigenvalues $-\mu_j^2$.

Then (*) has solutions of the form

$$u(t, x) = \frac{\sinh(\mu_j t)}{\mu_j} \varphi_j(x),$$

where $\varphi_j(x)$ solves $(-\Delta + V)\varphi_j = -\mu_j^2 \varphi_j$.

Integrating $\int_{-\infty}^{\infty} \sinh(\mu_j t) dt$ will go badly...
Theorem (Beceanu - G.): If $\lambda = 0$ is not an eigenvalue or resonance of $H$, then

$$\int_{-\infty}^{\infty} \left| K(t, x, y) - \sum_j \frac{\sinh(\mu_j t)}{\mu_j} P_j(x, y) \right| dt < \frac{C}{|x - y|}. $$

Also, $K(t, x, y)$ is supported inside the light cones $|t| \geq |x - y|$. 

Corollary: The resolvents $R_V(z) := (H - z)^{-1}$ are integral operators whose kernels are bounded pointwise by $\frac{C}{|x - y|}$ for all $z$ in a neighborhood of $\mathbb{R}^+$. 
**Sketch of Proof:** Like many linear dispersive bounds, it starts with the Stone formula for spectral measure of $H$.

\[
\frac{\sin(t\sqrt{H})}{\sqrt{H}} - \sum_j \frac{\sinh(\mu_j t)}{\mu_j} P_j = \frac{1}{2\pi i} \int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} (R_V^+(\lambda) - R_V^-(\lambda)) \, d\lambda
\]

\[
= \frac{1}{\pi i} \int_{-\infty}^\infty \sin(t\lambda) R_V^+(\lambda^2) \, d\lambda
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^\infty (e^{-it\lambda} - e^{it\lambda}) R_V^+(\lambda^2) \, d\lambda.
\]

Here $R_V^+(\lambda^2) := \lim_{\epsilon \to 0} (H - (\lambda + i\epsilon)^2)^{-1}$ has a meromorphic extension into the upper halfplane, with poles at $\lambda = i\mu_j$.

All we need is an $L^1$ estimate on the Fourier transform of $R_V^+(\lambda^2)(x,y)$. 
If $V \equiv 0$, there is an exact formula: $R_0^+(\lambda^2)(x,y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}$.

Then $\mathcal{F}(R_0^+(\lambda^2))(t,x,y) = \frac{\delta_0(t-|x-y|)}{4\pi|x-y|}$, whose integral (in $t$) is bounded by $\frac{1}{4\pi|x-y|}$. So far, so good.

Now $R_V^+(\lambda^2) = [I + R_0^+(\lambda^2)V]^{-1}R_0^+(\lambda^2) = G(\lambda) R_0^+(\lambda^2)$.

On the Fourier side, $\mathcal{F}(R_V^+(\lambda^2))(t) = \mathcal{F}(G(\lambda)) \ast \mathcal{F}(R_0^+(\lambda^2))(t)$.

It suffices to show that $I(x,w) = \int_{-\infty}^{\infty} |\mathcal{F}(G(\lambda))(t,x,w)| \, dt$ is a bounded operator on the space $L^\infty(\mathbb{R}^3, |\cdot - y|)$, uniformly in $y$. 
N. Wiener (1932): Suppose \( g(\lambda) \in C(\mathbb{T}) \) has \( \mathcal{F}g \in \ell^1(\mathbb{Z}) \), and \( g(\lambda) \neq 0 \) pointwise over \( \lambda \in \mathbb{T} \).

Then \( \mathcal{F}\left(\frac{1}{g(\lambda)}\right) \in \ell^1(\mathbb{Z}) \).

Beceanu (2010): Similar theorems for operator-valued functions \( G(\lambda) \) on the real line. In this case it’s operators in \( \mathcal{B}\left(\frac{L^\infty}{|\cdot - y|}\right) \).

One condition is that \( G(\lambda) = I + R_0^+(\lambda^2)V \) should be invertible for each \( \lambda \in \mathbb{R} \). This corresponds to the fact/our assumption that \( H \) has no eigenvalues in \([0, \infty)\).

A secondary issue is to get continuity with respect to \( y \in \mathbb{R}^3 \) and some sort of limit as \( |y| \to \infty \).
Brief Summary: The Fourier transform of $R_V^+(\lambda^2)$ is [almost] the “forward solution” of wave equation (*).

It satisfies an integrability condition

$$\int_{-\infty}^{\infty} \left| \mathcal{F}(R_V^+(\lambda^2))(t, x, y) \right| dt \leq \frac{C}{|x - y|}$$

and a support condition

$$\mathcal{F}(R_V^+(\lambda^2))(t, x, y) = \sum_j \frac{e^{\mu_j t}}{2\mu_j} P_j(x, y) \quad \text{for all } t < |x - y|. $$
Fourier Multipliers: Given a function $m : [0, \infty) \to \mathbb{C}$, one can define $m(\sqrt{-\Delta})$ to be the Fourier multiplier with symbol $m(|\xi|)$.

Operators of this type are well studied. In particular, we note the Hörmander-Mikhlin condition: Choose a smooth bump function $\phi$ supported on $[\frac{1}{2}, 2]$. Then if

$$\sup_{k \in \mathbb{Z}} \| \phi(\lambda) m(2^{-k}\lambda) \|_{H^s(\mathbb{R})}$$

for some $s > \frac{3}{2}$, then $m(|\xi|)$ is a Calderón-Zygmund operator.

If the condition holds for $s > 2$ then the integral kernel of $m(|\xi|)$ is bounded pointwise by $|x - y|^{-3}$. 
Given the same function \( m : [0, \infty) \to \mathbb{C} \), one can also define the spectral multiplier \( m(\sqrt{H}) \) in the functional calculus of \( H \).

**Theorem** (Beceanu - G.): If \( m \) satisfies the Hörmander-Mikhlin condition with \( s > \frac{3}{2} \), then \( m(\sqrt{H}) \) is bounded on \( L^p(\mathbb{R}^3) \) for \( 1 < p < \infty \).

If the condition holds for \( s > 2 \) then the integral kernel of \( m(\sqrt{H}) \) is bounded pointwise by \( |x - y|^{-3} \).

In particular \( H^{i\sigma} \) is well behaved, which is enough to deduce endpoint Strichartz estimates for \( (*) \).

It is not clear that the integral kernel of \( m(\sqrt{H}) \) satisfies

\[
\int_{|x-y|>2|y-y'|} |K(x,y) - K(x,y')| \, dx < C
\]

so our results do not include weak \((1,1)\) bounds for now.
Why these theorems are closely related:
The Stone formula for spectral measure gives us

$$m(\sqrt{H}) = \frac{1}{2\pi i} \int_0^\infty m(\sqrt{\lambda})(R^+_V(\lambda) - R^-_V(\lambda)) \, d\lambda$$

$$= \frac{1}{\pi i} \int_{-\infty}^{\infty} \lambda m(|\lambda|)R^+_V(\lambda^2) \, d\lambda$$

$$= \frac{1}{\pi i} \int_{-\infty}^{\infty} \mathcal{F}^{-1}\left(\lambda m(|\lambda|)\right)(t) \, \mathcal{F}\left(R^+_V(\lambda^2)\right)(t) \, dt.$$

If we assume \(s > 2\), then the Fourier transform of \(\lambda m(\lambda)\) decays like \(|t|^{-2}\). And \(\mathcal{F}(R^+_V(\lambda^2))\) is [mostly] supported where \(t > |x-y|\).

When \(t < |x-y|\) there is an explicit description of \(\mathcal{F}(R^+_V(\lambda^2))\) as a sum of exponential functions. This makes it easier to handle the \(|t|^{-2}\) singularity near \(t = 0\).
Questions for further study:

• Can you integrate the solution of (*) along time-like paths $(t, x(t))$ and still get a bound in terms of $\frac{1}{|x(0)-y|}$?

• Does $m(\sqrt{H})$ satisfy a weak $(1, 1)$ bound, even for really nice multipliers $m$?

• Is there a Hardy space theory of these multipliers?

• What happens for $s \leq \frac{3}{2}$? Is there a robust $L^p$ theory for Bochner-Riesz spectral multipliers?

• Do you have any good questions to contribute?