ASYMPTOTIC PROPERTIES OF THE VECTOR CARLESON EMBEDDING THEOREM

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Abstract. The dyadic Carleson embedding operator acting on $\mathbb{C}^n$-valued functions has norm at least $C \log n$. Thus the Carleson Embedding Theorem fails for Hilbert space valued functions.

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$, and $\{I\}_{I \in D}$ its collection of dyadic arcs. Let $w_I$ be nonnegative real numbers indexed by $I \in D$. For integrable functions $f$ on $\mathbb{T}$, denote by $\langle f \rangle_I$ the average $|I|^{-1} \int_I f(y)dy$. The classical Carleson embedding theorem [1] is equivalent to the following dyadic result:

**Theorem 0.** If $\sum_{I \subset K} w_I \leq |K|$ for all $K \in D$, then $\sum_{I \in D} w_I \langle f \rangle_I^2 \leq C \|f\|^2$ for all $f \in L^2(\mathbb{T})$.

The converse is also true (up to the placement of constants) and is verified by considering functions of the form $f = \chi_J, J \in D$.

An analogous statement may be made for functions taking values in $\mathbb{C}^n$ with matrix-valued weights $W_I \geq 0$ in the sense of quadratic forms. We wish to consider the following $n$-dimensional embedding theorem:

**Proposition.** If $\|\sum_{I \subset K} W_I\| \leq |K|$ for all $K \in D$, then $\sum_{I \in D} (W_I \langle f \rangle_I, \langle f \rangle_I) \leq C_n \|f\|^2$ for all $f \in L^2(\mathbb{T}; \mathbb{C}^n)$.

The space $\mathbb{C}^n$ here is viewed as a finite-dimensional Hilbert space. One might ask whether a similar result still holds when $f$ takes values in a general Hilbert space $\mathbb{H}$ and $W_I$ are positive selfadjoint operators. This is answered in the negative by [4], which proves that $C_n$ must be bounded from below by $c \log n$. In the current paper we will use the construction in [4] to verify the stronger bound $C_n \geq c (\log n)^2$, which is also proved in [5]. A precise statement is as follows:

**Theorem 1.** There exist a function $f \in L^2(\mathbb{T}; \mathbb{C}^n)$ and matrix weights $W_I \geq 0$ such that $\|\sum_{I \subset K} W_I\| \leq |K|$ and $\sum_{I \in D} (W_I \langle f \rangle_I, \langle f \rangle_I) \geq c (\log n)^2 \|f\|^2$, where $c > 0$ is independent of $n$.

Remarks. The example presented here is due to Nazarov, Treil, and Volberg [4]. It is further shown in [3] and [4] that the best possible $C_n$ is bounded above by $C (\log n)^2$, making these results sharp up to a constant factor.

**Proof of Theorem 1.** Let $e_0, e_1, \ldots, e_n$ be the standard basis for $\mathbb{C}^{n+1}$. Define the Rademacher functions $r_j(e^{2\pi i t}) = (-1)^{2jt}$. For a dyadic interval $I, |I| \leq 2^{-j}$, $r_j$ is seen to be constant along $I$. Its value throughout the interval will be called $r_j(I)$.

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Let \( f(x) = \sum_{j=0}^{n} r_j(x)e_j \). Clearly \( \|f\|^2 = n + 1 \). The averages of \( f \) over dyadic intervals are also easy to compute. When \( |I| = 2^{-i} \), \( f(I) = \sum_{j=0}^{i} r_j(I)e_j \).

Let \( W_I, |I| \geq 2^{-n} \) be the rank-one operator satisfying \( W_Iv = |I|(v, \phi_I)\phi_I \), where \( \phi_I = \sum_{j=0}^{i} \frac{1}{i+j-1} r_j(I)e_j \). Define \( \phi_I \) to be 0 when \( |I| < 2^{-n} \). Already we can estimate the sum

\[
\sum_{I \in D} (W_I(f)_I, (f)_I) = \sum_{I \in D} |I|((f)_I, \phi_I)^2 = \sum_{I \in D} \left( \sum_{j=0}^{i} \frac{1}{i + 1 - j} \right)^2 \geq cn(\log n)^2
\]

The only task remaining is to show that \( \| \sum_{I \subseteq K} W_I \| \) is controlled by \( |K| \). We will prove the estimate \( \sum_{I \subseteq K} W_Iv, v = \sum_{I \subseteq K} |I|(v, \phi_I)^2 \leq C|K||v|^2 \) for all \( v \in C^{n+1} \).

For each interval \( I \) with \( |I| = 2^{-i} \), split the vector \( \phi_I \) into the sum of two parts, \( \phi_I = \sum_{j=0}^{k} \frac{1}{i + j - 1} r_j(K)e_j + \sum_{j=k+1}^{n} \frac{1}{i + j - 1} r_j(I)e_j \). Denote the first sum, which depends only on the length of \( I \subseteq K \), by \( g_i \). Summing over all \( I \) with \( |I| = 2^{-i} \), and exploiting the orthogonality of the Rademacher functions,

\[
\sum_{I \subseteq K} |I|(v, \phi_I)^2 = |K|(v, g_i)^2 + \sum_{j=k+1}^{n} \frac{1}{(i + 1 - j)^2} |v_j|^2
\]

Thus

\[
\sum_{I \subseteq K} (W_Iv, v) = |K|(\sum_{i=k}^{n} (v, g_i)^2 + \sum_{j=k+1}^{n} |v_j|^2 \sum_{i=j}^{n} \frac{1}{(i + 1 - j)^2})
\]

The second sum is less than \( C|K| \sum_{j=0}^{n} |v_j|^2 = C|K|^2 |v|^2 \). To estimate the first sum, let \( G \) represent the \((n - k + 1) \times (k + 1)\) matrix whose \( i \) entry is the coefficient of \( e_{j-1} \) in \( g_{i+k-1} \). Then \( \sum_{i=k}^{n} (v, g_i)^2 \leq \|G\|^2 |v|^2 \). Here \( \|G\| \) is taken as an operator from \( C^{k+1} \) to \( C^{n-k+1} \). Under a suitable permutation of indices, however, \( G \) is seen to be a restriction of the Hilbert matrix \( A_i(A_{ij} = \frac{1}{i+j}) \) to finite-dimensional subspaces. It is well known [2] that \( A \) is bounded from \( \ell^2(\mathbb{N}) \) to itself. Thus the first sum is less than \( |K||v|^2 = C|K|^2 |v|^2 \). Dividing all weights \( W_I \) by an appropriate constant proves the theorem.

References


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