THE HELMHOLTZ EQUATION WITH $L^p$ DATA AND
BOCHNER-RIESZ MULTIPLIERS

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Abstract. We prove the existence of $L^2$ solutions to the Helmholtz equation
\((-\Delta - 1) u = f\) in $\mathbb{R}^n$ assuming the given data $f$ belongs to $L^{(2n+2)/(n+5)}(\mathbb{R}^n)$
and satisfies the “Fredholm condition” that $\hat{f}$ vanishes on the unit sphere. This
problem, and similar results for the perturbed Helmholtz equation $(-\Delta - 1) u =
-V u + f$, are connected to the Limiting Absorption Principle for Schrödinger
operators.

The same techniques are then used to prove that a wide range of $L^p \mapsto L^q$
bounds for Bochner-Riesz multipliers are improved if one considers their action
on the closed subspace of functions whose Fourier transform vanishes on the
unit sphere.

We consider the existence of a well-defined solution map for the Helmholtz equa-
tion in Euclidean space
\begin{equation}
\begin{cases}
(-\Delta - 1)u = f \text{ in } \mathbb{R}^n \\
u \in L^2(\mathbb{R}^n)
\end{cases}
\end{equation}

By conjugating with dilations, the same problem can be posed with an operator
\((-\Delta - \lambda^2), \lambda > 0\) with minimal modification. These equations are translation
invariant, so it would be desirable to choose $f$ from a function space whose norm is
also translation invariant. Our goal is to establish existence of solutions and a norm
bound for $u$ in terms of the $L^p$-norm of the given data $f$, provided $f$ is formally
orthogonal to all plane waves of unit frequency.

The Fourier dual formulation of (1) is
\begin{equation}
\begin{cases}
\hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 - 1} \\
\hat{u} \in L^2(\mathbb{R}^n)
\end{cases}
\end{equation}

with respect to the definition $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx$. The corresponding Plancherel
identity is $\|\hat{u}\|_2 = (2\pi)^{n/2} \|u\|_2$.

It is immediately clear from (2) that solutions should be unique, as $|\xi|^2 - 1$ is
nonzero almost everywhere and the Fourier Transform is (a scalar multiple of) a
unitary map between $L^2(dx)$ and $L^2(d\xi)$. One can also infer that solutions exist
only if $\hat{f}$ vanishes on the unit sphere in a suitable sense, and also the restrictions
of $\hat{f}$ to the sphere of radius $r$ must be controlled as $r$ approaches 1.

It would be sufficient, for example, if the map $S(r) = \hat{f}(r \cdot)\big|_{S^{n-1}}$ (taking $\mathbb{R}_+$
into $L^2(S^{n-1})$) was Hölder continuous of order $\gamma > \frac{1}{2}$ at $r = 1$ and vanished there.

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Theorem 2. for the Helmholtz equation exists for data in a narrow range of L, is Hölder continuous of any positive order. Nevertheless an function u vanishes on the unit sphere in the L, regardless of the exponent, as an L, is our main technical result.

Theorem 1 (Agmon). Suppose (1 + |x|)βf ∈ L2(Rn) for some β > 1/2, and ˆf vanishes on the unit sphere in the L2-trace sense. Then there exists a unique function u such that (−Δ − 1)u = f and (1 + |x|)β−1u ∈ L2(Rn).

It is not obvious that a similar result should hold for Lp(Rn) without weights, regardless of the exponent, as an Lp condition typically doesn’t guarantee that S(r) is Hölder continuous of any positive order. Nevertheless an L2 solution operator for the Helmholtz equation exists for data in a narrow range of Lp spaces.

Theorem 2. Let n ≥ 3 and max(1, 2n/(n+4)) ≤ p ≤ 2n/(n+2), with (n, p) ≠ (4, 1). Suppose f ∈ Lp(Rn) and ˆf vanishes on the unit sphere in the L2-trace sense.

There exists a unique u ∈ L2(Rn) such that (−Δ − 1)u = f. Furthermore, ||u||2 ≤ Cn,p||f||p.

There is no statement in dimensions 1 or 2 because 2n/(n+4) < 1. When n = 1 it should suffice to allow e±|x|f to belong to the Hardy space H2/3(ℝ). It is less clear what cancellation conditions might be required for n = 2.

The lower exponent bound of 2n/(n+4) comes from Sobolev embedding. It can be disregarded if one applies any sort of cutoff to remove high frequencies. As a special case, the sharp cutoff at |ξ| = 1 leaves a Bochner-Riesz multiplier of order -1. For further discussion of these operators we adopt the definition

(5) (Sαf)̂(ξ) = (1 − |ξ|^2)^α f(ξ).

For α ≤ −1 we define Sα by (formal) positivity of the operator rather than by analytic continuation. This preserves the multiplicative structure SαSβ = Sα+β, however it comes at the cost that Sα will not have a bounded action on general Schwartz functions once α ≤ −1.

Never the less, Sα may behave well when applied to functions whose Fourier transform vanishes on the unit sphere, as stated below.

Theorem 3. Let n ≥ 2 and 1/2 ≤ α ≤ 3/2. Suppose f ∈ Lp(Rn), 1 ≤ p ≤ 2n/(n+1+4α) with (α, p) ≠ (1/2, 2n/(n+3)), and suppose ˆf vanishes on the unit sphere.

Then ||S−αf||2 ≲ ||f||p.

Both Theorems 2 and 3 are easily derived from the following statement, which is our main technical result.
Proposition 4. Let \( n \geq 2 \) and \( \frac{1}{2} < \alpha < \frac{3}{2} \). Suppose \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \frac{2n+2}{n+1+4\alpha} \). There is a constant \( C_\alpha \) such that

\[
\left| \int_{\frac{1}{2}<|\xi|<\frac{1}{2}} \frac{|\hat{f}(\xi)|^2}{((1-|\xi|^2)^2 + \varepsilon^2)^\alpha} d\xi - \frac{C_\alpha}{\varepsilon^{2\alpha-1}} \|\hat{f}\|_{L^2(\mathbb{R}^n)}^2 \right| \lesssim \|f\|_p^2
\]

with a constant that remains bounded in the limit \( \varepsilon \to 0 \).

In both theorems, it is given that \( \hat{f} \) vanishes on the unit sphere, eliminating the \( \varepsilon^{1-2\alpha} \|f\|^2 \) term from the left side of (6). Assuming Proposition 4 holds, the same inequality is then true with \( \varepsilon = 0 \) by monotone convergence. The Hausdorff-Young inequality is more than sufficient to bound the left-side integral over the center region \( \{ |\xi| < \frac{1}{2} \} \) for any \( f \in L^p, 1 \leq p \leq 2 \).

For Theorem 2, let \( \chi \in C_c^\infty(\mathbb{R}^n) \) be any smooth cutoff that is identically 1 in the ball \( \{ |\xi| \leq \frac{2}{3} \} \) and has support in the ball of radius \( \frac{1}{2} \). Theorem 2 then reduces to the \( \alpha = 1 \) case of Proposition 4 combined with a Sobolev embedding estimate for the high frequency tail \( \frac{1}{\varepsilon^{2\alpha-1}} \hat{f} \). In a similar manner, all cases of Theorem 3 with \( \alpha > \frac{1}{2} \) follow from the Proposition by applying the multiplier of the unit ball, which is bounded on \( L^2(\mathbb{R}^n) \).

Finally, if \( \alpha = \frac{1}{2} \) and \( p \in [1, \frac{2n+2}{n+3}] \), we have already established Theorem 3 for the pair \((\beta, p)\) with \( \beta = \min\left( (n+1)(\frac{4}{2n} - \frac{1}{4}), 1 \right) \). Since \( \beta > \frac{1}{2} \), it follows that \( \|S^{-1/2}f\|_2 \leq \|S^{-\beta}f\|_2 \) by Plancherel’s formula.

Sharpness of the upper exponent \( \frac{2n+2}{n+1+4\alpha} \) is verified using a Knapp-type example. Let \( \hat{f} \) be a smooth compactly supported function, suitably scaled to have support in the slab \( \{ |\xi'| \leq \delta, 1-2\delta^2 \leq |\xi| \leq 1-\delta^2 \} \), where \( \xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1} \) and unit height. Then \( |f(x)| \sim \delta^{n+1} \) over the dual region \( \{ |x'| \leq \delta^{-1}, |x_n| \leq \delta^{-2} \} \) and has rapid decay elsewhere. It follows that \( \|f\|_p \sim \delta^{(n+1)(1-p^{-1})} \) and \( \|S^{-\alpha}f\|_2 \sim \delta^{\frac{n+1}{2}-2\alpha} \). If \( p > \frac{2n+2}{n+1+4\alpha} \) then \( (n+1)(1-p^{-1}) > \frac{n+1}{2} - 2\alpha \) and Theorem 3 fails by taking \( \delta \) to zero.

Remark 1. When \( 0 < \alpha < \frac{1}{2} \), no vanishing condition on the unit sphere is needed in the statement of Theorem 3. The range of viable exponents is once again \( p \in [1, \frac{2n+2}{n+3}] \) including the endpoints [3]. The full range of \( L^p \to L^q \) mappings in this regime is established in [2].

Remark 2. The statement of Proposition 4 is not true for \( \alpha > 1 \) if integration is limited to the inner annulus \( \{ \frac{1}{2} < |\xi| < 1 \} \). An additional remainder term of order \( \varepsilon^{2-2\alpha} \) is present in that case. The same remainder term appears with the opposite sign if one integrates over the outer annulus \( \{ 1 < |\xi| < \frac{3}{2} \} \). This illustrates a difference in behavior between “one-sided” and “two-sided” Bochner-Riesz multipliers of order below –1, with the former being modestly more singular than the latter.

Remark 3. The endpoint case \( \alpha = \frac{1}{2}, p = \frac{2n+2}{n+3} \) is quite delicate. The conclusion is certainly false if one does not assume that \( \hat{f} \) vanishes on the unit sphere. In one dimension it remains false even with the vanishing condition. Since \( S^{-\frac{1}{2}} \) in one dimension is closely related to the fractional integral operator \( I_{1/2} \), a stronger condition that \( \varepsilon^{2+\frac{1}{2}}f \) belongs to the Hardy space \( H^1(\mathbb{R}) \) is needed to guarantee that \( S^{-1/2}f \in L^2(\mathbb{R}) \).
The one-dimensional counterexamples do not generalize well to \( n \geq 2 \). We believe it is an open problem whether Theorem 3 is true in these endpoint cases.

Theorem 1 plays an important role in the spectral theory of Schrödinger operators \( H = -\Delta + V(x) \) with a short-range potential. Namely, it is used in a bootstrapping argument to show that any singular part of the essential spectrum of \( H \) must contain embedded eigenvalues. Thus the spectral measure on compact subsets of \([0, \infty) \setminus \sigma_{pp}(H)\) is absolutely continuous and satisfies an assortment of uniform mapping properties. In Section 2 we present a similar bootstrapping application using Theorem 2 as the primary device. These results are contained within the more general Limiting Absorption Principle of Ionescu and Schlag [4], and serve as an instructive special case.

The discussion of perturbed Schrödinger operators naturally raises the question of whether there is a similar existence theorem for the Helmholtz equation \((-\Delta + V - 1)u = f\). In Section 3 we use resolvent identities to obtain an affirmative answer.

**Theorem 5.** Let \( n \geq 3 \), \( p_0 = \frac{2n+2}{n+5} \), and suppose \( V \in L^{\frac{n+1}{2}}(\mathbb{R}^n) \). There is a subspace \( X \subset L^{p_0}(\mathbb{R}^n) \), isomorphic to the subspace \( X_0 \subset L^{p_0}(\mathbb{R}^n) \) of functions whose Fourier transform vanishes on the unit sphere, with the following property: For each \( f \in X \) there exists a unique \( u \in L^2(\mathbb{R}^n) \) such that \((-\Delta + V - 1)u = f\). Furthermore, \( \|u\|_2 \leq C_n\|f\|_{p_0} \).

The spaces \( X \) and \( X_0 \) are formally linked via the action of the wave operators for \(-\Delta + V\). For potentials where the wave operators are known to be bounded on \( L^{p_0}(\mathbb{R}^n) \) one can prove Theorem 5 directly via the intertwining relations. However the given integrability condition \( V \in L^{\frac{n+1}{2}}(\mathbb{R}^n) \) is a sharp threshold for the absolute continuity of spectral measure in both [4] and [6], and it seems unlikely that wave operators are well behaved outside of \( L^2 \) in this generality.

**Remark 4.** Versions of Theorem 5 can be stated for \( 1 \leq p < \frac{2n+2}{n+5} \) as well. The conditions on \( V \) are dictated by the range of resolvent estimates available on \( L^p(\mathbb{R}^n) \), and will be more restrictive than when \( p = \frac{2n+2}{n+5} \).

Finally, it is possible to extend Theorem 3 further by interpolation with other known estimates for Bochner-Riesz operators, subject to a few technical limitations. In this paper we do not assemble a full catalog of such estimates but instead consider a family of bounds that are sharp with respect to Knapp counterexamples. The following result is proved in Section 4.

**Theorem 6.** Let \( n \geq 2 \) and \( \beta \in \left(\frac{1}{2}, \frac{3}{2}\right) \) with \( \beta \leq \frac{n+1}{4} \). Suppose \( f \in L^{\frac{2n+2}{n+1+\beta}}(\mathbb{R}^n) \) with \( f \) vanishing on the unit sphere, and \( \alpha \in [\beta, 2\beta] \) with \( \alpha < 2 \). Then
\[
\|S^{-\alpha}f\|_{\frac{2n+2}{n+1+\beta}} \lesssim \|f\|_{\frac{2n+2}{n+1+\beta}}.
\]

**Remark 5.** The restriction \( \alpha < 2 \) may be removed if one instead considers the analytic family of operators \( S^{-\alpha} = \Gamma(1-\alpha)^{-1}S^{-\alpha} \). This will be evident in the proof.

1. **Proof of Proposition 4**

The proof of Proposition 4 mirrors that of the sharp Stein-Tomas restriction theorem. We follow the exposition in [7] most closely.
Let $\sigma_r$ denote the surface measure on $rS^{n-1}$ inherited from its embedding in $\mathbb{R}^n$. The main estimate will be a bound on $\langle K_1^r * f, f \rangle$, where

$$
K_1^r = \int_{-\frac{r}{2}}^{\frac{r}{2}} \frac{\sigma_r - \sigma_1}{((1 - r^2)^2 + \varepsilon^2)^\alpha} \, dr = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sigma_{1+s} - \sigma_1}{(s^2(2 + s^2 + \varepsilon^2))^{\alpha}} \, ds.
$$

This is almost equal to the left-hand expression in (6), with the only discrepancy arising in the coefficient of $\langle \sigma_1 * f, f \rangle$. More precisely,

$$
[\text{Left side of (6)}] - \langle K_1^r * f, f \rangle = \left( \int_{-\frac{r}{2}}^{\frac{r}{2}} ((1 - r^2)^2 + \varepsilon^2)^{-\alpha} \, dr - \frac{C_\alpha}{\varepsilon^{2\alpha - 1}} \right) \langle \sigma_1 * f, f \rangle

= O(1) \langle \sigma_1 * f, f \rangle.
$$

Since $f \in L^p(\mathbb{R}^n)$ with $p \leq \frac{2n+2}{n+1}$, the $O(1)$ term can be absorbed into the right side of (6) by the Stein-Tomas theorem.

The integrand in (8) may become highly singular at $s = 0$ as $\varepsilon$ decreases. However the denominator is approximately an even function of $s$, while the leading order behavior of the numerator is an odd function. To be precise, let

$$
A_{\text{even}}(s) = \frac{1}{2} \left( \frac{1}{(s^2(2 + s^2 + \varepsilon^2))^{\alpha}} + \frac{1}{(s^2(2 - s^2 + \varepsilon^2))^{\alpha}} \right),
$$

$$
A_{\text{odd}}(s) = \frac{1}{2} \left( \frac{1}{(s^2(2 + s^2 + \varepsilon^2))^{\alpha}} - \frac{1}{(s^2(2 - s^2 + \varepsilon^2))^{\alpha}} \right).
$$

Then

$$
K_1^r = \frac{1}{2} \int_{-\frac{r}{2}}^{\frac{r}{2}} \left( A_{\text{even}}(s)(\sigma_{1+s} - 2\sigma_1 + \sigma_{1-s}) + A_{\text{odd}}(s)(\sigma_{1+s} - \sigma_{1-s}) \right) \, ds.
$$

The main size bounds for $A_{\text{even}}$ and $A_{\text{odd}}$ are:

$$
|A_{\text{even}}(s)| \lesssim s^{-2\alpha}, \quad |A_{\text{odd}}(s)| \lesssim s^{1-2\alpha} \quad \text{uniformly in } \varepsilon > 0.
$$

It is a common practice to estimate the inverse Fourier transform of a surface measure by decomposing the surface into smaller regions where stationary phase methods can be applied. Consider a conical decomposition $\sum_{j=1}^{n} \eta_j(\frac{\xi}{|\xi|}) = 1$ where each smooth cutoff $\eta_j$ is supported in the region where $|\xi_j| \sim |\xi|$. One may symmetrize so that each $\eta_j$ is invariant under reflections across any one of the coordinate planes. Then (8) splits into a directional sum

$$
K_1^r = \sum_{j=1}^{n} \int_{-\frac{r}{2}}^{\frac{r}{2}} \frac{\eta_j(\sigma_{1+s}) - (\eta_j\sigma_1)}{(s^2(2 + s^2 + \varepsilon^2))^{\alpha}} \, ds.
$$

Let $K_2^r$ denote the $j = n$ term of this sum and write coordinates in $\mathbb{R}^n$ as $(x', x_n)$ or $(\xi', \xi_n)$. We will make further estimates on $K_2^r$ as a representative element.

Inside the support of $\eta_n\sigma_r$, the relationship $\xi_n = \pm (r^2 - |\xi'|^2)^{1/2}$ expresses $\xi_n$ as a smooth function of $\xi'$ on each hemisphere. Then the inverse Fourier transform of $\eta_n\sigma_r$ takes the form

$$
(\eta_n\sigma_r)(x', x_n) = (2\pi)^{-n} \sum_{\pm} \int_{\mathbb{R}^{n-1}} \frac{r \eta_n(\xi', \pm \sqrt{r^2 - |\xi'|^2})}{\sqrt{r^2 - |\xi'|^2}} e^{i(x' \cdot \xi' \pm x_n \sqrt{r^2 - |\xi'|^2})} \, d\xi'.
$$
For $\frac{1}{2} < r < \frac{3}{2}$, the Hessian of the phase function is bounded below by $x_n$ times the $(n-1)$-identity matrix and the initial fraction is a uniformly smooth function. This leads to the pointwise bound

$$|\eta_n \sigma_r(x', x_n)| \lesssim (1 + |x_n|)^{\frac{1-n}{2}}.$$ 

for $r$ in this range. Furthermore one can differentiate with respect to $r$ under the integral sign to obtain bounds

$$|\partial_r^k(\eta_n \sigma_r)(x', x_n)| \lesssim (1 + |x_n|)^{\frac{1-n}{2} + k}.$$ 

Taylor remainder estimates then imply that

$$|\eta_n \sigma_{1+s} - \eta_n \sigma_{1-s} - (x', x_n)| \lesssim \min(1 + |x_n|, 1) (1 + |x_n|)^{\frac{1-n}{2}},$$

$$|\eta_n \sigma_{1+s} - 2\eta_n \sigma_1 - (x', x_n)| \lesssim \min(1 + |x_n|, 1) (1 + |x_n|)^{\frac{1-n}{2}}$$

while $s < \frac{1}{2}$. Plugging these and (10) into the appropriately modified version of (9), one concludes that

$$|K_2(x)| \lesssim (1 + |x_n|)^{4n-1-n}.$$ 

In other words, for a fixed choice of $x_n$, the restricted convolution operator

$$Tg(x') = \int_{\mathbb{R}^{n-1}} K_2(x' - y', x_n) g(y')\,dy'$$

maps $L^1(\mathbb{R}^{n-1})$ to $L^\infty(\mathbb{R}^{n-1})$ with operator norm controlled by $(1 + |x_n|)^{4n-1-n}$.

One can also determine the size of $T$ as an operator on $L^2(\mathbb{R}^{n-1})$. This bound is given by the essential supremum of the $x'$-Fourier transform of the convolution kernel $K_2$. Since $K_2$ is a superposition of the inverse Fourier transforms of $(\eta_n \sigma_s)$ as in (11), the $x'$-Fourier transform reverses the procedure in all except the $x_n$ variable. More precisely,

$$\int_{\mathbb{R}^{n-1}} e^{-i \xi' \cdot x'} K_2(x', x_n)\,dx' = \frac{1}{2\pi} \int_{\xi' \times \mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{e^{ix_n \xi_n (\sigma_1 + s - \sigma_1)\eta_n}}{(s^2 + s_1 + s_2 + \varepsilon)^n} \,ds \,d\xi_n.$$

If the integral over $s$ is split into even and odd contributions as in (9), the result is

$$\int_{\mathbb{R}^{n-1}} e^{-i \xi' \cdot x'} K_2(x', x_n)\,dx'$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^{n-1}} \left( A_{\text{even}}(s) \int_{\xi' \times \mathbb{R}} e^{ix_n \xi_n (\sigma_1 + s - \sigma_1)\eta_n} \,d\xi' 
+ A_{\text{odd}}(s) \int_{\xi' \times \mathbb{R}} e^{ix_n \xi_n (\sigma_1 + s - \sigma_1)\eta_n} \,d\xi' \right) \,ds$$

For a fixed choice of $\xi' \in \mathbb{R}^{n-1}$ and radius $r > 0$, the line $\{\xi'\} \times \mathbb{R}$ intersects the support of $\sigma_r$ only when $\xi_n = \pm \sqrt{r^2 - |\xi'|^2}$. Thus for $|\xi'| < r$

$$\int_{\xi' \times \mathbb{R}} e^{ix_n \xi_n \sigma_r \eta_n} \,d\xi' = \frac{2 \cos(x_n \sqrt{r^2 - |\xi'|^2}) \eta_n(\xi'_r, \sqrt{1 - (|\xi'|/r)^2})}{\sqrt{1 - (|\xi'|/r)^2}}$$

and is zero otherwise. The denominator accounts for the angle of intersection between the line and surface. It is bounded away from zero within the support of $\eta_n$, so the integral expression is a smooth bounded function of $\xi'$ and $r$. Within
the range $\frac{1}{2} < r < \frac{3}{2}$, its first two derivatives with respect to $r$ are bounded by $(1 + |x_n|)$ and $(1 + |x_n|)^2$ respectively. It follows that
\[
\left| \int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} K_2^{\epsilon}(x', x_n) dx' \right| \lesssim \int_{-\frac{1}{2}}^{\frac{1}{2}} A_{\text{even}}(s) \max(s^2(1 + |x_n|)^2, 1) \nonumber \\
\quad + A_{\text{odd}}(s) \max(|s|(1 + |x_n|), 1) \, ds \nonumber \\
\lesssim (1 + |x_n|)^{2\alpha - 1}
\]
and therefore $T$ is a bounded operator on $L^2(\mathbb{R}^{n-1})$ with norm comparable to $|x_n|^{2\alpha - 1}$. Interpolating with the previous $L^1 \to L^\infty$ bound shows that
\[
\|Tg\|_{L^{p'}(\mathbb{R}^{n-1})} \lesssim (1 + |x_n|)^{2\alpha + \frac{n-3}{2} + \frac{1-n}{p}} \|g\|_{L^p(\mathbb{R}^{n-1})}, \quad 1 \leq p \leq 2.
\]
Returning to the action of $K_2^{\epsilon}$ on functions in $\mathbb{R}^n$, these estimates imply that
\[
\|K_2^{\epsilon} * f\|_{p'} \lesssim \left\| \int_{-\infty}^{\infty} (1 + |x_n - y_n|)^{2\alpha + \frac{n-3}{2} + \frac{1-n}{p}} \|f(\cdot, y_n)\|_{L^p(\mathbb{R}^{n-1})} \, dy_n \right\|_{L^{p'}(\mathbb{R})}
\]
provided $2\alpha + \frac{n-3}{2} + \frac{1-n}{p} \leq \frac{2}{p} - 2$, or more simply $1 \leq p \leq \frac{2n+2}{4\alpha + n+1}$. The last step is a restatement of the Hardy-Littlewood-Sobolev inequality in one dimension.

Summing over the $n$ pieces of the conical decomposition concludes the proof.

2. Application to embedded resonances of $-\Delta + V$

Statements like Theorem 2 are useful for constraining the spectral measure of Schrödinger operators $-\Delta + V$ with a scalar perturbation $V \in L^r(\mathbb{R}^n)$. We present a version of the Limiting Absorption Principle here; one can find a more extensive set of tools and results in [4].

**Proposition 7.** Suppose $n \geq 3$ and $V \in L^{\frac{n+1}{n+1}}(\mathbb{R}^n)$ is real valued. The Schrödinger operator $H = -\Delta + V$ has absolutely continuous spectrum on the interval $(0, \infty)$.

**Proof.** It is known that the free resolvent $R_0^+ (\lambda) = \lim_{\epsilon \to 0^+} (-\Delta - (\lambda + i\epsilon))^{-1}$ maps $L^{\frac{2n+2}{n+1}}(\mathbb{R}^n)$ to $L^{\frac{2n+2}{n+1}}(\mathbb{R}^n)$ for each $\lambda > 0$, with operator norm proportional to $\lambda^{-1/n+1}$ [5]. One may present the resolvent of $H$ using identities such as
\[
R_V^+ (\lambda) := \lim_{\epsilon \to 0^+} (H - (\lambda + i\epsilon))^{-1} = (I + R_0^+ (\lambda)V)^{-1} R_0^+ (\lambda).
\]
The spectral measure of $H$ is given by the imaginary part of $R_V^+ (\lambda)$, and this measure is absolutely continuous on intervals where the perturbed resolvent maps between the dual $L^p$ spaces named above with uniform bound. Note that $\lambda^{-1/n+1}$ is bounded on each compact set $E \subset (0, \infty)$.

Under the given condition $V \in L^{\frac{n+1}{n+1}}(\mathbb{R}^n)$, the factor $(I + R_0^+ (\lambda)V)$ is a compact perturbation of the identity on $L^{\frac{2n+2}{n+1}}(\mathbb{R}^n)$ except when $\lambda = 0$. Furthermore, it varies continuously (in operator norm) with respect to $\lambda$. It follows that the norm of $(I + R_0^+ (\lambda)V)^{-1}$ is bounded over $\lambda \in E$ unless there exists $\lambda_0 \in E$ and a function $g \in L^{\frac{2n+2}{n+1}}$ such that $g = -R_0^+ (\lambda_0)Vg$.

Such a function also has the property $(R_V^+ (\lambda_0) Vg, Vg) = -(g, Vg)$, where $(\cdot, \cdot)$ is the sesquilinear pairing between $L^{\frac{2n+2}{n+1}}$ and its dual. The imaginary part of the left-hand pairing is equal to $\epsilon \lambda_0^{-1/2} \|Vg\|^2 \langle d\sigma, \sqrt{\frac{\sigma}{\lambda_0^2}} \rangle$, whereas the right-hand pairing is real valued, hence the Fourier transform of $Vg$ vanishes on the sphere
radius $\sqrt{\lambda_0}$. Furthermore, $g$ is a solution of the Helmholtz equation $(-\Delta - \lambda_0)g = -Vg$.

The statement of Theorem 2 can be modified to accommodate any operator $-\Delta - \lambda_0$, $\lambda_0 > 0$, by conjugating with dilations of order $\sqrt{\lambda_0}$. Write $V = V_1 + V_2$, where $V_1$ is bounded and compactly supported, and $\|V_2\|_{n+2} < \delta$ for a quantity $\delta > 0$ to be chosen in a moment. We have

$$\|g\|_2 \leq C_{n,\lambda_0} \|Vg\|_{n+2} \leq C_{n,\lambda_0} (\|V_1\|_{n+1} \|g\|_{n+2} + \delta \|g\|_2).$$

If $C_{n,\lambda_0} \delta < \frac{1}{2}$, the last term can be moved to the left side of the inequality so that $\|g\|_2 \lesssim \|g\|_{n+2}$. In other words $g$ is an eigenfunction of $H$ in $L^2_0(\mathbb{R}^n)$ with eigenvalue $\lambda_0 > 0$.

The conclusion is that $H$ has absolutely continuous spectrum on each compact subset of $(0, \infty)$ that contains no eigenvalues. However it is also known that embedded eigenvalues do not exist if the potential is real and belongs to $L^{\frac{2n+2}{n+2}}(\mathbb{R}^n)$ [6], so in fact the spectrum of $H$ is absolutely continuous on the entire halfline. \(\square\)

### 3. Perturbed Helmholtz equation

Theorem 2 admits a relatively easy extension to the equation $(-\Delta + V - 1)u = f$. Factorize the perturbed Helmholtz operator as

$$-\Delta + V - 1 = (I + VR_0^+ (1))(-\Delta - 1)$$

where $R_0^+ (1) = \lim_{\varepsilon \to 0^+} (-\Delta - (1 + i\varepsilon))^{-1}$. In this case the choice of resolvent continuations is unimportant, as both $R_0^+ (1)$ and $R_0^- (1)$ act the same when applied to functions in the range of $-\Delta - 1$. Then there should exist $L^2$ solutions of $(-\Delta + V - 1)u = f$ whenever $f = g + VR_0^+ (1)g$ and the unperturbed equation $(-\Delta - 1)u = g$ has solutions in $L^2_0(\mathbb{R}^n)$.

Let $X_0$ be the subspace of functions in $L^p_0(\mathbb{R}^n)$ whose Fourier transform vanishes on the unit sphere, as defined in the statement of Theorem 5. For $p = \frac{2n+2}{n+3}$, Theorem 2 indicates that the latter problem admits solutions precisely when $g \in L^p(\mathbb{R}^n)$ also belongs to $X_0 \subset L^p(\mathbb{R}^n)$. The substance of Theorem 5 is that the correspondence between $g$ and $f$ is an isomorphism of subspaces of $L^p(\mathbb{R}^n)$. This statement is proved below.

**Proposition 8.** Assume the conditions of Theorem 5, namely that $p = \frac{2n+2}{n+3}$ and $V \in L^{\frac{2n+2}{n+2}}(\mathbb{R}^n)$. Let $J : X_0 \to L^p(\mathbb{R}^n)$ be the inclusion map. The linear operator $J + VR_0^+ (1) : X_0 \to L^p(\mathbb{R}^n)$ is an isomorphism onto its range.

The space $X \subset L^p(\mathbb{R}^n)$ in Theorem 5 is precisely the range of $J + VR_0^+ (1)$.

**Proof.** The fact that it is a bounded operator is a direct consequence of Theorem 2, which effectively states that $R_0^+ (1)$ is a bounded map from $X_0$ to $L^2_0(\mathbb{R}^n)$. It is injective by the result in [6], as $R_0^+ (1)g$ would be an $L^2$ eigenfunction of $-\Delta + V - 1$ for any $g$ in the nullspace of $J + VR_0^+ (1)$.

In fact $VR_0^+ (1)$ is a compact operator from $X_0$ into $L^p(\mathbb{R}^n)$. For smooth compactly supported $V$ it acts compactly on the larger domain $L^p(\mathbb{R}^n)$. Approximating $V \in L^{\frac{2n+2}{n+2}}(\mathbb{R}^n)$ preserves compactness of $VR_0^+ (1)$ over the restricted domain $X_0$.

The argument that $(J + VR_0^+ (1))X_0 \subset L^p(\mathbb{R}^n)$ is closed is nearly identical to the analogous statement in the Fredholm Alternative. Let $f_n = (J + VR_0^+ (1))g_n$ be a sequence converging to $f \in L^p$. If $g_n$ has a bounded subsequence, then
by compactness $VR^+_{0}(1)g_n$ has a convergent subsequence and so does $g_n = f_n - VR^+_{0}(1)g_n$. The limit point $g \in X_0$ satisfies $(J + VR^+_{0}(1))g = f$.

If $\lim_{n \to \infty} \|g_n\|_p = +\infty$, consider the normalized functions $\tilde{g}_n = g_n/\|g_n\|_p$. This sequence satisfies $(J + VR^+_{0}(1))\tilde{g}_n \to 0$, and by compactness there is a convergent subsequence of $VR^+_{0}(1)\tilde{g}_n$ with limit $-g$. Then the same subsequence of $\tilde{g}_n$ converges to $g$, which has unit norm and belongs to the nullspace of $J + VR^+_{0}(1)$. That would violate the injectivity property of the map.

Having ruled out unbounded (subsequences of) $g_n$, it follows that $f \in (J + VR^+_{0}(1))X_0$ as in the first case, making the range closed. By the closed graph theorem, $J + VR^+_{0}(1)$ is then an isomorphism onto its range.

4. Extensions via Interpolation

The subspace of $L^p$ consisting of functions whose Fourier transform vanishes on the unit sphere is not particularly well suited to interpolation. The Fourier-vanishing condition not preserved by lattice operations or by the complex-analytic families used in the Riesz-Thorin theorem. As a further impediment, it is not obvious that one can approximate each element by a sequence of simple functions (or compactly supported functions, or Schwartz functions) whose Fourier transforms also vanish on the sphere.

Fortunately, Theorem 6 can be proved by complex interpolation of operators and of the target $L^q$ space, where none of these problems arise. The argument proceeds as follows. Suppose $f$ and $g$ are simple functions with compact support. Let $\tilde{S}^z$ be the “analytic” Bochner-Riesz operators defined by

$$\tilde{S}^z = \frac{1}{\Gamma(z+1)}S^z$$

for real-valued $z > -1$, and by analytic continuation to $z \in \mathbb{C}$.

The key observation is that for $\Re z > -2$, and for functions whose Fourier transform vanishes on the unit sphere, $\Gamma(z+1)\tilde{S}^z f = S^z f$ (The singularity at $z = -1$ is removable in this case). Proposition 4 establishes the same observation about “two-sided” Bochner-Riesz operators over the larger range $\Re z > -3$.

It is true by construction that the function $\langle \tilde{S}^z f, g \rangle$ is holomorphic in $z$ for any pair of simple functions $f$ and $g$. This remains true by uniform convergence in the halfplane $\Re z > -\frac{1}{2} - \frac{2n}{n+1}$ if we take limits to a generic element $f \in L^{\frac{n+2}{n+1}}$. Then

$$G(z) := \Gamma(z+1)\langle \tilde{S}^z f, g \rangle$$

is meromorphic over the same domain, with residues at the negative integers determined by $\langle \tilde{S}^{-k} f, g \rangle$. Since $\tilde{S}^{-1}$ agrees (up to a scalar multiple) with convolution against $\delta_1$, if we further assume that $\hat{f}$ vanishes on the unit sphere then in fact the singularity of $G(z)$ at $z = -1$ is removable. The slightly modified function

$$\tilde{G}(z) := (z+2)\Gamma(z+1)\langle \tilde{S}^z f, g \rangle = (z+2)G(z)$$

is meromorphic with poles at the negative integers $k \leq -3$.

Assuming once again that $\hat{f}$ vanishes on the unit sphere, Theorem 3 provides a bound on the line $z = -\beta + i\mu$,

$$\|\Gamma(1 - \beta + i\mu)\tilde{S}^{-\beta + i\mu} f\|_2 \lesssim \|f\|_2^{\frac{2n+2}{n+1}}.$$
The constant does not depend on $\mu$ because any one of the Fourier multipliers 
$(1 - |\xi|^2)^{i\beta}$ is an isometry on $L^2$. Therefore

$$|\hat{G}(-\beta + i\mu)| \lesssim (1 + |\mu|)\|f\|_{\frac{n+2}{n+1-n\beta}}\|g\|_2.$$ 

On the line $z = -2\beta + i\mu$, we need the following estimates.

**Proposition 9.** Let $\frac{1}{2} < \beta < \frac{3}{2}$ and $\beta \leq \frac{n+1}{4}$. The inequality

$$(17) \quad \|(2-2\beta + i\mu)\Gamma(1-2\beta + i\mu)\tilde{S}^{-2\beta+i\mu}f\|_{\frac{n+2}{n+1-\beta}} \lesssim (1 + |\mu|)\|f\|_{\frac{n+2}{n+1-n\beta}}$$

holds uniformly for all $\mu \in \mathbb{R}$ and all $f \in L^{\frac{2n+2}{n+1-\beta}}(\mathbb{R}^n)$.

**Sketch of Proof.** Proposition 9 follows from the same argument as the endpoint Stein-Tomas theorem, using the fact that the convolution kernel of $\tilde{S}(z)$ has an asymptotic description

$$\tilde{S}^\circ(|x|) \sim \frac{C_n}{|x|^\frac{n}{2} + z} \cos \left( \frac{(n-3)\pi}{4} + \frac{\pi}{2}z \right)$$

for large $|x|$. Note that for complex $z$, oscillations of the cosine function in this formula have amplitude approximately $e^{(\pi/2)\imath \text{Im} z}$.

For $z = -2\beta + i\mu$, the prefactor $(z+2)\Gamma(z+1)$ is dominated by $(1+|z|)e^{-(\pi/2)\imath \text{Im} z}$, using Stirling’s approximation when $\mu$ is large, and the absence of poles for $\text{Re} \ z > -3$ when $\mu$ is small. Hence the product $(z+2)\Gamma(1+z)\tilde{S}^\circ$ enjoys mapping estimates that are uniform in $\mu$ along this line. \hfill \Box

Consequently $|\hat{G}(-2\beta + i\mu)| \leq (1 + |\mu|)\|f\|_{\frac{n+2}{n+1-\beta}}\|g\|_{\frac{n+2}{n+1-n\beta}}$. If one constructs $g_z$ to be a holomorphic family of simple functions (as in Riesz-Thorin interpolation) that belong isometrically to $L^2(\mathbb{R}^n)$ along the line $\text{Re} \ z = -\beta$, and to $L^{\frac{2n+2}{n+1-\beta}}(\mathbb{R}^n)$ along the line $\text{Re} \ z = -2\beta$ it follows from the Three-Lines Theorem that

$$|(z+2)\Gamma(z+1)\langle \tilde{S}^\circ f, g_z \rangle| \lesssim (1 + |\imath \text{Im} z|)\|f\|_{\frac{n+2}{n+1-\beta}}\|g_z\|_{\frac{n+2}{n+1-\beta}}$$

For a fixed real value $\beta \leq \alpha \leq 2\beta$, one can arrange for $g_{-\alpha}$ to be any simple function. It follows by duality and density of simple functions that

$$\|\Gamma(z+1)\tilde{S}^{-\alpha}f\|_{\frac{n+2}{n+1+\alpha-\beta}} \lesssim \frac{1}{|2-\alpha|}\|f\|_{\frac{n+2}{n+1-\beta}}.$$ 

For $\alpha < 2$ and with the assumption that $\hat{f}$ vanishes on the unit sphere, the left hand function is exactly $S^{-\alpha}f$. This is the norm bound claimed in Theorem 6.

As a final note, we observe that estimates can also be made for $\alpha < \beta$ by interpolating between Theorem 2 and other known bounds for Bochner-Riesz operators. To give a simple example, $S^{-\alpha}$ maps $L^p(\mathbb{R}^2)$ to itself for each $\alpha < -\frac{1}{2}$, and if $\hat{f}$ vanishes on the unit circle for a function $f \in L^1(\mathbb{R}^2)$ it is also true that $S^{-\frac{1}{2}}f \in L^2(\mathbb{R}^2)$. Complex interpolation then suggests that $S^{\alpha}f \in L^q(\mathbb{R}^2)$ for all $q > \frac{4}{3}$. This is a modest improvement over the generic $L^1 \rightarrow L^{4/3+}$ bound for the ball multiplier. We do not claim that the exponent $q = \frac{3}{4}$ is sharp, and suspect that the range of exponents can be extended further toward 1 by other methods.
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References


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