Mathematical Sciences, University of Cincinnati.

Wright Patterson Air Force Base.

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From the Numerical Analysis Bench:

Error Analysis of Overset Finite Element Methods for Aerospace Problems

High Order Finite Element Methods (FEM) are becoming popular in CFD. **Overset Methods** are important in problems with complex geometry (design). **Schwarz Iteration** is usually used to produce the overset approximation.

Talk Outline:

I. Survey known theory for Schwarz/Overset/FEM methods.

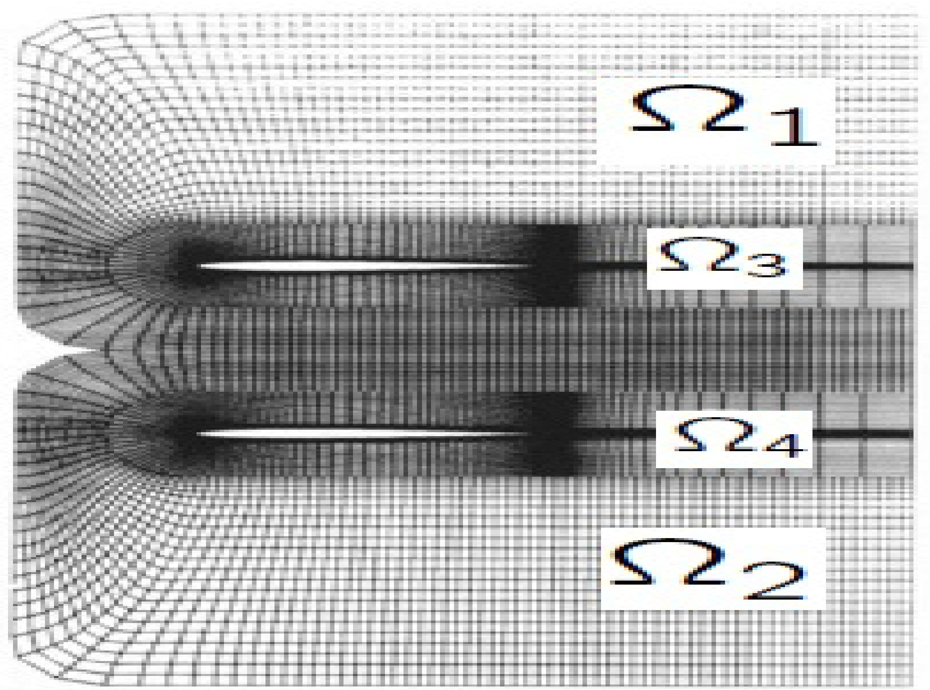
II. Theoretical Analysis of Direct Solve FEM/Overset.

III. Theoretical Analysis of Least Squares FEM/Overset

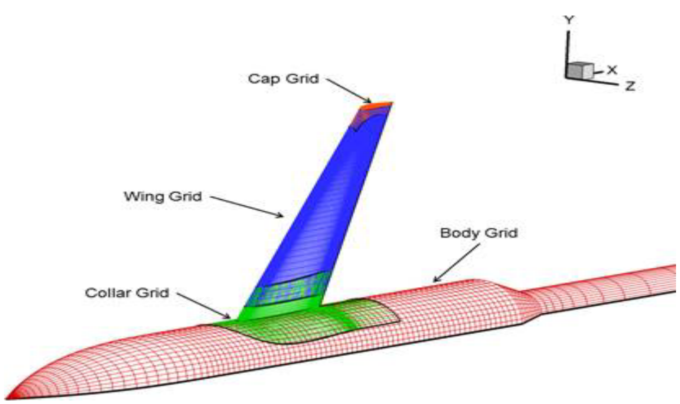
AIAA Aviation Denver Co. Thursday, June 8, 2017 (2:00-5:30 PM).

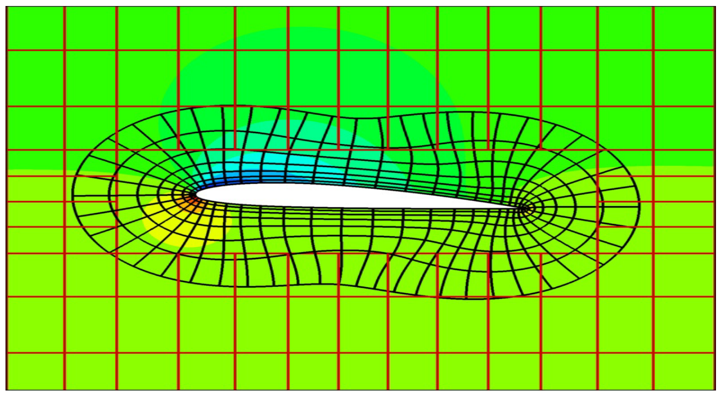
(*Don French* in collaboration with *Jack Benek* and *Chris Schrock*).

Inviscid Supersonic Flow Computation in a Complex Geometry



Four Overset Grids – Steger & Benek, 1986.



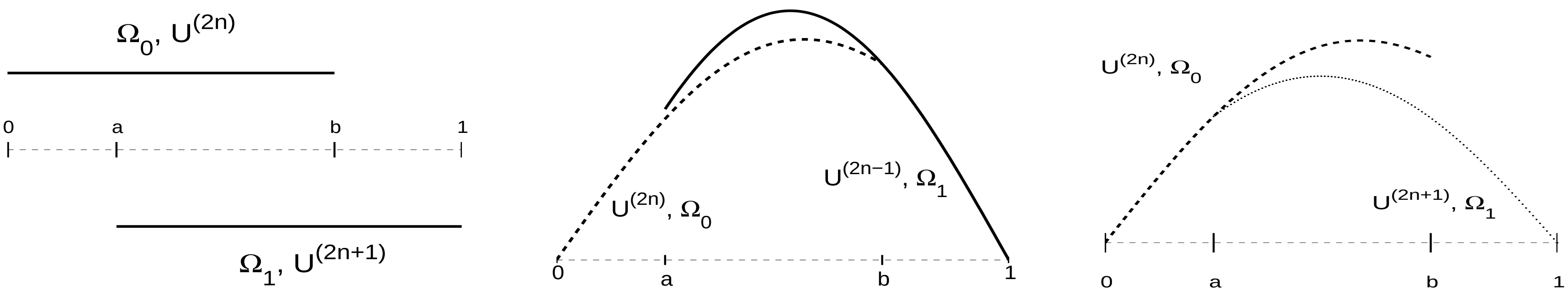


Classical Schwarz Iteration (1D Description):

$$\left\{ \text{Solve } LU = f \text{ on } \Omega = [0,1] \text{ with } U(0) = U(1) = 0. \right.$$

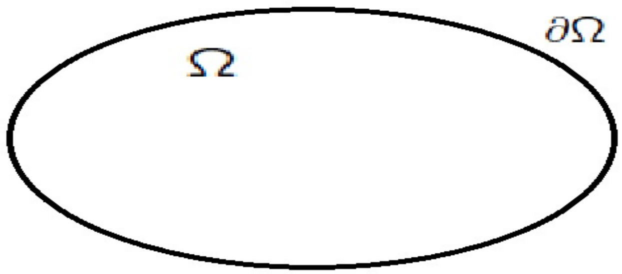
Schwarz Iteration on Overlapping Grids:

Guess $U^{(1)}(b)$.
Do $n = 1, 2, 3, \ldots$.
  Solve $LU^{(2n)} = f$ on Ω_0 with $U^{(2n)}(0) = 0$ and $U^{(2n)}(b) = U^{(2n-1)}(b)$.
  Solve $LU^{(2n+1)} = f$ on Ω_1 with $U^{(2n+1)}(a) = U^{(2n)}(a)$ and $U^{(2n+1)}(1) = 0$.
End.



Overlapping domains $\Omega_0 = [0, b]$ and $\Omega_1 = [a, 1]$ and successive iterates $U^{(2n)}$ and $U^{(2n+1)}$.
Guess $U^{(2n-1)}(a)$ to generate $U^{(2n-1)}$ on Ω_1 in middle figure.

Basic PDE Problems and Overset:



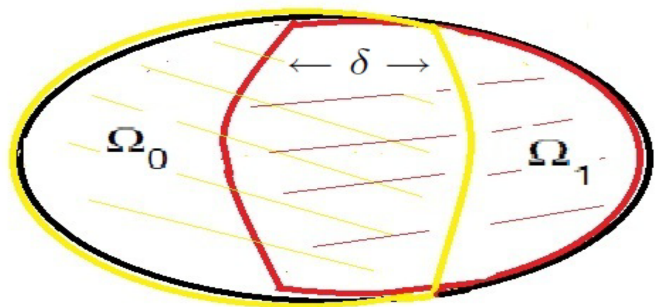
Abstract PDE Problem:

Solve $LU = f$ on Ω with BCs on $\partial\Omega$.

Example Differential Operator:

$LU = -\nu\Delta U + \beta\nabla \cdot F(U) + \gamma U$ with $0 < \nu << 1$, $\gamma \sim 1/\Delta t$.

Overset PDE Problem with Domains Ω_0 & Ω_1



Solve $LW = f$ on Ω_0 and $LV = f$ on Ω_1

BCs for W on $\partial\Omega_0$ and BCs for V on $\partial\Omega_1$

W & V "match" on "overlap".

FEM/Overset:

FEM functions w on Ω_0 and v on Ω_1 to approximate W and V.

Mesh parameters H on Ω_0 and h on Ω_1 (δ independent of H and h).

Theoretical Convergence Analyses:

(i) Overset/PDE/Schwarz using Maximum Principles: Many, many papers ... (Dolean & Natif (2005), Clerc (1998), Gander (2008), Gander & Rohde (2005), ...).

(ii) Overset/PDE/Schwarz using Sobolev Spaces: P.L. Lions (1989), Zhang & Jiang (2012).

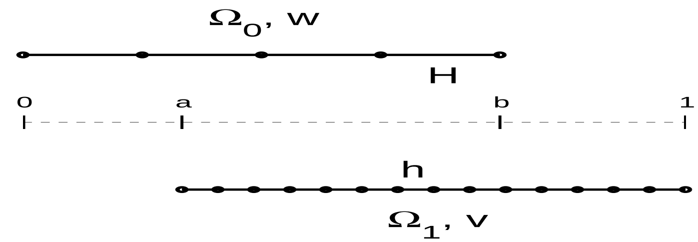
(iii) Numerical/Overset/PDE/Schwarz (δ independent of H & h) Mathew & Russo (FDs – 2002), Kopteva & Pickett (FEM – 2002), Canuto & Funaro (Spectral – 1988) and DF et al (2014 & 2016).

Observations:

(i) Maximum Principle proofs usually do not extend to FEMs.

(ii) Sobolev proofs appear to rely on symmetry of PDE & mesh extensions.

Direct FEM/Overset:



Prototype Problem: Find U so

$-\nu U'' + \beta U' + \gamma U = g$ on $(0,1)$

BCs $U(0) = 0$ & $U(1) = 0$ and $\nu > 0$, $\gamma >> 1$.

Direct Matrix Solve:

$$\left[\begin{array}{c|c} A_H & 0 \\ \vec{e}_{N_0}^{\,T} & \vec{\mathcal{I}}_h \\ \hline 0 & A_h \\ \vec{\mathcal{I}}_H & \vec{e}_0^{\,T} \end{array}\right] \left(\begin{array}{c} \vec{w} \\ \\ \vec{v} \\ \\ \end{array}\right) = \left(\begin{array}{c} \vec{G}_H \\ 0 \\ \hline \vec{G}_h \\ 0 \end{array}\right)$$

Computations: Galbraith et al (2014) used Direct Solve schemes on many CFD problems.

Theoretical Analysis (DF et al $\Rightarrow$ AIAA Aviation June 2016):

Block Matrix System is invertable.

Convergence: $(w, v) \to (U\big|_{\Omega_0}, U\big|_{\Omega_1})$ as h & $H \to 0$.

Likely Extensions: Time-dependent case, 1-D Systems, ...

Least Squares Overset Finite Element Scheme in 2D

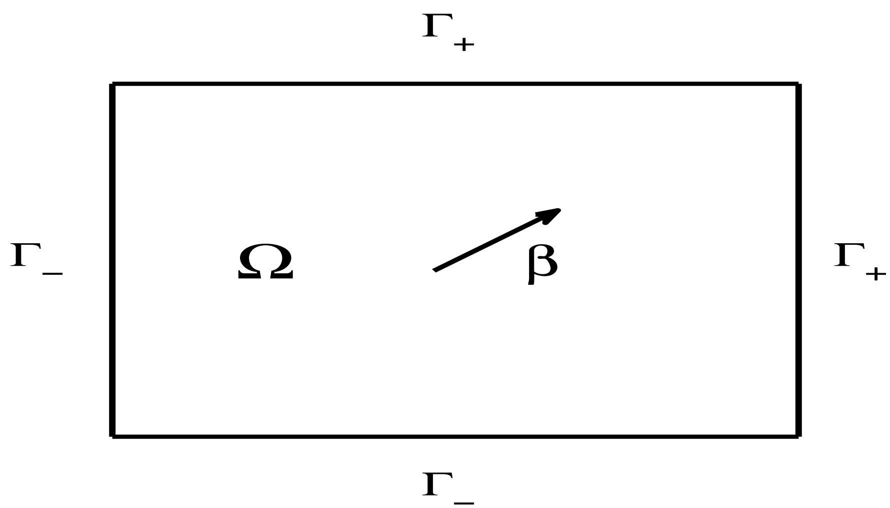
Model Problem:

$$\begin{cases} \text{Find } U = U(x,y) \text{ such that} \\ \quad \mathcal{L}U = \beta \cdot \nabla U + \gamma U = f \quad \text{in } \Omega \subset \mathbf{R}^2 \quad \text{with} \quad U = g \quad \text{on } \Gamma_- \text{ and } \Gamma = \partial\Omega. \end{cases}$$

Where $\Gamma_- = \{x \in \Gamma : n(x) \cdot \beta(x) < 0\}$ inflow boundary & $\Gamma_+ = \partial\Omega - \Gamma_-$ is outflow boundary.

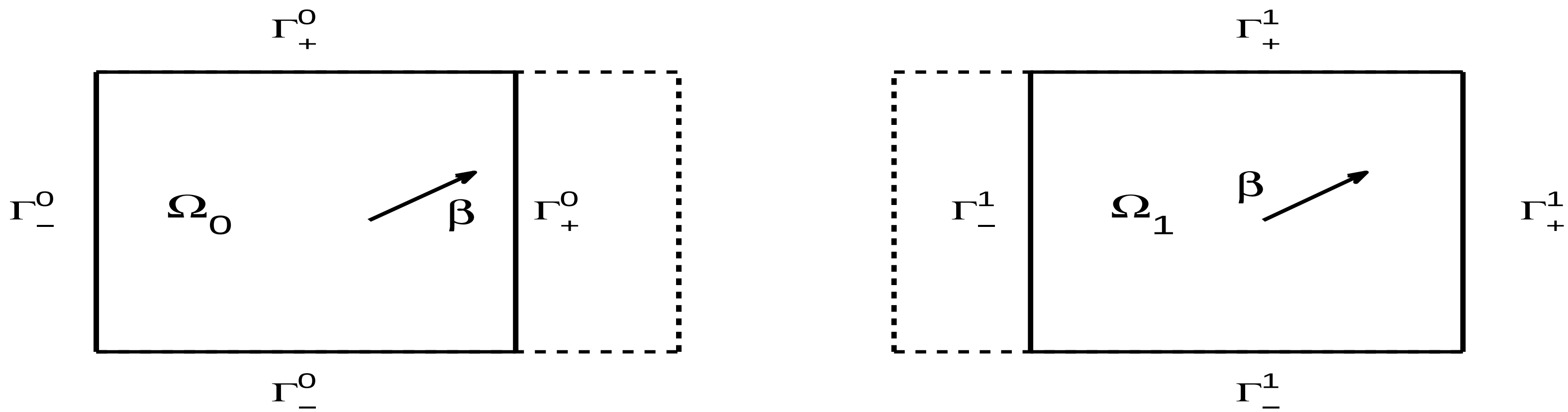
Assumption: Vector function β and scalar function $\gamma > 0$ are smooth & $\exists$ number $\gamma_0 > 0$ so that

$$-\frac{1}{2}\nabla \cdot \beta + \gamma \geq \gamma_0 > 0 \quad \text{on } \Omega.$$



LS for First-Order Hyperbolic (No Overset): Bochev & Choi (2000 & 2001), Bochev & Gunzburger (2006) and Guermond (2004 − L^1-Minimization).

Overlapping Overset Domains:



S_H^0 is PW Polynomials on Ω_0 and S_h^1 is PW Polynomials on Ω_1.

LSFEM Overset Scheme: S_H^0 and S_h^1: Continuous PW polynomials of degree $\leq q$.

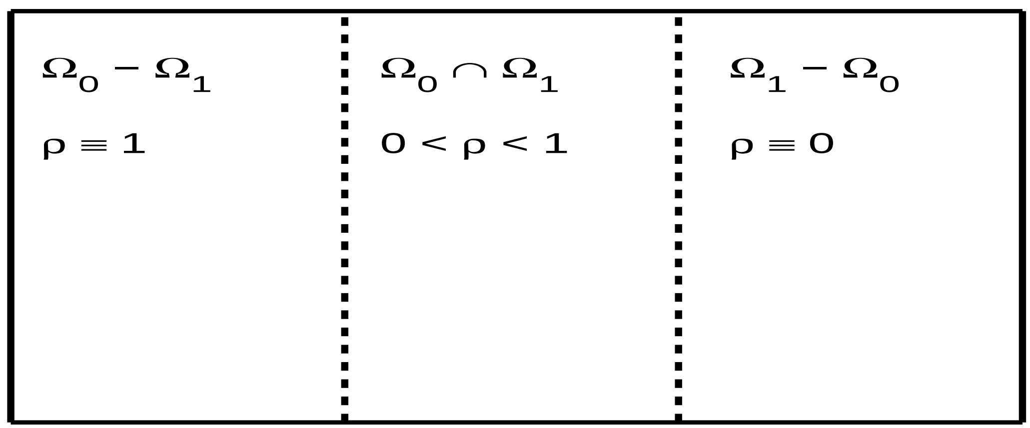
$$\begin{cases} \text{Find } (w,v) \in S_H^0 \times S_h^1 \text{ that minimizes} \\ \Phi(p,q) = \int_{\Omega_0} (\mathcal{L}p - f)^2 \, dA + \int_{\Omega_1} (\mathcal{L}q - f)^2 \, dA + \int_{\Omega_0 \cap \Omega_1} (p-q)^2 \, dA \\ \qquad + \int_{\Gamma_-^0 \cap \partial\Omega} (p-g)^2 \, ds + \int_{\Gamma_-^1 \cap \partial\Omega} (q-g)^2 \, ds \\ \text{over all } (p,q) \in S_H^0 \times S_h^1. \end{cases}$$

Partition of Unity Analysis:

Cutoff functions ρ and $1-\rho$ where $\rho \in C^\infty(\bar{\Omega})$, $\rho \equiv 1$ on $\Omega_0 - \Omega_1$, $\rho \equiv 0$ on $\Omega_1 - \Omega_0$ and $0 < \rho < 1$ on $\Omega_0 \cap \Omega_1$.

$$\nabla \rho \neq 0 \text{ on } \Omega_0 \cap \Omega_1 \quad \text{and} \quad \|\rho\|_{1,\infty,\Omega_0\cap\Omega_1} \leq \rho_0 \sim \Big(\text{Minimum Width } \Omega_0 \cap \Omega_1\Big)^{-1}.$$

Distribution of ρ



Define for LSFEM solution (w, v) function $u \cong U$

$$u = \rho w + (1-\rho)v \qquad \text{or} \qquad u = \begin{cases} w \text{ on } \Omega_0 - \Omega_1 \\ \rho w + (1-\rho)v \text{ on } \Omega_0 \cap \Omega_1 \\ v \text{ on } \Omega_1 - \Omega_0. \end{cases}$$

$$\Rightarrow \quad \mathcal{L}u = \Big[\rho \mathcal{L}w + (1-\rho)\mathcal{L}v\Big] + (\beta \cdot \nabla \rho)(w - v).$$

Key Proof Steps:

Energy Estimate:

$$\gamma_0 \int_\Omega Z^2 \, dA + \frac{1}{2}\int_{\Gamma_+} Z^2 |\beta \cdot n| \, ds \le \frac{1}{2}\int_{\Gamma_-} Z^2 |\beta \cdot n| \, ds + \int_\Omega (\mathcal{L}Z) Z \, dA$$

$$\gamma_0 \int_\Omega (u-U)^2 \, dA + \frac{1}{2}\int_{\Gamma_+} (u-U)^2 |\beta \cdot n| \, ds \le \frac{1}{2}\int_{\Gamma_-} (u-U)^2 |\beta \cdot n| \, ds + \int_\Omega (\mathcal{L}u - \mathcal{L}U)(u-U) \, dA$$

Interpolants $\mathcal{I}_H$ on S_H^0 and $\mathcal{I}_h$ on S_h^1

$$\Phi(w,v) \le \Phi(\mathcal{I}_H U, \mathcal{I}_h U) \le C\Big(H^{2q} + h^{2q}\Big).$$

Since $f = \mathcal{L}U$ and $U = g$ on Γ_-;

$$\begin{aligned} \|\mathcal{L}u - f\|_{0,2,\Omega} &= \|\rho(\mathcal{L}w - f) + (1-\rho)(\mathcal{L}v - f) + (\beta \cdot \nabla \rho)(w - v)\|_{0,2,\Omega} \\ &\le \|\mathcal{L}w - f\|_{0,2,\Omega_0} + \|\mathcal{L}v - f\|_{0,2,\Omega_1} + C\|w - v\|_{0,2,\Omega_0 \cap \Omega_1} \\ &\le C\sqrt{\Phi(w,v)} \le C\Big(H^{2q} + h^{2q}\Big)^{1/2}. \end{aligned}$$

For Prescribed BC on Γ_-:

$$\|u - g\|_{0,2,\Gamma_-} \le C\Big(H^{2q} + h^{2q}\Big)^{1/2}$$

Conclude:

$$\gamma_0 \int_\Omega |u-U|^2 \, dA + \frac{1}{2}\int_{\Gamma_+} |u-U|^2 |\beta \cdot n| \, ds \le C\Big(H^{2q} + h^{2q}\Big) \quad \Rightarrow \quad u \to U \text{ at rate } O(h^q + H^q).$$

Implementation Details:

$$\begin{cases} \text{Find } (w,v) \in S_H^0 \times S_h^1 \text{ so that} \\ \quad \int_{\Omega_0} (\mathcal{L}w - f)\mathcal{L}\eta \; dA + \int_{\Omega_0 \cap \Omega_1} (w - v)\eta \; dA + \int_{\Gamma_-^0 \cap \partial\Omega} (w - g)\eta \; ds = 0 \quad \forall \eta \in S_H^0 \\ \quad \int_{\Omega_1} (\mathcal{L}v - f)\mathcal{L}\xi \; dA - \int_{\Omega_0 \cap \Omega_1} (w - v)\xi \; dA + \int_{\Gamma_-^1 \cap \partial\Omega} (v - g)\xi \; ds = 0 \quad \forall \xi \in S_h^1 \end{cases}$$

$$w(x) = \sum_{j=0}^{N_0} w_j \phi_j^0(x) \text{ and } v(x) = \sum_{j=0}^{N_1} v_j \phi_j^1(x).$$

(Basis $\{\phi_0^0, \ldots, \phi_{N_0}^0\}$ for S_H^0 and $\{\phi_0^1, \ldots, \phi_{N_1}^1\}$ for S_h^1).

$$\Rightarrow \left[\begin{array}{c|c} K^0 + M^0 + B^0 & -C \\ \hline -C^T & K^1 + M^1 + B^1 \end{array} \right] \begin{pmatrix} \vec{w} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \vec{F}^0 \\ \vec{F}^1 \end{pmatrix}$$

Where

$$K_{ij}^0 = \int_{\Omega_0} \mathcal{L}\phi_j^0 \mathcal{L}\phi_i^0 \; dA, \quad M_{ij}^0 = \int_{\Omega_0 \cap \Omega_1} \phi_j^0 \phi_i^0 \; dA, \quad B_{ij}^0 = \int_{\Gamma_-^0 \cap \partial\Omega} \phi_j^0 \phi_i^0 \; ds$$

$$K_{ij}^1 = \int_{\Omega_1} \mathcal{L}\phi_j^1 \mathcal{L}\phi_i^1 \; dA, \quad M_{ij}^1 = \int_{\Omega_1 \cap \Omega_1} \phi_j^1 \phi_i^1 \; dA, \quad B_{ij}^1 = \int_{\Gamma_-^1 \cap \partial\Omega} \phi_j^1 \phi_i^1 \; ds,$$

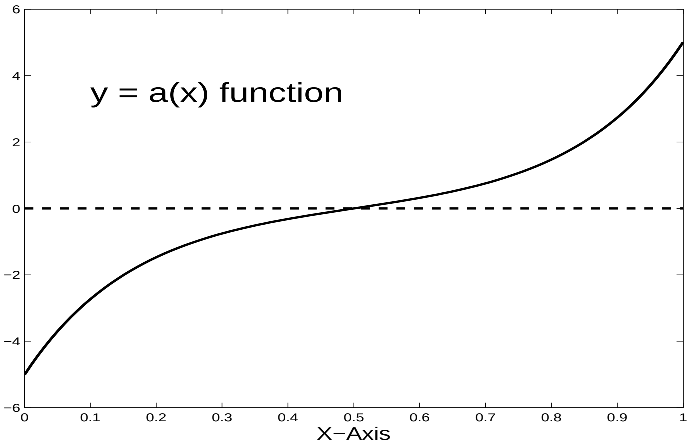
$$F_i^0 = \int_{\Omega_0} f \mathcal{L}\phi_i^0 \; dA + \int_{\Gamma_-^0 \cap \partial\Omega} g\phi_i^0 \; ds, \quad F_i^1 = \int_{\Omega_1} f \mathcal{L}\phi_i^1 \; dA + \int_{\Gamma_-^1 \cap \partial\Omega} g\phi_i^1 \; ds \text{ and } C_{ij} = \int_{\Omega_0 \cap \Omega_1} \phi_j^1 \phi_i^0 \; dA.$$

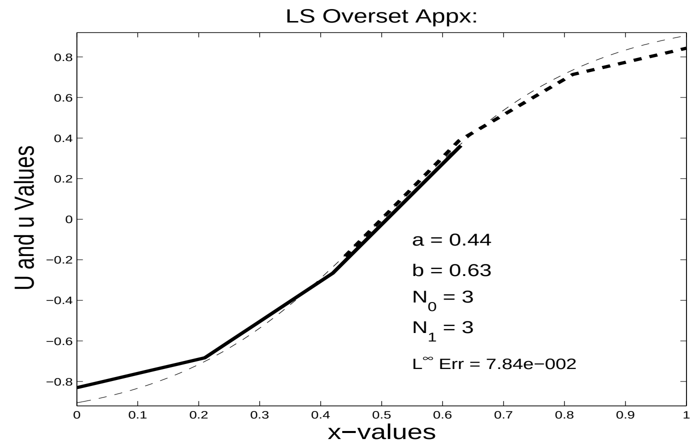
<u>Sample Computation:</u> Peculiar Problem:

$$aU' + \gamma U = f \quad \text{with} \quad a(x) = \sinh(3(x - 1/2))\cosh(3(x - 1/2)), \quad \gamma = \cosh(3/2)),$$

$$U(x) = \tanh(x - 1/2) \;\; \& \;\; f(x) = (\gamma + 1)\tanh(3(x - 1/2)).$$

No initial or boundary conditions – with γ as chosen, solution is unique.





One dimensional computation with piecewise linears.

Convergence Rate & Errors:

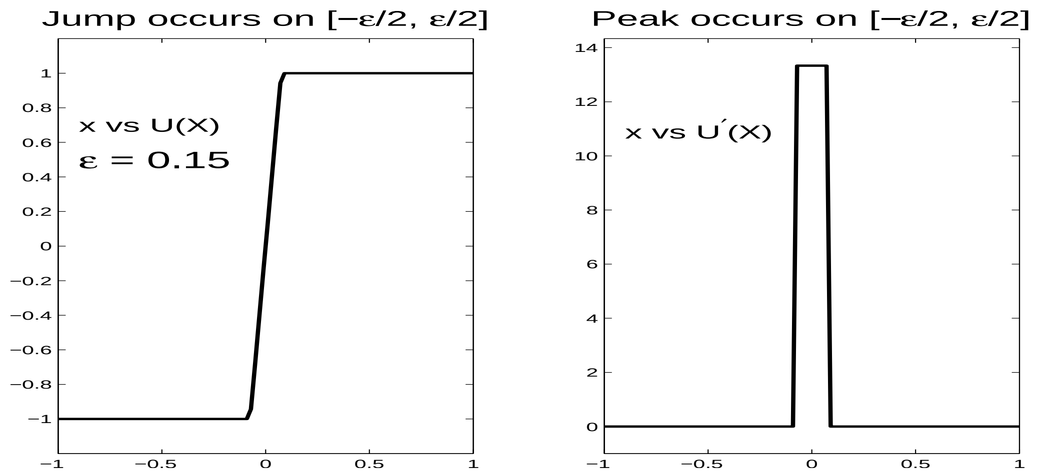
N_0	N_1	L^∞-Errors	Rate
5	5	3.00×10^{-2}	—
10	10	8.08×10^{-3}	1.89
20	20	2.24×10^{-3}	1.85
40	40	5.83×10^{-4}	1.94
80	80	1.47×10^{-4}	1.99
160	160	3.69×10^{-5}	1.99

L^1-Method? Shocks/Roughness/$W^{1,1}(\Omega)$:

Galerkin Schemes tend to provide *Best Approximations* in the H^1 − Norm:

$$\|Z\|_{1,2,\Omega} = \Big(\int_\Omega Z^2 \, dx + \int_\Omega (Z')^2 \, dx \Big)^{1/2} \qquad \Omega = (-1, 1).$$

Consider $U = U(x)$ with a slightly smoothed shock:



$$\||U\|^2_{1,2,\Omega} \geq \int_\Omega (U')^2 \, dx = \int_{-\epsilon/2}^{\epsilon/2} (1/\epsilon)^2 \, dx = 1/\epsilon.$$

While in the $W^{1,1}$ − Norm:

$$\|U\|_{1,1,\Omega} = \int_\Omega |U| \, dx + \int_\Omega |U'| \, dx \leq 2 \cdot 1 + \int_{-\epsilon/2}^{\epsilon/2} (1/\epsilon) \, dx = 3.$$

⇒ Guermond (2004) L^1-Minimization Scheme.

Future Directions:

Overset LSFEM Overset:

Computations/Extensions to more general Aerospace Problems.

DG & SUPG Approaches as well as estimates with Viscosity via Mixed Methods.

L^1-Method from Guermond for FEM/Overset.

Assumptions on Overlap Width δ: Dependence on Mesh Parameters $h\&H$.

The "Lab":

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