An Introduction to Mathematics

by A. Cole F.
Dedicated to my patient family and friends


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Note To The Reader

What follows is, as the title suggests, an introduction to mathematics, particularly for older readers that have already been “introduced” to the subject in some way; that is, perhaps their acquaintance is highly superficial, and was only entertained enough to meet the absolutely minimal requirements to pass a course; or, on the other hand, maybe they have performed quite well in their course(s), and yet still have a very shallow understanding of the subject; perhaps they are struggling with mathematics at this very moment, and need a “fresh perspective”; or, we might imagine, they simply do not “get” math, and are left with more “why?”s than they are comfortable with. It is also highly likely that your situation is not even slightly approximated by these assumptions, or it is some mix of them. This text, regardless, may very well be of use.

I suppose it may be better to consider this a “re-introduction” to mathematics for most. Mathematics is, like so many other fields of study, a highly diversified discipline full of beautiful and striking subtleties that frequently remain enshrouded by apparently esoteric theory and foreign formulae. Many students find the standard pedagogical approach to mathematics unpalatable for any number of reasons, and instead of being offered alternative avenues toward understanding, they are usually urged (in a most deplorably cavalier way) to press on with the current method, assured that if they try hard enough, it will eventually “click.” This dismissive attitude toward education is not endemic to math: many a struggling student in a wide array of subjects have become victims to this educational experiment, as it were. Some are lucky enough to finally “get” it, whatever the subject may be. Others are not so lucky, and they scrape by with whatever incipient “understanding” they are able to muster, until their ignorance or confusion is exacerbated by future material (which demands a firm foundation in the fundamentals) thus compounding the confusion, anxiety, embarrassment, depression – and a whole host of other negative emotions – they come to associate with studying and doing math. Therefore for some readers, part of my job here is to show you the illusory bonds that hold you back from doing math, and more importantly, enjoying and appreciating it.

Oftentimes in math, there is “a solution”; to wit, there is some answer, somewhere in some invisible, abstract space, and we are asked to find it by “solving” the problem. In this case, the context is usually that of a course, a tutoring session, &c. The fact that there is posited to exist some “right answer” that one must find, and that one will not be able to find it only in the absence of appropriate proficiency, is a source of fear for many students. This idea is also fundamentally flawed, and does not accurately represent the incredibly broad spectrum that is mathematical reasoning. This is the insidious face of schoolhouse computational
Note To The Reader

There is, you might be delighted to know, much more to mathematics than “finding $x$,” arcane strings of symbols, and mind-bogglingly sober pools of numbers. There is much room (and necessity!) for intuition, experimentation, curiosity, imagination, and creativity in mathematics. Anyone that does not tap into these to some degree will have a hard time doing much that is profound or fundamentally ground-breaking in mathematics; their familiarity is not much less superficial than someone who cannot be bothered to engage math again after leaving the classroom. They certainly do not exhibit any “genius.” They may receive a math olympiad trophy, or a whole collection of them – and in doing so, they have won their reward. Do not limit your reward to trophies or GPAs.

Now, you must not misinterpret what I am saying here. One does not need to fundamentally alter a subject to show they have a mastery over it, and neither do they need to initially possess all of the above qualities – perhaps there are some they will never feel they have truly cultivated! Just what sort of mindset and perspective that is needed to make a breakthrough, or at least master your coursework, cannot be perfectly described. To assume so is to make the same mistake, at bottom, that many primary school instructors do when they self-defeatingly try to train every student with the same approach. The point is that every dedicated and cognizant individual has the capacity to engage and contribute to mathematics in their own way. This I cannot emphasize enough. In this text, there will be a number of exercises which are designed not just to “check your understanding,” but also to construct your understanding. An important part of this process is a sort of candid experimentation process, except unlike in, say, a chemistry lab, we are at liberty here to be as sloppy and informal as we like.

I preemptively encourage you to stop fearing being wrong in any aspect of life. Always accept being incorrect as a possibility, and transform the experience of being wrong into one of learning what is right (or at least of narrowing down the options). Sometimes no one answer is currently known to be “right,” especially in fields such as physics and high level mathematics, wherein there lie many open questions and ostensibly insoluble problems. For my exercises, though, I will include a solution for each, usually with lots of explanation regarding my reasoning process. This, rather than the answer, is what is most important.

My style will at some points be very expository and philosophical, and at others it may become insultingly descriptive or strangely conversational. Some details you might find trivial, but I will include them nonetheless, for both the sake of completion and in response to the diversity of my readership. Professional mathematicians, physicists, enthusiasts, or more meticulous students may find some of my statements to be insufficient or cursory – that is, lacking the typical rigor of definition, or lacking the usual rhetoric that they have come to appreciate throughout their training and studies. In the spirit of the mathematician Morris Kline, I do feel that introducing new ideas in mathematics is oftentimes most fruitful when austere rigor and formality follow the student’s intuitive grasp of the topic. There may be some concepts or methods which do not grant me, and my limited powers of expression, this luxury; I apologize in advance if I cause undue confusion in my attempt at unconventional pedagogy.

Math is a discipline of rigor, sometimes to apparently pedantic degrees. When you work with abstract objects (i.e., numbers), you can oftentimes do no wrong by being thorough and sometimes even a little obsessive. You are likely about to
encounter math and numbers in a totally new environment, so if you need to take
time to mull over anything I say, then you are on the right track. If you need to go to
outside sources for explanations of differing depth, then you are in good company.
Strive constantly to shrug off all the negative emotions you might associate with
mathematics, and begin reading this text as though you had barely “met” math
in your life. After all, if any of the above applies to you, it is very likely that you
really haven’t.
CHAPTER 1

On The Consideration of Numbers

1. The Natural Numbers

The most obvious and easy things in mathematics are not those that come logically at the beginning; they are things that, from the point of view of logical deduction, come somewhere in the middle. Just as the easiest bodies to see are those that are neither very near nor very far, neither very small nor very great, so the easiest conceptions to grasp are those that are neither very complex nor very simple (using simple in a logical sense).

– Bertrand Russell, An Introduction to Mathematical Philosophy

To begin our foray into the mathematical “world,” we must introduce those elements with which we shall be concerned. In the spirit of Lord Russell’s insights, we should not begin at the most highly abstracted elements that we might be able to conceive of (and thus the “most general”), but rather a particular and familiar system. The more general we make our investigations, the less familiar the constructs which shall emerge, fundamental and cardinal though they may be for mathematical reasoning; likewise, the less general we become, we discover particularities which are not very enlightening for coming to a broader understanding of mathematics.

These considerations, coupled with a few others which are omitted for brevity’s sake, have led me to begin, like Russell in his text cited above, with the so-called “natural” numbers. In doing so, we get to discuss many important ideas which will be necessary to understand later on. So, we begin by looking at a series:

\[0, 1, 2, 3, \ldots, n, n + 1, \ldots,\]

where this is the series of natural numbers, which is often symbolized (as a means of shorthand) by \(\mathbb{N}\). (Note that the article “the” is also boldface: that is because it is important to understand that this series is unique – that is, there is no other series of natural numbers. This is the only one! We will discuss uniqueness more as we progress through the text.) As seen above, it begins at 0, not at 1. The ellipses are a specific symbol in mathematics, which indicate a continuation of a pattern which precedes them. Here we began at 0, and then proceeded by incrementing by 1 for each iteration in the series: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and

\[\text{1This text by Lord Russel begins very similarly to the work you are now reading. It is a thin volume of great importance, and one which I recommend you read if you ever get the chance.}\]
so on. When reading “0, 1, 2, 3, ⋅⋅⋅” we understand this to mean, then, that
every subsequent number in the series may be obtained by adding 1 to the number
directly before it. There is no end to the series of natural numbers; it continues
indefinitely, so the symbol \( n \) was used to indicate this. In other words, you may
count to an arbitrary number – that is to say, to any number you like, no matter
how big – and you would still be “in” the set of natural numbers. The symbol \( n + 1 \)
indicates the aforementioned pattern of incrementation, so it is, strictly speaking,
not necessary for our understanding of the series; you likely would have noticed the
pattern in the series without needing to refer to the symbol \( n + 1 \). In fact, you
probably did just now, without explicitly realizing or analyzing it.

Below is an example exercise; play around with it for a bit before checking the
solution. The spirit of creatively “playing around” is one of vital importance in
mathematics, just like in many other fields of inquiry. When you check your an-
swer for any problem in this text, do not be so concerned with whether or not your
answer is identical to mine. Yours may very well look different, but mean the same
thing. Regardless, do study the answer and see if you comprehend every bit of it;
if you do not, play with the problem until the answer begins to make some sense.
Feel free to look up outside resources too: this is a great skill to learn, as it will
grant you access to an indefinite amount of knowledge in your life. Finally, unless
a problem and its associated solution skill set is totally obvious or rudimentary, I
advise you to revisit old problems and re-work them, even if you got them correct.
This is especially useful for problems that gave you trouble solving, but eventually
conquered.

**Exercise 1.** What is another, shorter way to rewrite the series

\[ 0, 1, 2, 3, \cdots, n, n + 1, \cdots \]

That is, write down another expression for the series which is equivalent to it.
There are multiple correct answers.

**Solution 1.** There are many other ways to write this series. The shortest
and simplest way is probably \( 0, 1, 2, \cdots \). After we specify the starting number and
include two terms after it, we have given enough information so that the reader
can pick out the pattern of the series. (More generally, three elements are usually
required to formally detect the simplest patterns in series.) Furthermore, we do
not need to include the terms \( n \) or \( n + 1 \) because the ellipsis indicates that the
pattern continues indefinitely, so technically these terms are redundant. Sometimes,
however, we specify otherwise redundant information for the sake of clarity, like I
did at the beginning of this section.

**1.1. What are (natural) numbers?** For now, it is easiest to understand a
series as an ordered list. But how do we understand natural numbers themselves?
What are they?

This is a difficult question. Many thinkers have proposed several distinct an-
swers, some better than others. One might imagine that it would be easier to first
address the question “what is a number, much less a natural one?”, but this is not
so. Imagine the Egyptian geometers, which developed much of their mathematics
for the purpose of land surveying. How were they able to come to quantifying shapes and objects – that is, how were they able to abstract from a particular, physical entity, a mathematical representation of it? They were not the first to understand numbers, nor the more general idea of number. The formal, explicit definition of number, however, would have to wait until 1884, when Gottlob Frege offered the first (correct) answer.

There is an important distinction to be made, in order to preemptively combat a fundamental misunderstanding about number. Number is distinct from plurality, which many philosophers of the past two centuries have grappled with rigorously defining. Whereas plurality is an instance of certain numbers (numbers greater than 1 in our system of natural numbers), number is a quality understood to be common to all instances of numbers. “Number” is what makes a number be a number. Videlicet, think of a pair of birds; we apprehend that here is an instance of 2, though this pair of birds is not “equal to” 2, which is a number. Plurality is present here though; and it would be the “same” plurality present in a trio of birds, though this set of birds could not be understood using the concept of number invoked by the number 2, but rather 3. Thus, plurality establishes that a quantity, say \( n \), is greater than 1: \( n > 1 \). But it does not specify just “how big” \( n \) precisely is.

We can turn back toward the “natural numbers,” and try to see what makes them special. Incorrectly we are often instructed that they comprise the “counting” numbers. Considering we can use the operation (which you may intuitively understand, for the moment, as being as process) of counting over any list of numbers, this is not so. (That is to say, you may invent your own numbers and count through them, instead of using the numbers with which you are already familiar. We will examine this idea later!) To be more precise, we might say “these are the numbers which we can count through, and meaningfully attribute to a physical object.” This is better, but not rigorous enough for a mathematician, a philosopher of mathematics, or someone that seeks to better understand the nuances of mathematical inquiry. A man named Peano offered some primitive axioms which help us define the natural numbers in a more “clear” way, though its clarity is in its mathematical fidelity, not in how easy it is to fully digest at first acquaintance. These axioms (five in total) depend upon three fundamental, or “primitive” ideas, namely the ideas of 0 (zero), number, and successor.

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Axiom 1. \( 0 \) is a number; to wit, it is a natural number.

Axiom 2. The successor to any natural number is also a natural number.

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2 Notice the change in case: plural for the first, singular for the second! By the “general idea” – or concept – “of number,” I am referring to the property that all numbers share, which makes them be numbers. E.g., a tree has a certain “tree-ness”, and all trees share this property; a square has a “square-ness” which makes it a square, and no other shape. Likewise, we say that numbers have the property “number,” which separates them ontologically from trees, squares, or mud-wrestling. Being “primitive” entails that these ideas cannot be explained in terms of something more general. Consider this example: we can explain that 3 is a natural number; and a natural number is a number such that we may attribute some form of physical significance to it; and a number is an abstract object which possesses the quality of number. However, since number is a primitive idea, we have to stop our explanation here, as we cannot describe it further unless we discover even more primitive ideas regarding number.
Axiom 3. There is no number to which 0 is a successor; i.e., 0 is the first natural number!

Axiom 4. If two natural numbers have the same successor, then those natural numbers are equal.

Axiom 5. Since 0 has the property of being a natural number, and so does its successor, and so too does the successor’s successor, &c. &c., then it follows that every object we recover in this process is also a number, and has the same property of number as 0.\footnote{This is not to say that every number after 0 is equal to 0, just that every number after 0 shares a categorical property with 0. Namely, every object following 0 in our list of natural numbers is also a natural number, simply by virtue of it belonging to the same list as 0.}

Recall that we may write the series (or set) of natural numbers as

\[0, 1, 2, 3, \cdots, n, n + 1, \cdots.\]

We define 0 to be a number, arbitrarily, in accordance with the 1st postulate. You will notice that many starting points in mathematical inquiry begin with an act of assumption. Next, we might notice that this set begins with 0 – therefore it (the whole set) satisfies Peano’s 3rd postulate. Looking at the series expression for our set of natural numbers, we can see that the 4th postulate’s demand is met, if we assume each number to possess an intrinsically distinct value (that is to say, \(n \neq n + 1\); for instance, 3 \(\neq 4\), and we assume it never shall). The 5th postulate and the 2nd postulate are also (relatively speaking, of course) easy to agree with; in the former case, this is just another assumption on Peano’s part which allows us to construct the natural number set – in the latter case, this simply establishes that we will never encounter a non-number while counting through the natural numbers.

Here, 0, number, and successor are primitive ideas; this means that they cannot be proven in a rigorous sense, though using methods available to us via abstract algebra, we can certainly “prove” that the system of natural numbers has a unique number 0 – but this is irrelevant for now.

You may have noticed – continuing on in the spirit of mathematics’ apparently arbitrary basic nature (basic meaning fundamental, not “easy” or “trivial”!) – that Peano’s primitive ideas and five postulates do not say anything about any particular number other than 0. They do point, however, to entities which occur in specific ways “after” 0. We know 0 has a successor according to this system we have set up; we can call it “one,” and write it numerically as 1. We could have just so happened to call it anything else, and symbolize it with a smiley face; the specific name and form of the number is not important, in this context. 1 happens to be a number with many interesting properties, and it is in fact “older” than 0, as the Greeks and Romans were not cognizant of 0, and this bore an insoluble challenge for the Pythagoreans. In any case, we know 1 has a successor as well, à la Peano, and we shall call it “two”, and write it numerically as 2. Next is three (3), which has so many fascinating properties that religions and cultures have revered it globally for millennia. We continue on in the familiar way until we reach nine (9), and then we get tired of inventing new numerals. If we wish to continue, we just flip 9 “back” to
0, and create a new column to the left of our new 0, where we now insert a 1. That is: 1, 2, \cdots, 8, 9, 10. This is the essence of our base-10 number system, whose ten numerals are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Thus, after we reach 19 and can no longer “fit” any more numbers in the first column, that column flips back to 0 again, and we insert another 1 into the second column, yielding 20. Every other numbering system that I am aware of follows this convention: we exhaust all of our options for numbers in a column, and then begin anew in another column.

As a closing remark, notice how we can actually interpret Peano’s primitive ideas in an infinite number of ways, each time satisfying his five postulates, such that the natural numbers do not have any “necessary” appearance. An illustration of this is left to be explored as an exercise below (see Exercise 3).

**Exercise 2.** Recall that I implied that the inequality \( n \neq n + 1 \) represents the essence of Peano’s 3rd postulate, where \( n \) is any one of the natural numbers that we like. Peano says no two distinct numbers can have the same successor, but \( n \neq n + 1 \) seems like an awfully limited expression for such a broad statement! Peano’s postulate also implies that all numbers before \( n \) have unique successors, but \( n \neq n + 1 \) seemingly refers only to numbers directly after \( n \). Prove explicitly that the statement \( n \neq n + 1 \) does in fact imply that, for all \( n \) (sometimes written \( \forall n \), where \( \forall \) means “for all”), no two numbers can have the same successor. You can use algebra or purely linguistic reasoning to reach your answer.

**Solution 2.** If we take \( n \neq n + 1 \) to be a valid mathematical statement, we can use the rules of algebra to rewrite this equation as \( n - 1 \neq n \) by subtracting the constant, 1, from both sides. So not only does the original statement say that, for any imaginable number \( n \), \( n \neq n + 1 \), it also simultaneously says that \( n \neq n - 1 \). You could reason this out with words only, as well, but here I have chosen to demonstrate an answer using algebraic rules.

**Exercise 3.** As I said above, Peano’s primitive ideas can be subjected to an infinite number of interpretations in a wholly consistent manner. Since Peano used these as the apparatus for reconstructing the natural numbers, we will machinate our own number system according to his ideas and postulates.

First, let “0” have its usual meaning with which you are familiar, and thus also fulfilling Peano’s first primitive idea. For his second primitive idea, define “number” to only mean “even numbers.”

a) What does this mean “successor” must be defined as?
b) Write down the first five terms, starting from and including 0, of this new natural number system.
c) After obtaining your answer for part b, consider your terms. Do you think they are the same as their counterparts in the “traditional” natural number system? That is, is your new numbering system’s 4 the “same” as the traditional 4? If not, what is?

**Solution 3.**

a) The successor would also have to be even, given our definition of number, which means that for any \( n \), the successor would be \( n + 2 \) (and thus its predecessor would be \( n - 2 \)).
b) 0, 2, 4, 6, 8.
c) *Strictly speaking, no.* Here, “2” takes on the roll of “1” – that is, as 0’s immediate successor. Likewise, 4 is analogous to 2, 6 to 3, and 8 to 4. Note that this is ultimately a question about the values of 2, 4, 6, and 8 in our new system, rather than their appearances on paper!

### 1.2. Value

Now some overall final remarks should be made before moving on to the next topic. I think it is a good time to briefly discuss value in sort of a philosophical sense, as it will encourage a deeper understanding for the natural numbers in particular, and also flesh out the ideas behind my solution to Exercise 3.c.

Consider the decimal system, which, as we know, has 10 unique numerals, starting with 0 and ending with 9, or more compactly, \(0, 9\). There are many other number systems besides decimal, however. The most famous alternative number system today probably belongs to the realm of digital computing: the **binary system**, or base-2 system. Just behind binary in fame/infamy is the **hexadecimal system**, or base-16 system. All of these systems share the same concept of zero, though everywhere else they may differ in general:

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Binary</th>
<th>Hexadecimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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<td>10</td>
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<td>11</td>
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<td>1111</td>
<td>f</td>
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<tr>
<td>16</td>
<td>10000</td>
<td>10</td>
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<td>...</td>
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<td>...</td>
</tr>
</tbody>
</table>

If we ignore the specific symbols used for each number (that is, if we ignore each number’s numeral!) then we might gain some insight into the concept of value. 0’s successor in decimal is 1, as in binary and hexadecimal. But 1’s successor is 2 in decimal and hexadecimal, and 1’s successor is 10 in binary. Although they appear quite different, their value is the same, and we can obtain one number from the other by translating from one number system to the other, using a prescribed method. (Here we omit such methods as they are pedagogically superfluous.) I
might be tempted to write some quaint equalities which express the relationships between the representation of arbitrary numbers, in these three numbering systems:

\[ 2_D = 2_H = 10_B \]
\[ 13_D = c_H = 1101_B \]

and so on, if I wish, where the subscripts refer to decimal, hexadecimal, and binary, respectively.

Ultimately, I advise you to revisit the above table and pay more attention to the orientation of the boxes which house the numbers rather than to the numbers themselves. Disavowing one’s natural romance with the surface appearance of things is a critical ritual of the mathematician, philosopher, scientist, or scholar; in doing so, we oftentimes augment our awareness of the *res explicanda*, the object under study. At the present moment we are concerned with value, and I invite you to consider the essence of value being manifest yet tacit in the ordering of the boxes. Each number points to a value, though this value is logically separable from the form of the number we write on the page: 2’s value in decimal or hexadecimal could just as well be considered the same as 10 in binary. The impression of an underlying, unarticulated order emerges from beneath the endless sea of numbers.

2. The Integers

Now that we have sufficiently explored the set of natural numbers, or the so-called “counting” and “ordering” numbers, we are ready to construct a new set of numbers based on what we have learned. This new set will contain more numbers than the set of natural numbers. That’s right: if an infinite amount of natural numbers was not enough for you, you can be happy, because now we are tacking more onto our list!

That last comment deserves some elucidation, no? If we start with an infinitely large set, how can we add more to it? For, if we add more to it, then that means that what we are left with is something bigger. How can this possibly be? There seems to exist an inconsistency in our reasoning.

In fact, there is no inconsistency in our reasoning: the set of natural numbers does indeed have an infinite amount of elements. The discrepancy is in my language. When I say that the set of natural numbers is infinite, I mean to say that if we begin with 0 and continue counting “up” forever, I will never reach a number at which I have to stop. The “end” to my counting merely occurs when I tire of the counting process. I can always pick it back up later, beginning from where I left off before. So, I can always just add 1 to the number I am currently “at” in my counting, and thereby find the next number. If you program a computer to iterate through the series of natural numbers and do not tell it to stop at some specific number, then your computer will be stuck in an infinite loop, and will continue printing numbers on your screen until you abort the program. So, by demonstration, we have an infinite amount of natural numbers.

This operation of counting is limited, though. Allow me to draw your attention to the word “up,” which I used above to designate “adding 1 to the number I am currently ‘at’ in my series.” What about going backwards? If you worked on Exercise 2 in the previous section, you will know that we could go “down” (or
“backwards”) by subtracting 1 from our current number \( n \), as well – that is, until we reach 0. Here, we have to stop: Peano’s axioms say that 0 is not the successor to any natural number \( n \); in particular, \( 0 \neq n + 1, \forall n \).

Another way to phrase the above problem, in the language appreciated by mathematicians, is that the set of natural numbers is semi-infinite. A semi-infinite object (like a set of numbers) is an object which is infinite in only some possible ways, but finite (“not infinite”) in others. When an object in mathematics is infinite “in some way,” we say that it is unbounded in that way. Otherwise, when it is finite “in some way,” we say that it is bounded in that way. So, the set of natural numbers is semi-infinite because it is unbounded for counting in the incremental sense (or “positive direction,” if you are spatially inclined), but bounded for counting in the decremental sense (or “negative direction,” conversely).

Now enters the set of integers, symbolized by \( \mathbb{Z} \): this set of numbers will be unbounded in both directions. This feature inspires us to now define two additional terms to help us navigate this interesting new set of numbers: positive and negative. We define positive integers, symbolized by \( \mathbb{Z}^+, \mathbb{Z}_+, \) or \( \mathbb{Z}^\geq \), to be all the numbers greater than 0. 0 we take to be neither negative nor positive, so we call it non-negative. All positive integers are non-negative, but not all non-negative integers are positive, because 0 is not positive! The shorthand for non-negative integers is written \( \mathbb{Z}^\geq \).

How should we write the numbers less than 0? The most convenient way would be to recycle the numerals \( 0, 9 \) used in constructing the natural numbers, but use a special symbol to distinguish between the positive and negative versions of any number formed using these numerals. I say this is the “most convenient” way because the non-negative integers are the same as the natural numbers, so we need not invent new symbols to represent the natural numbers in the set of integers. Thus, we define the negative sign: \( - \) and the positive sign: \( + \). For integers greater than 0, we preface them with \( + \): +1, +2, +3, \( \cdots \); similarly, for integers less than 0, we write -1, -2, -3, \( \cdots \). Since the natural numbers, or the non-negative integers, continue on indefinitely, we can be assured that the same is true for the negative integers, as we attribute to every positive integer a negative counterpart! By convention, we typically do not write the positive sign when we write down individual, positive numbers.

**Exercise 4.** Does \( \mathbb{Z} \) satisfy Peano’s axioms? Are Peano’s primitive ideas still applicable to \( \mathbb{Z} \)?

**Exercise 5.** Examining the different symbols for the different sets of integers (i.e., for the positive and non-negative integers), what are some ways you might write the set of negative integers?

**Solution 4.** Peano’s 2nd axiom is violated in \( \mathbb{Z} \) (but not in \( \mathbb{Z}^\geq \)). For, if we were to count “up” from any negative number \( n \), we would eventually reach 0, meaning 0 is some number’s successor (namely, -1’s successor). Thus, this axiom

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5You may even pick up on a subtler inconsistency here if you are familiar with the finer points of the operation of subtraction. Not to worry, this will be ironed out shortly. Additionally, you may have noted that we never really defined numerical operations to begin with; this will also be addressed later.

6\( \mathbb{Z} \) comes from the German “Zahlen,” which translates as “number.”
does not apply. This isn’t a weakness of Peano’s axioms, though, which were only enumerated to explain natural numbers. Equivalently – since the \( 2^{\text{nd}} \) axiom fixes (or sets) the starting point for the natural numbers at 0 – if we subtract a natural number from a smaller natural number (e.g., in \( 5 - 7 \)), we get a number that doesn’t even exist in the set of natural numbers! (In this case, the result is \(-2\), which by definition cannot exist in \( \mathbb{N} \) because therein 0 is the first number, and \(-2\) is less than 0. When this kind of phenomenon occurs, we say that the set of numbers is “not closed under subtraction,” a concept we will explore later in the text.)

**Solution 5.** Although alternative notation may be in use somewhere, the two ways that come to my mind are \( \mathbb{Z}^- \) and \( \mathbb{Z}_- \). By the way, you can also symbolize “the set of non-zero integers” (i.e., all the negative and positive integers) by \( \mathbb{Z}^* \), pronounced “zee star,” or by \( \mathbb{Z}^\neq \). Note that the superscripts and subscripts used in the different sets of \( \mathbb{Z} \) refer to the relation of the set to the number 0: in the case of \( \mathbb{Z}^\neq \), the \( \neq \) symbol says that the set does not contain 0; for \( \mathbb{Z}^\geq \), it does include (and thus starts with) 0; and in \( \mathbb{Z}^- \), it solely contains the numbers less than 0.

Recall that I found fault with identifying the natural numbers as “counting numbers,” because we could count through any sort of numbers we might define. Really, since we generally only count using natural numbers, this isn’t so bad a term; however, I much prefer the term “ordering numbers,” because it is the sense of “order” that is imparted by the natural numbers that allows us to ascertain the order of other types of numbers. Negative numbers can be made intelligible to us intuitively by comparing them to the positive integers, or the natural numbers: as you count from 1 upwards to 100, you are counting through numbers with values that are becoming greater, or, as we might say, positively increasing (or increasingly positive – you will find both of these used in mathematical literature). Similarly, as you count from \(-1\) “downwards” to \(-100\), you are arithmetically doing the same thing: incrementing the current number you are at by 1, but now you prepend the number with a negative sign \(-\). You can describe this using several terms: “negatively increasing,” “increasingly negative,” &c. If you visualize the positive and negative numbers extending away from 0 in opposite directions, you are relying on the sense of order implicit in the organization of the natural numbers. The order of the negative integers \(-1, -2, -3, \cdots\) employs a numerical substructure analogous to that of the natural numbers, the non-negative integers.

**2.1. Comparing the sets of numbers.** What follows is a brief introduction to some symbols used and topics discussed in set theory, which we will explore later in the text, in its own chapter. The language of set theory will be partially embraced here such that we may recast the previous discussion in a more precise way. By now you probably have noticed I like to introduce things gradually – the earlier you see certain things, the better. This approach will slowly allow you to think about numbers more like a mathematician, and force you to connect the abstract with your intuition. Though it may be tricky or look strange, grappling with and trying to understand the notation slowly and carefully can start introducing you to new methods of thinking. Playing with symbols and notation has a lot of pedagogical benefit. The more you think explicitly about notation and its meaning, the more you will get out of the text (and all other math or physics texts).
Reflecting back, we see that the set of natural numbers is wholly contained in the set of integers, which we can write as \( \mathbb{N} \subset \mathbb{Z} \), where the symbol \( \subset \) means that the set on the left of it is “contained in” (or, in other words, “belongs to”) the set on the right. Formally we read this as “\( \mathbb{N} \) is a proper subset of \( \mathbb{Z} \).” We use this symbol often when working with sets, as we are frequently interested in the sizes of sets and how they relate to the sizes of other sets, and also how elements of one set relate (if at all) to members of other sets. You can remember the orientation of \( \subset \) by realizing the opening is faced toward the bigger set, so the criterion for its orientation is the same as for the greater-than and less-than signs for numbers, \( >, < \). The larger of the two sets is known as a proper superset of the smaller one, and hence \( \mathbb{Z} \supset \mathbb{N} \) reads as “\( \mathbb{Z} \) is a proper superset of \( \mathbb{N} \).”

The idea of sets can be further expanded upon. The symbols \( \subset, \supset \) are used above to designate proper subsets and supersets. A proper subset is contained fully within the larger set: since one set is larger, it contains elements that the smaller set does not. In other words, suppose we have one set, \( \mathfrak{A} \), and another set \( \mathfrak{B} \) larger than \( \mathfrak{A} \); since \( \mathfrak{B} \) is larger than \( \mathfrak{A} \), there exist elements in set \( \mathfrak{B} \) that are “not common to both \( \mathfrak{B} \) and \( \mathfrak{A} \): \( \mathfrak{A} \subset \mathfrak{B} \). A concrete example will follow below.

But suppose we have another set \( \mathfrak{C} \) which has the same amount of elements as \( \mathfrak{B} \), and, upon checking all of the elements in both sets, we find that all of \( \mathfrak{C} \)’s elements are also found in \( \mathfrak{B} \). In this case, \( \mathfrak{C} \) isn’t a proper subset of \( \mathfrak{B} \), nor vice versa; rather, they are merely subsets or supersets of one another, which we write as \( \mathfrak{B} \subseteq \mathfrak{C} \). As you might have guessed, we can also say these two sets are equal to each other. If all of their members are equal, and both sets have the same size (the same number of members), how could they not be equal? You can consider the subset symbols analogous to the “less than or equal to” sign for numbers, \( \leq \), except now we can flip the symbol indiscriminately without also flipping the terms of comparison (the sets): \( \mathfrak{B} \supseteq \mathfrak{C} \) is equally true. To see this, simply make your own two sets consisting of the same three numbers (or any other amount of numbers: even one number will do the trick though it may not be as convincing).

**Example 1.** I can define a set by listing all of its members. For instance, I will define set \( A \) as a set containing the first 5 non-zero integers, which I may write as \( A = \{1, 2, 3, 4, 5\} \). I will define another set named \( B \), and it will contain the first 7 non-zero integers: \( B = \{1, 2, 3, 4, 5, 6, 7\} \). In this way, \( A \) is a proper subset of \( B \): \( B \) fully contains \( A \), but also contains elements (6 and 7) that do not belong to \( A \). \( A \) cannot be a subset of \( B \); if it were, then all of \( B \)’s members would also belong to \( A \). So we can write \( A \subset B \) but not \( A \subseteq B \), nor \( B \subset A \).

As a tangential point, you can rewrite both sets solely in terms of their members: \( A \subset B \leftrightarrow \{1, 2, 3, 4, 5\} \subset \{1, 2, 3, 4, 5, 6, 7\} \), where the double-sided arrow \( \leftrightarrow \) says you may proceed from one representation (like our arbitrarily-used set symbols \( A \)
and $B$) to the other (our set elements, contained in curly brackets {})) no matter which representation you start with.)

Note that if one set is a proper subset of another, the former cannot simultaneously be a subset of the latter, because a subset must contain all the elements of the set it is a subset of. But, if one set is the subset of another, it is possible for the former to be a proper subset of the latter.

Since it is true that $\mathbb{N} \subset \mathbb{Z}$, I feel it is natural to continue along this path of exploring broader sets of numbers. Our next set, the set of rational numbers, will include even more numbers, and will also contain the integers and therefore the natural numbers as well. In the language of set theory, we could write $\mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$, where $\mathbb{Q}$ (you will see why we use $\mathbb{Q}$ soon) represents the set of rational numbers.

**Summary 1.**
1. $A \subset B$ says $A$ is a proper subset of $B$, or that $A$ is fully contained in $B$, and $B$ has members not found in $A$.
2. $A \subseteq B$ says $A$ is a subset of $B$, or that $A$ is fully contained in $B$, and $B$ has no members exterior to $A$. Thus, $A = B$.

**Exercise 6.** Write statements, using the set theory comparison symbols ($\subset$, $\supset$, $\subseteq$, $\supseteq$), relating
a. $\mathbb{Z}^+$ to $\mathbb{N}$
b. $\mathbb{Z}^-$ to $\mathbb{Z}$
c. The set of natural numbers to the set of non-negative integers
d. $\Phi = \{-3, -1, 0, 1, 3\}$ and $\Psi = \{-5 + 2, -3 + 2, -2 + 2, -1 + 2, 1 + 2\}$
e. $\{\diamondsuit, \heartsuit, \clubsuit\}$, $\{\heartsuit, \diamondsuit, \clubsuit\}$, $\{\heartsuit, \diamondsuit, \spadesuit\}$, and $\{\spadesuit\}$.

**Exercise 7.** For an arbitrary set $S$ (that is to say, we may make $S$ contain whatever we like), is it true that $S \subset S$ or $S \subseteq S$, or both? Carefully consider the definition of the comparison symbols in the summary above.