Spherical Shell on a Ramp

The set-up

Suppose we have a spherical shell of mass \( m \) and radius \( R \) on a ramp, such that the shell’s point of contact is located at a height \( h \) above the flat surface on which the ramp rests. The ramp is oriented at an angle of \( \theta \) with respect to this surface. The ramp is not frictionless. We are asked to solve for the velocity, \( v_f \), of the spherical shell at the precise moment it touches the horizontal surface, after rolling without slipping.

Force decomposition

We begin by breaking the expected forces into their individual Cartesian components. I will orient my \( x \)-axis along the direction in which the shell will roll. I will orient \( y \) parallel to the normal force, \( F_N \), which acts on the shell. I arbitrarily draw my diagram such that the shell will roll down and leftward.

Our forces then become:

\[
\begin{align*}
\text{x Forces} & : mg \sin \theta - f_s = ma_x \\
\text{y Forces} & : F_N - mg \cos \theta = 0
\end{align*}
\]

It is important to realize why I chose to employ static friction instead of kinetic friction. The distinction is subtle yet critical. Since the spherical shell rolls without slipping, its displacement is purely due to rotation. Each point which comes into contact with the surface of the ramp is, at that infinitesimal moment, stationary. The shell’s center of mass moves only because the mass distributed along its outer surface rolls due to the static frictional force.

Frictional force

We must obtain more information about this frictional force, to understand how it influences the motion. If we understand \( R \) to be the effective moment
arm, then the rotational motion is totally described by

\[ f_s R = I \alpha \rightarrow f_s R = \frac{2}{3} m R^2 \alpha, \]

where \( I \) is the moment of inertia for the spherical shell. We see that, given
\( \alpha = \frac{a_x}{R}, \)
\( f_s = \frac{2}{3} m a_x. \)

Plugging this into our equation for the x-component of the sphere's motion, we find that
\( a_x = \frac{(3g \sin \theta)}{5}. \) From trigonometry, we know that \( x_f = \frac{h}{\sin \theta}. \) From kinematics, this quantity must also be equal to \( \frac{1}{2} a_x t^2. \) Recall too that \( v_f = a_x t, \) which implies that \( t = \frac{v_f}{a_x}. \) Plugging this expression for \( t \) into our equation for \( x_f \) yields

\[ x_f = \frac{1}{2} a_x \left( \frac{v_f}{a_x} \right)^2 \rightarrow \frac{v_f^2}{2a_x} \Rightarrow v_f = \sqrt{2a_x x_f} \rightarrow \sqrt{\frac{3}{5} g \sin \theta \left( \frac{h}{\sin \theta} \right)}. \]

Thus, \( v_f \approx 10 \frac{m}{s}. \)

**Concluding remarks**

Some students were confused by this problem because they focused on the fact that numerical values were not assigned to \( R \) or \( \theta. \) Carefully listing the relevant forces and considering the rotational character of the sphere’s motion allowed us to cancel out these nuisance variables, and thus obtain a numerical answer. Notice that the result is independent of both the radius of the shell and the ramp’s angle of inclination.