We are asked to work with equation 1.50, or

\[ H = -(\frac{eB}{mc^2}), \]

writing the Heisenberg equations of motion in order to solve for \( S_{x,y,z} \). Note that in Heisenberg’s treatment, the \( S_{x,y,z} \rightarrow S_{x,y,z}(t) \). This contrasts with Schrödinger’s treatment, where the state ket \(|\Psi\rangle_s\) are functions of time, and the operators of observables are fixed. Heisenberg’s method seems to provide a “closer analogy” to classical physics, where the concept of state kets does not canonically exist.

We may think of the \( S_{x,y,z}(t) \) as \( \mathcal{W}^\dagger S_{x,y,z} \mathcal{W} \), where the S superscript denotes the Schrödinger picture of the operator. (Note that the operator in this case is unchanged; when coupled to state kets, the unitary operators act upon those.)

Letting the multiplicative constant equal \( \omega \), and using 2.19 in Sakurai, we get the following set of equations:

\[
\begin{align*}
\frac{d}{dt} S_x &= \frac{1}{i\hbar} [S_x, H] \\
\frac{d}{dt} S_y &= \frac{1}{i\hbar} [S_y, H] \\
\frac{d}{dt} S_z &= \frac{1}{i\hbar} [S_z, H] = 0
\end{align*}
\]

The most important part of the Hamiltonian is the \( S_z(t) \) component, as it will determine which equation(s) we can ignore. Immediately we see that our third equation is 0. Now that we have written the equations of motion, we can attempt to solve them:

\[
\frac{d}{dt} S_x + \frac{d}{dt} S_y = \frac{\omega}{i\hbar} ([S_x, S_z] + [S_y, S_z])
\]

. Using equation 4.20 in Sakurai, we note

\[
[S_i, S_j] = i\epsilon_{ijk}\hbar S_k \Rightarrow [S_i, S_k] = (-1)i\epsilon_{ijk}\hbar S_j,
\]
where \( x \rightarrow i, y \rightarrow j, z \rightarrow k \). Therefore \( \frac{d}{dt} S_x = (-i)(\omega) \hbar S_y = -\omega S_y \). Similarly, \( \frac{d}{dt} S_y = \omega S_x \).

The forms \( S_x + i S_y = S_+ \) and \( S_x - i S_y = S_- \) are familiar. Euler’s identity can be invoked to solve:

\[
S_x(t) = S_x(0) \cos(\omega t) - S_y(0) \sin(\omega t), \quad S_y(t) = S_x(0) \sin(\omega t) + S_y(0) \cos(\omega t).
\]

Differentiating both equations will reproduce the differential equations we derived above.