CARDAN'S FORMULAS AND BIQUADRATIC EQUATIONS

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1. Introduction. In 1956, Dr. Chao-Hui Yang showed me an interesting way to use a $3 \times 3$ cyclic matrix to recall Cardan's formulas for the roots of a cubic equation. Recently, by using a $4 \times 4$ cyclic matrix in a similar manner, I discovered analogous formulas for the roots of a biquadratic equation. All the details for a cubic are given in Section 2. My results for a biquadratic are presented in Theorem 1 of Section 4; they are related to other solution techniques in Sections 5 and 6.

As coefficient domain, suppose $F$ is a field of characteristic $\neq 2, 3$ with the property: for each element $\gamma$ in $F$, $x^2 = \gamma$ has a solution in $F$ and $x^3 = \gamma$ has a solution in $F$. In particular, the quadratic formula is applicable, and each second-degree polynomial over $F$ has a root in $F$; thus, $F$ contains a principal cube root $\omega$.
of unity and a principal fourth root \( i \) of unity. For definiteness, \( F \) can be the field \( C \) of complex numbers.

2. Cardan's formulas. Starting with the cyclic matrix

\[
\begin{bmatrix}
X & Y & Z \\
Z & X & Y \\
Y & Z & X
\end{bmatrix}
\]

we can easily remember to write the identity

\[
X^3 + (-3YZ)X + (Y^3 + Z^3) = (X + Y + Z)(X + \omega Y + \omega^2 Z)(X + \omega^2 Y + \omega Z).
\]

The left member of (2) equals the determinant \( D \) of (1). The right member of (2) follows via row operations on (1). Thus, by addition to the first row \((X, Y, Z)\) of the second row \((Z, X, Y)\) and the third row \((Y, Z, X)\), we see \( X + Y + Z \) is a factor of \( D \).

With \( \omega^3 = 1 \), addition to \((X, Y, Z)\) of \( \omega^3(Z, X, Y) \) plus \( \omega(Y, Z, X) \) shows \( X + \omega Y + \omega^2 Z \) is a factor of \( D \); etc.

By replacing \( Y \) by \(-Y\) and \( Z \) by \(-Z\) in (2), we obtain

\[
X^3 + (-3YZ)X + (-Y^3 - Z^3) = \prod_{s=0}^{2} (X - \omega^s Y - \omega^{2s} Z).
\]

Suppose \( \alpha \) and \( \beta \) are elements of \( F \). To solve the cubic equation

\[
X^3 + \alpha X + \beta = 0,
\]

we seek a solution \((y_0, z_0)\) of

\[
-3YZ = \alpha \text{ and } -Y^3 - Z^3 = \beta.
\]

There are two cases.

(i) Suppose \( \alpha \neq 0 \) or \( \beta \neq 0 \). Let \( t_0 \) be a nonzero solution in \( F \) of

\[
T^2 + \beta T + \left(-\frac{\alpha}{3}\right)^3 = 0,
\]

and let \( y_0 \) be a solution in \( F \) of \( Y^3 = t_0 \). With \( y_0 \neq 0 \), set \( z_0 = -\alpha/3y_0 \). Then, \((y_0, z_0)\) is a solution of

\[
YZ = -\frac{\alpha}{3} \text{ and } (Y^3)^2 + \left(-\frac{\alpha}{3}\right)^3 = -\beta Y^3.
\]

Thus, with \( y_0 \neq 0 \), we see \((y_0, z_0)\) is a solution of (5).

(ii) Suppose \( \alpha = 0 \) and \( \beta = 0 \). Then, we set \( y_0 = 0 \) and \( z_0 = 0 \).

We substitute \( y_0 \) for \( Y \) and \( z_0 \) for \( Z \) in (3) to obtain

\[
X^3 + \alpha X + \beta = \prod_{s=0}^{2} (X - \omega^s y_0 - \omega^{2s} z_0).
\]
Consequently, equation (4) has three roots $x_1, x_2, x_3$ in $F$ given by

$$x_{s+1} = \omega^s y_0 + \omega^{2s} x_0, \text{ for } s = 0, 1, 2.$$

Given a cubic equation $\bar{X}^3 + \alpha_1 \bar{X}^2 + \alpha_2 \bar{X} + \alpha_3 = 0$ over $F$, the substitution $\bar{X} = X - (x_i/3)$ reduces it to the form (4). Thus, any cubic equation over $F$ is solvable in $F$.

3. The determinantal of a cyclic matrix. We shall use the following modification of topics in [2].

**Lemma.** Suppose $K$ is a field which contains a primitive $n$th root of unity; let $K[X_1, \ldots, X_n]$ be a polynomial ring over $K$ in $n$ variables; and set

$$A = \begin{bmatrix}
X_1 & X_2 & \cdots & X_{n-1} & X_n \\
& X_1 & \cdots & X_{n-2} & X_{n-1} \\
& & \ddots & \ddots & \ddots \\
& & & X_1 & X_2 \\
& & & & X_1
\end{bmatrix}.$$

Then, $\det A = f_0 f_1 \cdots f_{n-1}$, where

$$f_s = X_1 + \rho^s X_2 + \rho^{2s} X_3 + \cdots + \rho^{(n-1)s} X_n, \text{ for } s = 0, 1, \ldots, n - 1.$$

**Proof.** For $k = 1, 2, \ldots, n$, let $R_k$ denote the $k$th row of $A$. Set

$$R = R_1 + \sum_{k=2}^{n} \rho^{(a-k+1)s} R_k.$$

We find $R = (f_0, \rho^{(a-1)s} f_0, \rho^{(a-2)s} f_0, \ldots, \rho^s f_0)$. Let $B$ denote the matrix obtained when the first row of $A$ is replaced by $R$. We observe $\det B = \det A$ and $f_j$ divides $\det A$ in $K[X_1, \ldots, X_n]$. The polynomial ring is factorial [1], and the elements $f_0, f_1, \ldots, f_{n-1}$ are irreducible. If $j$ and $k$ are integers with $0 \leq j < k \leq n - 1$, then $\rho^j \neq \rho^k$ and each common divisor of $f_j$ and $f_k$ is a unit. Thus, the product $f_0 f_1 \cdots f_{n-1}$ divides $\det A$. We set

$$\det A = q f_0 f_1 \cdots f_{n-1}.$$

In terms of total degree, we find

$$n = \deg(\det A) = \deg q + \sum_{s=0}^{n-1} \deg f_s = \deg q + n$$

and $\deg q = 0$. Hence, $q$ belongs to $K$. In $\det A$ and in $f_0 f_1 \cdots f_{n-1}$, the coefficient of $X_1^*$ is 1. This yields $q = 1$ and completes the proof.
4. Formulas for the roots of a biquadratic equation. With $K = F, n = 4$, and $\rho = i$ (where $i^2 = -1$), the Lemma gives

\[
\begin{vmatrix}
X & U & V & W \\
W & X & U & V \\
V & W & X & U \\
U & V & W & X \\
\end{vmatrix} = \prod_{s=0}^{3} (X + i^sU + i^{2s}V + i^{3s}W).
\]

We expand the determinant and replace $U, V, W$ by $-U, -V, -W$ to obtain

\[
X^4 + (-2V^2 - 4UW)X^2 + (-4U^2V - 4VW^2)X
\]

\[
+ (-U^4 + V^4 - W^4 + 2U^2W^2 - 4UVW^2)
\]

\[
= \prod_{s=0}^{3} (X - i^sU - i^{2s}V - i^{3s}W).
\]

This identity leads to the following result.

**Theorem 1.** Suppose $a, b, c$ are elements of $F$. When $a = b = c = 0$, set $u_0 = v_0 = w_0 = 0$; otherwise, let $v_0$ be a nonzero solution in $F$ of

\[
(4V^2)^3 + 2a(4V^2)^2 + (a^2 - 4c)(4V^2) - b^2 = 0,
\]

and let $u_0, w_0$ be elements of $F$ which satisfy

\[
UW = -\frac{v_0^2}{2} - \frac{a}{4} \quad \text{and} \quad U^2 + W^2 = -\frac{b}{4v_0}.
\]

Then, the biquadratic equation

\[
X^4 + aX^2 + bX + c = 0
\]

has four roots $x_1, x_2, x_3, x_4$ in $F$ given by

\[
x_{s+1} = i^s u_0 + i^{2s} v_0 + i^{3s} w_0, \quad \text{for} \quad s = 0, 1, 2, 3.
\]

**Proof.** First, we verify $(u_0, v_0, w_0)$ is a solution of

\[
4UW = -2V^2 - a,
\]

\[
4V(U^2 + W^2) = -b,
\]

and

\[
(U^2 + W^2)^2 + 8UWV^2 = \frac{a^2 - 4c}{4}.
\]

For $a = b = c = 0$, this is clear. For $v_0 \neq 0$, we rewrite (7) as
\(4^2 V^2 \left(\frac{a^2 - 4c}{4} - 2(-2V^2 - a)V^2\right) = b^2;\)

then, \((u_0, v_0, w_0)\) is a solution of (7), (8), (11), (12), (14), and

\[4^2 V^2 \left(\frac{a^2 - 4c}{4} - 8UW V^2\right) = 4^2 V^2 (U^2 + W^2)^2;\]

hence, with \(v_0 \neq 0\), \((u_0, v_0, w_0)\) is also a solution of (13).

We use (11) to eliminate \(a\) from (13); thus, \((u_0, v_0, w_0)\) satisfies

\[-2V^2 - 4UW = a,\]
\[-4U^2 V - 4VW^2 = b,\]

and

\[-U^4 + V^4 - W^4 + 2U^2 W^2 - 4UV^2 W = c.\]

We substitute \((u_0, v_0, w_0)\) in (6), (15), (16), and (17) to obtain

\[X^4 + aX^2 + bX + c = \prod_{s=0}^{3} (X - i^s u_0 - i^{2s} v_0 - i^{3s} w_0).\]

Consequently, (9) has four roots in \(F\) given by (10).

5. Further information. In Theorem 1, the element \(4v_0^2\) is a root of

\[Y^3 + 2aY^2 + (a^2 - 4c)Y - b^2 = 0.\]

We proceed to relate all three roots of (18) to Theorem 1.

**Proposition.** Suppose \(u_0, v_0, w_0\) in \(F\) satisfy (11), (12), and (13). Set

\[r_1 = (1 + i)u_0 + (1 - i)w_0, r_2 = 2v_0, r_3 = (1 - i)u_0 + (1 + i)w_0.\]

Then, \(r_1 r_2 r_3 = -b\) and (18) has three roots in \(F\) given by \(r_1^2, r_2^2,\) and \(r_3^2.\)

**Proof.** We use (19), (11), (13), and (12) to obtain

\[(Y - r_1^2)(Y - r_2^2)(Y - r_3^2) = (Y - r_1^2)(Y^2 - 8u_0w_0 Y + 4(u_0^2 + w_0^2)^2)\]
\[= Y^3 + (-4v_0^2 - 8u_0w_0)Y^2 + (4(u_0^2 + w_0^2)^2 + 32u_0w_0v_0^2)Y\]
\[-16v_0^2(u_0^2 + w_0^2)^2\]
\[= Y^3 + 2aY^2 + (a^2 - 4c)Y - b^2\]

and

\[r_1 r_2 r_3 = 4v_0(u_0^2 + w_0^2) = -b.\]

In [4], equation (18) was given as a cubic resolvent for equation (9), and a dif-
Different solution procedures were established. Next, we derive the solution formulas of [4] from Theorem 1.

**Theorem 2.** Suppose elements $r_1, r_2, r_3$ of $F$ satisfy

\begin{equation}
(Y - r_1^2)(Y - r_2^2)(Y - r_3^2) = Y^3 + 2aY^2 + (a^2 - 4c)Y - b^2
\end{equation}

and

\begin{equation}
r_1r_2r_3 = -b.
\end{equation}

Then, equation (9) has four roots $x_1, x_2, x_3, x_4$ in $F$ given by

\begin{align*}
x_1 &= (r_1 + r_2 + r_3)/2, \\
x_2 &= (r_1 - r_2 - r_3)/2, \\
x_3 &= (-r_1 + r_2 - r_3)/2, \\
x_4 &= (-r_1 - r_2 + r_3)/2.
\end{align*}

**Proof.** We define $u_0, v_0, w_0$ in $F$ through

\begin{equation}
4u_0 = (1 - i)r_1 + (1 + i)r_3, 2v_0 = r_2, 4w_0 = (1 + i)r_1 + (1 - i)r_3.
\end{equation}

Using (23), (20), and (21), we find:

\begin{align*}
4u_0w_0 + 2v_0^2 &= r_1^2 + r_2^2 + r_3^2 = -a, \\
4v_0(u_0^2 + w_0^2) &= r_1r_2r_3 = -b, \text{ and} \\
(u_0^2 + w_0^2)^2 + 8u_0w_0v_0^2 &= r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 = \frac{a^2 - 4c}{4}.
\end{align*}

Thus, $(u_0, v_0, w_0)$ is a solution of (11), (12), and (13). From the proof of Theorem 1, the roots of (9) are given by (10). For $s = 0, 1, 2, 3$, we use (10) and (23) to obtain (22).

6. Several observations. Given $\lambda, \mu$ in $F$, we specify a solution of

\[ UW = \lambda \text{ and } U^2 + W^2 = \mu. \]

For $\lambda = \mu = 0$, set $u_0 = w_0 = 0$; otherwise, let $t_0$ be a nonzero solution in $F$ of $T^2 - \mu T + \lambda^2 = 0$, let $u_0$ satisfy $U^2 = t_0$, and set $w_0 = \lambda/u_0$. In this way, (8) can be satisfied.

When $b = 0$ in equation (9), the conditions

\[ V = 0, \quad UW = -\frac{a}{4}, \text{ and } (U^2 + W^2)^2 = \frac{a^2 - 4c}{4}, \]

also specify a solution of (11), (12), and (13) as well as a solution procedure (10) for (9). We can take $\lambda = -a/4$ and $\mu$ so $\mu^2 = (a^2 - 4c)/4$. Of course, with $b = 0$, equation (9) can be solved directly as a quadratic in $X^2$. 
Let $S_1$ be the set of arrangements (first, second, third, fourth) for the four roots of (9); let $S_2$ be the set of solutions of (11), (12), and (13); and, let $S_3$ be the set of triples $(r_1, r_2, r_3)$ which satisfy (20) and (21). There exists a bijection of $S_2$ onto $S_1$. Namely, the mapping from $S_2$ to $S_1$ given by (10) is clearly injective. To prove it surjective, suppose $(x_1, x_2, x_3, x_4)$ is an element of $S_1$; then, $x_1 + x_2 + x_3 + x_4 = 0$; by solving linear equations, we find unique elements $u_0, v_0, w_0$ in $F$ which satisfy (10); by (6), $(u_0, v_0, w_0)$ is a solution of (15), (16), and (17); hence, $(u_0, v_0, w_0)$ is an element of $S_2$; etc. Similarly, there exists a bijection of $S_3$ onto $S_1$. Directly from (19) and (23), the sets $S_2$ and $S_3$ are in one-to-one correspondence.

Another procedure to deduce a cubic resolvent for (9) is given in [3]. Based upon Galois theory, the solution formulas of [3] are analogous to (18), (20), (21), and (22). The differences of notation are due to the changes necessitated when $Y$ is replaced in (18) by $-Y$.

Given a biquadratic equation $X^4 + a_1X^3 + a_2X^2 + a_3X + a_4 = 0$ over $F$, the substitution $X = X - (a_1/4)$ reduces it to the form (9). Thus, any biquadratic equation over $F$ is solvable in $F$.

7. An example. In the equation

\[(24)\quad X^4 - 2\beta^2X^2 - 4\alpha^2\beta X + (\beta^4 - \alpha^4) = 0,\]

we suppose $\alpha$ and $\beta$ are nonzero elements of $F$. With

\[a = -2\beta^2, \quad b = -4\alpha^2\beta, \quad c = \beta^4 - \alpha^4, \quad \text{and} \quad (a^2 - 4c)(-2a) - b^2 = 0,\]

the corresponding cubic resolvent (18) has $-2a$ as a root.

To solve (24) by Theorem 1, we take

\[4v_0^2 = -2a = 4\beta^2, \quad v_0 = \beta, \quad UW = 0, \quad U^2 + W^2 = \alpha^2, \quad u_0 = \alpha, \quad \text{and} \quad w_0 = 0.\]

Thus, by (10), the roots of (24) are

\[(25)\quad x_{s+1} = t^s\alpha + i^{2s}\beta, \quad \text{for} \quad s = 0, 1, 2, 3.\]

In this situation, Theorem 2 is less direct. First, all three roots

\[2i\alpha^2, 4\beta^2, -2i\alpha^2\]

of (18) are required. Then, elements $r_1, r_2, r_3$ of $F$ are needed to satisfy

\[r_1^2 = 2ix^2, \; r_2^2 = 4\beta^2, \; r_3^2 = -2ix^2, \quad \text{and} \quad r_1r_2r_3 = 4x^2\beta;\]

one choice is $r_1 = (1 + i)x, \; r_2 = 2\beta$, and $r_3 = (1 - i)x$. At this point, we can use (22) to obtain (25).

References