Our purpose here is to illustrate how numerous research ideas implicit in the work of Richard Dedekind about circulant matrices and their most natural generalizations have been systematically ignored by later expositors.

**Theorem 1.** Suppose that $A$ and $B$ are $n \times n$ circulant matrices having components in a ring $R$. Then, the matrix product of $A$ and $B$ is a circulant matrix and, when $R$ is a commutative ring, $BA = AB$.

Proof. Let $g$ be a generator for a cyclic group $G$ of order $n$. Then, with $e$ as the unit element of $G$ and $g^n = e$, Figure 1 provides a multiplication table for the elements of $G$. Consequently, there is a function $\alpha: G \rightarrow R$ such that the $(i, j)$-component of $A$ is given by

$$[A]_{i,j} = \alpha(g^{-i}g^j), \quad \text{for } i, j = 1, 2, \ldots, n$$

and there is a function $\beta: G \rightarrow R$ such that the $(i, j)$-component of $B$ is given by

$$[B]_{i,j} = \beta(g^{-i}g^j), \quad \text{for } i, j = 1, 2, \ldots, n.$$ 

Let $\gamma$ denote the function $\gamma: G \rightarrow R$ defined for each $x$ in $G$ by

$$\gamma(x) = \sum_{r=1}^{n} \alpha(g^{-r})\beta(g^{r}x).$$

For any fixed integers $i$ and $j$, the elements $g^r$ of $G$ for $1 \leq r \leq n$ are also given by means of $g^{i-k}$, for $1 \leq k \leq n$ and by $g^{j-k}$, for $1 \leq k \leq n$. Hence, (3) yields

$$\gamma(x) = \sum_{k=1}^{n} \alpha(g^{i-k})\beta(g^{k}x) = \sum_{k=1}^{n} \alpha(g^{j-k})\beta(g^{k}x), \quad \text{for each } x \in G.$$
We use (1), (2), and (4) to see that the \((i, j)\)-component of \(AB\) is given by

\[
[AB]_{i,j} = \sum_{k=1}^{n} [A]_{i,k} [B]_{k,j} = \sum_{k=1}^{n} \alpha (g^{k-i}) \beta (g^{j-k})
\]

\[
= \sum_{k=1}^{n} \alpha (g^{k-i}) \beta (g^{j-k}(g^{-i}g^{j})) = \gamma (g^{-i}g^{j}),
\]

for \(i, j = 1, 2, \ldots, n\). Hence, \(AB\) is a circulant matrix.

Suppose that the ring \(R\) is commutative. Then, for \(i, j = 1, 2, \ldots, n\), we use (2), (1), (4), and (5) to obtain

\[
[BA]_{i,j} = \sum_{k=1}^{n} [B]_{i,k} [A]_{k,j} = \sum_{k=1}^{n} \beta (g^{k-i}) \alpha (g^{j-k})
\]

\[
= \sum_{k=1}^{n} \alpha (g^{j-k}) \beta (g^{k-j}(g^{-i}g^{j}))
\]

\[
= \gamma (g^{-i}g^{j}) = [AB]_{i,j}.
\]

Thus, we have \(BA = AB\). This completes the proof.

In the research of Richard Dedekind typified by portions of his collected works found by clicking here, various \(n \times n\) matrices are considered in which the pattern of their components is that of a multiplication table for a finite group \(G\) of order \(n\). With the implied context that the components are complex numbers, his writing shows that he knew all such matrices having the same pattern have the properties of a ring under matrix addition and multiplication. He also knew that such matrices could be diagonalized when the corresponding group is abelian. For a mathematician of his ability, the supplying of detailed proofs for such results in a general context would have been a routine matter unnecessarily wasting time from the more difficult questions he wished to answer when the group is nonabelian. Thus, it is natural to attribute the following definition of a group- matrix to Richard Dedekind and to regard it as the standard definition when there is no further qualification. In particular, later uses of the term group matrix by George Frobenius and others to describe different concepts should be qualified in some manner such as by the name of the person using it or the place of usage.

**Definition 1.** Let \(A\) be an \(n \times n\) matrix having components in a set \(S\) and let \(G\) be a group of order \(n\) for which its \(n\) elements are denoted by \(g_1, g_2, \ldots, g_n\). Then, \(A\) is a group matrix for \(G\) and the ordering \(g_1, g_2, \ldots, g_n\) of its elements if and only if there is a function \(\sigma: G \to S\) such that the \((i, j)\)-component of \(A\) is given by

\[
[A]_{i,j} = \sigma (g_i^{-1}g_j), \quad \text{for } i, j = 1, 2, \ldots, n.
\]

To illustrate how various algebraic properties of circulant matrices can be transferred directly to group matrix, we supply a proof for the following result.

**Theorem 2.** Suppose that \(n \times n\) matrices \(A\) and \(B\) with components in a ring \(R\) are both group matrices for a group \(G\) relative to a designation of its elements by \(g_1, g_2, \ldots, g_n\). Then, the matrix product \(AB\) is a group matrix of that same kind. Moreover, when \(R\) is a commutative ring and \(G\) is abelian, \(BA = AB\).
We use (7), (8), and (10) to see that the \((i, j)\)-component of \(A\) is given by
\[
[A]_{i,j} = \alpha(g_i^{-1}g_j), \quad \text{for } i, j = 1, 2, \ldots, n
\]
and there is a function \(\beta: G \to R\) such that the \((i, j)\)-component of \(B\) is given by
\[
[B]_{i,j} = \beta(g_i^{-1}g_j), \quad \text{for } i, j = 1, 2, \ldots, n.
\]
Let \(\gamma\) denote the function \(\gamma: G \to R\) defined for each \(x\) in \(G\) by
\[
\gamma(x) = \sum_{r=1}^{n} \alpha(g_r^{-1})\beta(g_rx).
\]
For any fixed integers \(i\) and \(j\), the elements \(g_r\) of \(G\) for \(1 \leq r \leq n\) are also given by means of \(g_k^{-1}g_i\), for \(1 \leq k \leq n\) and by \(g_k^{r-j}\), for \(1 \leq k \leq n\). Hence, (9) yields
\[
\gamma(x) = \sum_{k=1}^{n} \alpha(g_i^{-1}g_k)\beta(g_k^{-1}g_jx) = \sum_{k=1}^{n} \alpha(g_i^{-k})\beta(g_j^{k-j}x), \quad \text{for each } x \in G.
\]
We use (7), (8), and (10) to see that the \((i, j)\)-component of \(AB\) is given by
\[
[AB]_{i,j} = \sum_{k=1}^{n} [A]_{i,k}[B]_{k,j} = \sum_{k=1}^{n} \alpha(g_i^{-1}g_k)\beta(g_k^{-1}g_j)
\]
\[
= \sum_{k=1}^{n} \alpha \left( (g_i^{-1}g_k)^{-1} \right) \beta \left( (g_j^{-1}g_k)(g_i^{-1}g_j) \right) = \gamma(g_i^{-1}g_j),
\]
for \(i, j = 1, 2, \ldots, n\). Hence, \(AB\) is a group matrix of that same type.

Suppose that the ring \(R\) is commutative and \(G\) is abelian. Then, for \(i, j = 1, 2, \ldots, n\), we use (8), (7), (10), and (11) to obtain
\[
[BA]_{i,j} = \sum_{k=1}^{n} [B]_{i,k}[A]_{k,j} = \sum_{k=1}^{n} \beta(g_i^{-1}g_k)\alpha(g_k^{-1}g_j)
\]
\[
= \sum_{k=1}^{n} \alpha \left( (g_j^{-1}g_k)^{-1} \right) \beta \left( (g_j^{-1}g_k)(g_i^{-1}g_j) \right) = \gamma(g_j^{-1}g_i) = [AB]_{i,j}.
\]
Thus, we have \(BA = AB\). This completes the proof.

Definition 1 is closely related to the following formulation.

**Definition 2.** Let \(A\) be an \(n \times n\) matrix having components in a set \(S\) and let \(G\) be a group of order \(n\) for which its \(n\) elements are denoted by \(h_1, h_2, \ldots, h_n\). Then, \(A\) is a group matrix for \(G\) and the ordering \(h_1, h_2, \ldots, h_n\) of its elements if and only if there is a function \(\tau: G \to S\) such that the \((i, j)\)-component of \(A\) is given by
\[
[A]_{i,j} = \tau(h_i h_j^{-1}), \quad \text{for } i, j = 1, 2, \ldots, n.
\]

In fact, when \(h_i = g_i^{-1}\), for \(i = 1, 2, \ldots, n\), we see that an \(n \times n\) matrix \(A\) is a group matrix for \(G\) and the ordering \(g_1, g_2, \ldots, g_n\) of its elements according to Definition 1 if and only if \(A\) is a group matrix for \(G\) and the ordering \(h_1, h_2, \ldots, h_n\) according to Definition 2.

To be developed further.