CHAPTER 16

Computer Algebra with Formulas (15.9)–(15.18)

The research presented in [19, 20, 21] was made possible when (15.16) was discovered and systems of computer algebra could then be used to find several key identities through trial-and-error experimentation. Similarly, one can make interesting discoveries or rediscoveries merely by using the formulas for \( c_i^*(z) \) and \( c_i^{**}(\zeta) \) with a few basic commands in a system of computer algebra. Here, we illustrate how that can be done by selecting a version of Mathematica from [55, 56, 57, 58, 59] as the system. The names of its commands indicate well what they do.

16.1. Computer representations for \( c_i^*(z) \) and \( c_i^{**}(\zeta) \)

We apply (15.9), (15.12), (15.17), (15.18), and (15.16) with the selected version of Mathematica to conclude that successive notebook evaluations of

\[
\begin{align*}
c[m_,0][z_] := & 1 \\
cS[m_,i_][z_] := & \text{Sum}[\text{Binomial}[m-j,i-j]*
(D[rho[z],{z,i-j}]/rho[z])*c[m,j][z],{j,0,i}] \\
alpha[0,j_][zeta_] := & 1 \\
alpha[i_,j_][zeta_] := & (\text{Sum}[alpha[i-1,k]’[zeta]-(i-1+k)(f''[zeta]/f’[zeta])*alpha[i-1,k][zeta],{k,1,j}] ) /; i >= 1 \\
cSS[m_,i_][zeta_] := & \text{Sum}[alpha[i-j,m-i][zeta]*
(f’[zeta])^j*c[m,j][f[zeta]],{j,0,i}]
\end{align*}
\]

enable Mathematica to then give computer representations for \( c_i^*(z) \) and \( c_i^{**}(\zeta) \), when \( i = 0, 1, 2, \ldots \) and \( m \) can remain a symbol for any positive integer \( \geq i \).

For instance, the computer representations for the evaluations of \( cS[m,1][z] \) and \( cSS[m,1][\zeta] \) show that \( c_1^*(z) \) and \( c_1^{**}(\zeta) \) are respectively given by

\[
c_1^*(z) \equiv c_1(z) + m\frac{\rho'(z)}{\rho(z)} \quad \text{and} \quad c_1^{**}(\zeta) \equiv f'(\zeta) c_1(f(\zeta)) - \left( \frac{m}{2} \right) \frac{f''(\zeta)}{f'(\zeta)}.
\]

Also, the computer representation for the evaluation of \( cS[m,2][z] \) yields

\[
c_2^*(z) \equiv c_2(z) + (m - 1)c_1(z)\frac{\rho'(z)}{\rho(z)} + \left( \frac{m}{2} \right) \frac{\rho''(z)}{\rho(z)}.
\]
16.2. Applications based on the representations for \( c_i^*(z) \) and \( c_i^{**}(\zeta) \)

Example 16.1. With \( m \geq 2 \) and symbols \( r_1, r_2 \) for rational numbers, we set

\[
P_{m,2} \equiv w_2^{(0)} + r_1 w_1^{(0)} + r_2 w_1^{(1)}.
\]

In regard to the function \( P_{m,2}(z) \) on \( \Omega \) that is obtained by replacing each \( w_i^{(j)} \) in \( P_{m,2} \) with the corresponding \( c_i^{(j)}(z) \) from (15.9), we see that the evaluation of

\[
P[z_] := c[m,2][z] + r_1*c[m,1][z]^2 + r_2*c[m,1]'[z]
\]

represents \( P_{m,2}(z) \). Also, for the function \( P_{m,2}^*(z) \) on \( \Omega \) that is obtained by replacing each \( w_i^{(j)} \) in \( P_{m,2} \) with the corresponding \( c_i^{*(j)}(z) \) from (15.12), the evaluation of

\[
PS[z_] := cS[m,2][z] + r_1*cS[m,1][z]^2 + r_2*cS[m,1]'[z]
\]

represents \( P_{m,2}^*(z) \). There are eight terms in the output for the evaluation of

\[
dif1[z_] = \text{Expand}[ PS[z] - P[z] ]
\]

and in those terms the parts not involving \( m, r_1, r_2 \) are equal to the evaluations of

\[
b[1] = c[m,1][z]*\text{rho}'[z]/\text{rho}[z];
\]

\[
b[2] = (\text{rho}'[z]/\text{rho}[z])^2;
\]

\[
b[3] = \text{rho}''[z]/\text{rho}[z];
\]

while the evaluations of

\[
a[1] = \text{Coefficient}[dif1[z],b[1]];\]

\[
a[2] = \text{Coefficient}[dif1[z],b[2]];\]

\[
a[3] = \text{Coefficient}[dif1[z],b[3]];\]

then yield the respective coefficients \( a[1], a[2], a[3] \) of \( b[1], b[2], b[3] \) in \( \text{dif1}[z] \). Of course, if \( r_1 = r_1 \) and \( r_2 = r_2 \) are specific rational numbers, then we see that: \( a[1], a[2], a[3] \) are zero if and only if \( PS(z) - P(z) \) is zero and \( P_{m,2}^*(z) \equiv P_{m,2}(z) \). After the evaluation of

\[
\text{list1} = \{a[1]==0, a[2]==0, a[3]==0\}
\]

as a system of three linear equations in \( r_1 \) and \( r_2 \), the evaluation of

\[
\text{Solve}\left[\text{list1}, \{r_1,r_2\}\right]
\]

yields a unique solution that corresponds to

\[
r_1 \equiv -\frac{(m-1)}{2m} \quad \text{and} \quad r_2 \equiv -\frac{(m-1)}{2}.
\]

Thus, when \( r_1, r_2 \) for (16.1) are defined by (16.2), we have \( P_{m,2}^*(z) \equiv P_{m,2}(z) \) on \( \Omega \) as a valid identity for any (15.9) on \( \Omega \) having \( m \geq 2 \) and any transformation (15.10) of that (15.9) into a corresponding equation (15.11) on \( \Omega \).
Example 16.2. With \( m \geq 2 \) and symbols \( s_1, s_2 \) for rational numbers, we set

\[
Q_{m,2} \equiv w_2(0) + s_1(w_1(0))^2 + s_2 w_1(1).
\]

In regard to the function \( Q_{m,2}(z) \) on \( \Omega \) that is obtained by replacing each \( w_i^{(j)}(z) \) in \( Q_{m,2} \) with the corresponding \( c_i^{(j)}(z) \) from (15.9), we see that the evaluation of

\[
Q[z,_] := c[m,2][z] + s_1*c[m,1][z]^2 + s_2*c[m,1][z]'[z]
\]

represents \( Q_{m,2}(z) \). For the function \( Q_{m,2}^{**}(\zeta) \) on \( \Omega^{**} \) that is obtained by replacing each \( w_i^{(j)} \) in \( Q_{m,2} \) with the corresponding \( c_i^{*(j)}(\zeta) \) from (15.16), the evaluation of

\[
QSS[\text{zeta}_,_] := ( cSS[m,2][\text{zeta}] + s_1*cSS[m,1][\text{zeta}]^2 + s_2*cSS[m,1]'[\text{zeta}] )
\]

represents \( Q_{m,2}^{**}(\zeta) \). There are twenty terms in the output for the evaluation of

\[
dif2[\text{zeta}_,_] = \text{Expand}[ QSS[\text{zeta}] - (f'[\text{zeta}])^2*Q[f[\text{zeta}]] ]
\]

and in those terms the parts not involving \( m, s_1, s_2 \) are given by the evaluations of

\[
b[4] = c[m,1][f[\text{zeta}]] f''[\text{zeta}];
\]

\[
b[5] = (f''[\text{zeta}]/f'[\text{zeta}])^2;
\]

\[
b[6] = f'''[\text{zeta}]/f'[\text{zeta}];
\]

while the evaluations of

\[
a[4] = \text{Coefficient}[dif2[\text{zeta}],b[4]];\]

\[
a[5] = \text{Coefficient}[dif2[\text{zeta}],b[5]];\]

\[
a[6] = \text{Coefficient}[dif2[\text{zeta}],b[6]];\]

give the coefficients of \( b[4], b[5], b[6] \) in \( \text{dif2}[\text{zeta}] \). Naturally, if \( s_1 = s_1 \) and \( s_2 = s_2 \) are specific rational numbers, then we see that: \( a[4], a[5], a[6] \) are zero if and only if \( QSS[\text{zeta}] - (f'[\text{zeta}])^2*Q[f[\text{zeta}]] \) is zero and we have the identity \( Q_{m,2}^{**}(\zeta) \equiv (f'(\zeta))^2 Q_{m,2}(f(\zeta)) \). After the evaluation of

\[
\text{list2} = \{ a[4]==0, a[5]==0, a[6]==0 \}
\]

as a system of three linear equations in \( s_1, s_2 \), the evaluation of

\[
\text{Solve}[\text{list2}, \{s_1,s_2\}]
\]

yields a unique solution that corresponds to

\[
s_1 \equiv -\frac{(m - 2)(3m - 1)}{6m(m - 1)} \quad \text{and} \quad s_2 \equiv -\frac{m - 2}{3}.
\]

Thus, for \( s_1, s_2 \) in (16.3) defined by (16.4), we have \( Q_{m,2}^{**}(\zeta) \equiv (f'(\zeta))^2 Q_{m,2}(f(\zeta)) \) on \( \Omega^{**} \) as a valid identity for any equation (15.9) on \( \Omega \) having \( m \geq 2 \) and any transformation (15.14) of that (15.9) into a corresponding equation (15.15) on \( \Omega^{**} \).
Example 16.3. Here, we use the computer representations for $c_i^*(z)$ and $c_i^{**}(\zeta)$ in Section 16.1 to check that the expression for $I_{4,1,4}$ in (1.17) on page 4 is printed correctly. We find that the evaluation of

\[
\text{Simplify}\left[ \left( cS[4,4][z] -(1/4)cS[4,1][z]*cS[4,3][z] \right. \\
\left. - (1/2)cS[4,3]'[z] - (9/100)cS[4,2][z]^2 \right) \\
+(1/5)cS[4,2]'''[z] + (13/100)cS[4,1][z]^2*cS[4,2][z] \\
+(27/100)cS[4,1]'[z]*cS[4,2][z] + (1/4)cS[4,1][z]*cS[4,2]'[z] \\
-(39/1600)cS[4,1][z]^4 - (39/200)cS[4,1][z]^2*cS[4,1]'[z] \\
-(33/200)(cS[4,1]'[z])^2 -(3/20)cS[4,1][z]*cS[4,1]'''[z] \\
-(1/20)cS[4,1]''''[z] \right] \\
\text{Simplify}\left[ \left( cSS[4,4][\zeta] -(1/4)cSS[4,1][\zeta]*cSS[4,3][\zeta] \right. \\
\left. - (1/2)c[4,3]'[\zeta] - (9/100)c[4,2][\zeta]^2 \right) \\
+(1/5)c[4,2]'''[\zeta] + (13/100)c[4,1][\zeta]^2*c[4,2][\zeta] \\
+(27/100)c[4,1]'[\zeta]*c[4,2][\zeta] + (1/4)c[4,1][\zeta]*c[4,2]'[\zeta] \\
-(39/1600)c[4,1][\zeta]^4 - (39/200)c[4,1][\zeta]^2*c[4,1]'[\zeta] \\
-(33/200)(c[4,1]'[\zeta])^2 -(3/20)c[4,1][\zeta]*c[4,1]'''[\zeta] \\
-(1/20)c[4,1]'''''[\zeta] \right] \\
\text{Simplify}\left[ \left( f'[\zeta] \right)^4 \left( c[4,4][f[\zeta]] ight. \\
\left. - (1/4)c[4,1][f[\zeta]]*c[4,3][f[\zeta]] \right) \\
-(1/2)c[4,3]'[f[\zeta]] - (9/100)c[4,2][f[\zeta]]^2 \right) \\
+(1/5)c[4,2]'''[f[\zeta]] \\
+(13/100)c[4,1][f[\zeta]]^2*c[4,2][f[\zeta]] \\
+(27/100)c[4,1]'[f[\zeta]]*c[4,2][f[\zeta]] \\
+(1/4)c[4,1][f[\zeta]]*c[4,2]'[f[\zeta]] \\
-(39/1600)c[4,1][f[\zeta]]^4 \\
-(39/200)c[4,1][f[\zeta]]^2*c[4,1]'[f[\zeta]] \\
-(33/200)(c[4,1]'[f[\zeta]])^2 \\
-(3/20)c[4,1][f[\zeta]]*c[4,1]'''[f[\zeta]] \\
-(1/20)c[4,1]'''''[f[\zeta]] \right] \\
\right] \\
is zero and the evaluation of

\[
\text{Simplify}\left[ \left( cS[4,4][z] -(1/4)cS[4,1][z]*cS[4,3][z] \right. \\
\left. - (1/2)cS[4,3]'[z] - (9/100)cS[4,2][z]^2 \right) \\
+(1/5)cS[4,2]'''[z] + (13/100)cS[4,1][z]^2*cS[4,2][z] \\
+(27/100)cS[4,1]'[z]*cS[4,2][z] + (1/4)cS[4,1][z]*cS[4,2]'[z] \\
-(39/1600)cS[4,1][z]^4 - (39/200)cS[4,1][z]^2*cS[4,1]'[z] \\
-(33/200)(cS[4,1]'[z])^2 -(3/20)cS[4,1][z]*cS[4,1]'''[z] \\
-(1/20)cS[4,1]'''''[z] \right] \\
\right] \\
is zero. Consequently, $I_{4,1,4}$ as presented in (1.17) on page 4 is a relative invariant of weight $s = 4$ for the equations (15.9) on page 158 having order $m = 4$. 

16.2. Applications Based on the Representations for $c_i(z)$ and $c_i^*(\zeta)$

Example 16.4. With $m \geq 3$ and symbols $t_1, t_2, t_3, t_4, t_5$ representing rational numbers, we introduce

\[(16.5) \quad I_{m,3} \equiv w_3 + t_1 w_1 w_2 + t_2 (w_1)^3 + t_3 w_2^{(1)} + t_4 w_1 w_1^{(1)} + t_5 w_1^{(2)}.\]

For the function $I_{m,3}(z)$ on $\Omega$ that is obtained by replacing each $w_i^{(j)}$ in $I_{m,3}$ with the corresponding $c_i^{(j)}(z)$ from (15.9), the evaluation of

$$\text{Inv}_z := (c_{m,3}[z] + t_1 c_{m,1}[z] c_{m,2}[z] + t_2 c_{m,1}[z]^3 + t_3 c_{m,2}'[z] + t_4 c_{m,1}[z] c_{m,1}'[z] + t_5 c_{m,1}''[z])$$

represents $I_{m,3}(z)$. For the function $I_{m,3}^*(z)$ on $\Omega$ that is obtained by replacing each $w_i^{(j)}$ in $I_{m,3}$ with the corresponding $c_i^*(j)(z)$ from (15.12), the evaluation of

$$\text{InvS}_z := (c_{S,m,3}[z] + t_1 c_{S,m,1}[z] c_{S,m,2}[z] + t_2 c_{S,m,1}[z]^3 + t_3 c_{S,m,2}'[z] + t_4 c_{S,m,1}[z] c_{S,m,1}'[z] + t_5 c_{S,m,1}''[z])$$

represents $I_{m,3}^*(z)$. For the function $I_{m,3}^{**}(\zeta)$ on $\Omega^{**}$ that is obtained by replacing each $w_i^{(j)}$ in $I_{m,3}$ with the corresponding $c_i^{**(j)}(\zeta)$ from (15.16), the evaluation of

$$\text{InvSS}_{\zeta} := (c_{SS,m,3}[\zeta] + t_1 c_{SS,m,1}[\zeta] c_{SS,m,2}[\zeta] + t_2 c_{SS,m,1}[\zeta]^3 + t_3 c_{SS,m,2}'[\zeta] + t_4 c_{SS,m,1}[\zeta] c_{SS,m,1}'[\zeta] + t_5 c_{SS,m,1}''[\zeta])$$

represents $I_{m,3}^{**}(\zeta)$. We note that $t_1, t_2, t_3, t_4, t_5$ for (16.5) yield

\[(16.6) \quad I_{m,3}(z) \equiv I_{m,3}(z) \text{ on } \Omega, \quad I_{m,3}(z) \equiv (f'(\zeta))^3 I_{m,3}(f(\zeta)) \text{ on } \Omega^{**}.\]

if and only if their representations $t_1, t_2, t_3, t_4, t_5$ for (16.5) yield

\[\text{diff1}_z := \text{Expand}[\text{InvS}_z - \text{Inv}_z] \]

\[\text{diff2}_{\zeta} := \text{Expand}[\text{InvSS}_{\zeta} - (f'[\zeta])^3 \text{Inv}[f[\zeta]]] \]

identically zero. Among the thirty-eight terms in the expansion of $\text{diff1}_z$, there are eight parts that do not involve $m, t_1, t_2, t_3, t_4, t_5$. Let them be copied individually from the output, pasted into individual input cells, given the names $b3[1], b3[2], \ldots, b3[8]$, and then evaluated. Among the ninety-three terms in the expansion of $\text{diff2}_{\zeta}$, there are eight parts that do not involve $m, t_1, t_2, t_3, t_4, t_5$. Let them be copied from the output, pasted into input cells, given the names $b3[9], b3[10], \ldots, b3[16]$, and then be evaluated. We evaluate

\[\text{Do}[a3[k] = \text{Coefficient}[\text{diff1}_z, b3[k]], \{k,1,8\}]\]

\[\text{Do}[a3[k] = \text{Coefficient}[\text{diff2}_{\zeta}, b3[k]], \{k,9,16\}]\]

and then find that the evaluation of

\[\text{Solve}[\text{Table}[a3[k] == 0, \{k,1,16\}], \{t1,t2,t3,t4,t5\}]\]
yields a unique solution. As expressed for (16.5), it is given by

\[
\begin{align*}
 t_1 & = -\frac{m - 2}{m}, \\
 t_2 & = \frac{(m - 1)(m - 2)}{3m^2}, \\
 t_3 & = -\frac{m - 2}{2}, \\
 t_4 & = \frac{(m - 1)(m - 2)}{2m}, \quad \text{and} \\
 t_5 & = \frac{(m - 1)(m - 2)}{12}.
\end{align*}
\]

Thus, (16.6) is satisfied by (16.7) for each equation (15.9) having \( m \geq 3 \) as well as each transformation (15.10) of (15.9) into a corresponding (15.11) and each transformation (15.14) of (15.9) into a corresponding (15.15). In this regard, see (1.13) of page 3. If the definitions of \( \mathbf{b}_3[1], \mathbf{b}_3[2], \ldots, \mathbf{b}_3[16] \) give difficulty, use the Google browser Chrome to visit

http://homepages.uc.edu/~chalklr/Chapter-16.html

and then download the Mathematica notebook available there. Details are also given in that notebook for Examples 16.1, 16.2, 16.3, and 16.5.

**Example 16.5.** There are unique rational numbers \( u_1, u_2, \ldots, u_{12} \) for

\[
(16.8) \quad \mathbf{I}_{m,4} \equiv w_1 + u_1 w_1 w_3 + u_2 w_1^{(1)} + u_3 (w_2)^2 + u_4 w_2^{(2)} + u_5 (w_1)^2 w_2 \\
+ u_6 w_2^{(1)} w_2 + u_7 w_1 w_2^{(1)} + u_8 (w_1)^4 + u_9 (w_1)^2 w_1^{(1)} \\
+ u_{10} (w_1^{(1)})^2 + u_{11} w_1 w_1^{(2)} + u_{12} w_1^{(3)}, \quad \text{with} \ m \geq 4,
\]

such that the functions \( I_{m,4}(z) \) on \( \Omega \), \( I_{m,4}^*(z) \) on \( \Omega^* \), and \( I_{m,4}^{**}(z) \) on \( \Omega^{**} \) that are obtained by replacing each \( w_i^{(j)} \) in \( I_{m,4} \) with the corresponding \( c_i^{(j)}(z) \) from (15.9), with the \( c_i^{(j)}(z) \) from (15.11), and with the \( c_i^{**(j)}(z) \) from (15.15), satisfy both

\[
I_{m,4}^*(z) \equiv I_{m,4}(z) \quad \text{on} \ \Omega, \quad \text{and} \quad I_{m,4}^{**}(z) \equiv (f(z))^4 I_{m,4}(f(z)), \quad \text{on} \ \Omega^{**}.
\]

When the technique of Example 4.4 is repeated here, the main difference is that: in place of the copy and paste for Example 4.4 where \( \mathbf{b}_3[k] \) was obtained separately for \( 1 \leq k \leq 8 \) and \( 9 \leq k \leq 16 \), we now use copy and paste to obtain \( \mathbf{b}_4[k] \) separately for \( 1 \leq k \leq 20 \) and for \( 21 \leq k \leq 40 \). Of course, this requires more patience. However, when details similar to those of Example 4.4 are carried out, the coefficients for \( I_{m,4} \) in (16.8) are found to be

\[
(16.9) \quad u_1 = -\frac{m - 3}{m}, \quad u_2 = -\frac{m - 3}{2}, \quad u_3 = -\frac{(m - 2)(m - 3)(5m + 7)}{10(m + 1)m(m - 1)} ,
\]
\[
 u_4 = \frac{(m - 2)(m - 3)}{10}, \quad u_5 = \frac{(m - 2)(m - 3)(5m + 6)}{5(m + 1)m^2}, \quad u_6 = \frac{(m - 2)(m - 3)(5m + 7)}{10(m + 1)m} ,
\]
\[
 u_7 = \frac{(m - 2)(m - 3)}{2m}, \quad u_8 = -\frac{(m - 1)(m - 2)(m - 3)(5m + 6)}{20(m + 1)m^2} ,
\]
\[
 u_9 = -\frac{(m - 1)(m - 2)(m - 3)(5m + 6)}{10(m + 1)m^2}, \quad u_{10} = -\frac{(m - 1)(m - 2)(m - 3)(2m + 3)}{20(m + 1)m} ,
\]
\[
 u_{11} = -\frac{(m - 1)(m - 2)(m - 3)}{10m}, \quad u_{12} = -\frac{(m - 1)(m - 2)(m - 3)}{120}.
\]

By setting \( m = 4 \) in these formulas, we obtain the coefficients for (1.17) on page 4.

**Observation.** The basic relative invariants \( \mathbf{I}_{m,1,s} \) of weight \( s \geq 3 \) for the equations (15.9) of order \( m \geq s \) are given explicitly by the computer program in Section 6.1 on pages 53–54. We note that \( \mathbf{I}_{m,3} \) in (16.5) with the coefficients of (16.7) is equal to \( \mathbf{I}_{m,1,3} \). Also, \( \mathbf{I}_{m,4} \) in (16.8) with the coefficients of (16.9) is equal to \( \mathbf{I}_{m,1,4} \).