The Research about Invariants of Ordinary Differential Equations
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Roger Chalkley

Professor Emeritus of Mathematics
University of Cincinnati
Cincinnati, Ohio 45221-0025

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Abstract. Several basic relative invariants for homogeneous linear differential equations were discovered during the years shortly after 1878. Also, a basic relative invariant was found by Paul Appell in 1889 for a type of nonlinear differential equation. There was little progress during the years 1892–1988 as researchers who worked with homogeneous linear differential equations were unknowingly handicapped by the standard practice of introducing binomial coefficients in the writing of their equations. They thereby failed to develop adequate formulas for the coefficients of equations resulting from a change of the independent variable. Consequently, for relative invariants as the most important kind of invariant, progress was stymied.

The notation was simplified in 1989, adequate transformation formulas were developed, and explicit expressions were deduced in 2002 for all of the basic relative invariants of homogeneous linear differential equations. In 2007, explicit formulas were obtained for all of the basic relative invariants of a type of ordinary differential equation involving two parameters $m$ and $n$ that represent positive integers. When $n = 1$ and $m \geq 3$, the formulas specialize to provide all of the basic relative invariants for homogeneous linear differential equations of order $m$; and, when $m = n = 2$, they yield all three of the basic relative invariants for the equations of Paul Appell.

A general method developed in 2014 combines two relative invariants of weights $p$ and $q$ for the same type of equation to explicitly obtain a relative invariant of weight $p + q + r$, for any $r \geq 0$. With that, the principal problems about relative invariants have now been solved.

This monograph provides clear perspective about the reformulation begun after 1988 and recently completed. Chapters 15 and 18 show how the major difficulties confronting earlier researchers have been overcome.
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Preface

The subject of relative invariants for ordinary differential equations has been completely redeveloped in a series of publications begun in 1989. Now, there are satisfactory solutions to the principal unsolved problems that provided interest for researchers after Edmund Laguerre found a relative invariant in [37, 38] of 1879 for third-order homogeneous linear differential equations and the French Academy of Sciences encouraged extensions of his research. In particular, Georges-Henri Halphen won the 1880 Grand Prize of the French Academy of Sciences for research about invariants published in [32] and Henri Poincaré received honorable mention for his competitive submission to them in 1880.

Explicit formulas for all of the basic relative invariants of homogeneous linear differential equations of each fixed order were found and presented in [19] of 2002. For a type of nonlinear differential equation studied by Paul Appell in [4] of 1889, he discovered one of its three basic relative invariants. The other two were obtained for [20] of 2007 and all three appear in [20, page 13, Theorem 1.8] of 2007.

As a remarkable generalization not anticipated by earlier researchers, all of the basic relative invariants were discovered and presented by explicit formulas in [20, pages 257, 264, 275–276] for a type of ordinary differential equation involving two integral parameters \(m\) and \(n\), where \(m\) is the order of the equation and \(n\) is its degree when its left member is regarded as a homogeneous polynomial in the various derivatives of the dependent variable. In particular, when \(n = 1\), the formulas specialize to yield the ones in [19] for the basic relative invariants of homogeneous linear differential equations of each order \(m \geq 3\); and, when \(m = n = 2\), they specialize to yield the ones in [20] for the three basic relative invariants of the nonlinear equations Paul Appell studied in [4].

To complete the research involving the preceding results, a construction was developed in [21] of 2014 where, under general conditions, it combines two relative invariants of respective weights \(p\) and \(q\) for the same type of equation to produce a relative invariant of weight \(p + q + r\), for any integer \(r \geq 0\). Examples were also given in [21] to illustrate how, starting with the basic relative invariants for a given type of equation, that construction can be repeatedly applied to obtain linearly independent relative invariants of a given weight whose linear combinations yield all of the relative invariants having that weight.

This revision of [21] includes Chapters 15 and 18 as new ones to show why, after a flurry of intense interest during the years 1879–1891, the subject remained in limbo until 1989. In particular, these chapters make precise the principal difficulties earlier researchers failed to overcome.

Roger Chalkley