Conditional expectation with respect to dependent $\sigma$-fields

In this note we will deal with random variables $(X_k)$, which are the conditional expectation of some random variable $X$ with respect to $\sigma$-fields $(F_k)$. In the case where $(F_k)$ are independent $\sigma$-fields such random variables were characterized by Bryc and Kwapień [2] and by a similar technique part of their results was carried over to some dependent $\sigma$-fields by Bryc [3]. The asymptotic behaviour of $(X_k)$ for Markov dependent $\sigma$-fields $(F_k)$ was studied by Isaac [4].

Here we will concentrate on the case of two $\sigma$-fields $F_1, F_2$ and we will look for the conditions close to being necessary for all reasonable $X_1, X_2$ to be of the form $X_k = E^{F_k}(X), \ k = 1, 2$; cf. the theorem below.

We assume in the sequel that each $F_k, \ k = 1, 2$, is a $P$-complete $\sigma$-field. Let $G = F_1 \cap F_2$ and denote by $L^p_0(G)$ the Banach subspace of $L^p, 1 \leq p \leq \infty$, generated by $F_k$-measurable random variables $Y$ such that $E^G(Y) = 0$, $k = 1, 2$ ($L^p$ denotes the Banach space of all $p$-integrable random variables with the usual norm). Let us observe that the conjugated space to $L^p_0(G)$ is isometrically isomorphic to $L^p_0(G)$, where $p < \infty$, $p$ and $p'$ are conjugated numbers, i.e. $1/p + 1/p' = 1$ ($\infty = 1$).

First, let us observe that if there exists a random variable $X \in L^1$ such that $X_k = E^{F_k}(X), \ k = 1, 2$, then $E^G X_1 = E^G X_2$. Moreover it is possible to assume that $E^G X_k = 0, \ k = 1, 2$, since evidently the assertions

(i) $\forall \ X_k \in L^p_0(F_k), \ E^G(X_1) = E^G(X_2) \Rightarrow \exists Z \in L^p$, such that $X_k = E^{F_k}(Z), \ k = 1, 2$,

and

(ii) $\forall \ X_k \in L^p_0(F_k), \ \exists Z \in L^p$, such that $X_k = E^{F_k}(Z), \ k = 1, 2$.

Lemma 1. Let $1 \leq p < \infty$. Then the following conditions are equivalent

(A) For every $X_k \in L^p_0(F_k)$, $k = 1, 2$, there exists $Z \in L^p$ such that $X_k = E^{F_k}(Z), \ k = 1, 2$;

(B) There exists $C > 0$ such that for every $Y_k \in L^p_0(F_k)$,

$$\|Y_1\|_{p'} + \|Y_2\|_{p'} \leq C \|Y_1 + Y_2\|_{p'}.$$

Proof. We will prove the lemma only for $1 < p < \infty$. If $p = \infty$ some changes are needed in the reasoning (a conjugate version of the Banach theorem has to be used). Let $T : L^p \rightarrow L^p_0(F_1) \times L^p_0(F_2)$ be defined by $T(X) = (E^{F_1}(X), E^{F_2}(X)), k = 1, 2$. Then $T^* : L^p_0(F_1) \times L^p_0(F_2) \rightarrow L^{p'}$ is easily seen to be defined by $T^*((X_1, X_2)) = X_1 + X_2$. By the Banach theorem (see, e.g., Rudin [6], Th. 4.15) the linear operator $T$ is "onto" if the adjoint operator

409
\( T^* \) is an isomorphic embedding, i.e. there exists a constant \( C \) such that 
\[ \left\| (Y_1, Y_2) \right\| \leq C \left\| T^*((Y_1, Y_2)) \right\| \]
where \( \left\| \cdot \right\| \) is any of the equivalent norms on \( L_p^0(F_2) \times L_p^0(F_2) \). Choosing \( \left\| (Y_1, Y_2) \right\| = \left\| Y_1 \right\|_p + \left\| Y_2 \right\|_p \) we conclude the proof.

Let \( \rho = \sup \{ \left\langle E(X_1, X_2) : X_2 \in L_p^0(F_2), \left\| X_2 \right\|_2 = 1, k = 1, 2 \} \). Note that \( 0 \leq \rho \leq 1 \). In the case where \( G \) is the trivial \( \sigma \)-field \( \rho \) is known as the \textit{maximal correlation coefficient}. The rôle of \( \rho \) in the context of conditional expectations is partially explained by

**Corollary 1.** Let \( F_1, F_2 \) be \( \sigma \)-fields such that for every \( X_k \in L_p^0(F_2), k = 1, 2 \), there exists \( Z \in L_2 \) such that \( X_k = E^{F_k}(Z), k = 1, 2 \). Then \( \rho < 1 \).

**Proof.** On account of Lemma 1 the corollary follows from the inequality 
\[ \left\| X_1 + X_2 \right\|_2^2 \leq (1 - \rho) \left\langle E \left| X_1 \right|^2 + E \left| X_2 \right|^2 \right\rangle. \]

Note that since the inequality in the proof of Corollary 1 can be arbitrarily close to equality the converse implication holds, too. This will be reproved in more constructive a way in our main result

**Theorem.** Let \( 1 < p < \infty \) and assume \( F_1, F_2 \) are such that \( \rho < 1 \). Then for every \( X_1 \in L_p^0(F_1), X_2 \in L_p^0(F_2) \) there exists \( Z \in L_p \) such that \( X_k = E^{F_k}(Z), k = 1, 2 \).

As an immediate consequence of the theorem we have

**Corollary 2.** If \( F_1, F_2 \) are \( \sigma \)-fields such that for every \( X_k \in L_p^0(F_k) \) there exists \( Z \in L_2 \) such that \( X_k = E^{F_k}(Z), k = 1, 2 \), then for every \( X \in L_p^0(F_2) \) there exists \( Z \in L_p \) such that \( X_k = E^{F_k}(Z), k = 1, 2 \), \( \forall 1 < p < \infty \).

For the proof of the theorem we will need one more lemma which strengthens the convergence in the "alternierende Verfahren" (Rota [5]).

**Lemma 2.** Let \( 1 < p < \infty \), \( \rho < 1 \). If \( E^{F_k} \) is considered as a linear operator acting on \( L_p \), then \( (E^{F_k})^k \to E \) as \( k \to \infty \) in the norm topology. The same holds for \( E^{F_k}F_k \).

**Proof.** Let \( Q = E^{F_k}F_k \). First, let us observe that if \( Q \) is considered as acting on \( L_p^0 \), then \( \left\| Q \right\| < \rho \). Indeed, \( \left\| Qf \right\|^2 = E(\left\langle E^{F_k}F_k f, E^{F_k}F_k f \right\rangle) = E(\left\langle E^{F_k} f, E^{F_k} f \right\rangle) \leq \left\| Q \right\|_2 \left\| f \right\|_2 \). Therefore \( Q \to 0 \) as \( k \to \infty \) when \( Q \) is considered as an operator acting on \( L_p^0 \), and since \( \left\| Q^k \right\|\leq \left\| Q \right\|^k \to 0 \) as \( k \to \infty \) the lemma is proved for \( \rho = 2 \).

Now, let \( 1 < \rho < 2 \). Then, since \( \left\| Q^k \right\| < \rho \), \( \left\| L_{p-1} \right\| \leq 2 \), \( \left\| Q^k \right\|_{L_{p-1}} \to 0 \) as \( k \to \infty \) by the Riesz convexity theorem ([1], Th. 1.1.1.) \( \left\| Q^k \right\|_{L_{p-1}} \leq C \left\| Q^k \right\|_{L_2} \to 0 \) as \( k \to \infty \).

A similar argument with \( L_m \) instead of \( L_1 \) proves the lemma in the case \( 2 < \rho < \infty \).

By symmetry the same holds for \( E^{F_k}F_k \).

**Proof of the theorem.** By Lemma 2 both series \( \sum_{k=0}^{\infty} (E^{F_k}F_k)^k \) and \( \sum_{k=0}^{\infty} (E^{F_k})^k \) are convergent in the operator norm, when considered as linear operators acting on the Banach space \( L_p^0 \). Indeed, \( (E^{F_k}F_k)^k = (E^{F_k})^k - E \) on \( L_p^0 \), hence \( \left\| (E^{F_k}F_k)^k \right\| < 1 \) for some \( n_0 \) large and \( \left\| (E^{F_k})^k \right\| \leq \left\| (E^{F_k})^k \right\| \leq \left\| (E^{F_k})^k \right\| < 1 \).

To complete the proof it suffices now to notice that if \( X_k \in L_p^0(F_k), k = 1, 2 \), then \( X_1 = E^{F_1}(X_2) \) and \( X_2 = E^{F_2}(X_1) \) are in \( L_p^0 \). Therefore

\[
Z = \sum_{k=0}^{\infty} (E^{F_k}F_k)^k (X_1 - E^{F_k}(X_2)) + \sum_{k=0}^{\infty} (E^{F_k})^k (X_2 - E^{F_k}(X_1))
\]
is a well-defined \( \beta \)-integrable random variable. A direct computation shows that 
\[ E^\beta(Z) = X_n, \quad n = 1, 2. \]

Remark: During the Dragov Conference the author was convinced that the theorem holds at least for \( \beta = 1 \) without any assumptions on the dependence of \( \sigma \)-fields. This appeared not to be true. To see this let us consider the interval \([0, 1]\) with the uniform probability distribution and the following \( \sigma \)-fields: \( \mathcal{F}_2 \) is generated by a family \( I \) of intervals such that \( I = \bigcup_{n=1}^{\infty} I_n \), where 

\[ I_n \]

is any finite partition of \((2^{n+1}, 2^n)\) consisting of intervals of length less than \(2^{-n\beta} \); \( \mathcal{F}_1 \) is generated by a family \( J \) of intervals such that \( J = \bigcup_{n=1}^{\infty} J_n \), where \( J_n \) is a finite partition of \((2^{n+1}, 2^n)\) generated by the centers of intervals in \( I_n \) (thus the intervals in \( J_n \) are of length less than \(2^{-n\beta} \), too). Then the \( \sigma \)-field \( \mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2 \) is generated by the intervals \((2^{n+1}, 2^n)\). To prove that there are \( \mathcal{F}_2 \)-measurable random variables \( X_n \), \( \beta = 1, 2 \), such that \( E^\beta(X_n) = E^{\beta}(X_1) \) and which cannot be of the form \( X_n = E^\beta(Z) \), \( \beta = 1, 2 \), for any integrable \( Z \), we use Lemma 1. Namely, we will show that there is no constant \( C > 0 \) such that for every \( Y \in L^2(\mathcal{F}_1), Y \in L^2(\mathcal{F}_2) \),

\[ \|Y\|_1 + \|Y\|_2 < C \|Y - Y_1\|_1. \]

To this end let \( n \in N \) be fixed and define

\[ Y(w) = \begin{cases} 2^{n\beta}w - 1 & \text{if } 2^{-\beta} < w < 2^n, \\ 0 & \text{otherwise.} \end{cases} \]

and

\[ Y_n = E^\beta(Y), \quad n = 1, 2. \]

Then, since \( I_n \) consists of intervals of length less than \(2^{-n\beta} \) and \( Y \) is a Lipschitz function with constant \(2^n \beta \), we have \( \|Y - Y_n\|_2 < \sigma^{-1} \) and, similarly, \( \|Y - Y_n\|_1 < \sigma^{-1} \). Thus, \( \|Y - Y_2\|_2 < 2\sigma^{-1} \) and \( \|Y - Y_2\|_1 < 2\sigma^{-1} \), which contradicts (1) since \( n \) is arbitrary and \( \|Y\|_1, \|Y\|_2 \geq (n - 1)/n. \)

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