Chapter 9

Weak convergence

This is based on [Billingsley, Section 25]

1. Convergence in distribution

A cumulative distribution function $F$ can have at most a countable number of discontinuity points. In fact, the set $\{x : F(x) - F(x^-) \geq 1/n\}$ can have at most $n$ points. (This observation will be used in many proofs.)

**Definition 9.1.** Let $F_n$, $F$ be cumulative distribution functions. We say that $F_n \xrightarrow{D} F$ if $F_n(x) \to F(x)$ for every point $x$ of continuity of $F$.

We say that $X_n \xrightarrow{D} X$ if $F_n \xrightarrow{D} F$.

We first show how to use the definition to prove weak convergence.

**Example 9.1.** Suppose $\{X_k\}$ are independent exponential. Then $\max_{1 \leq k \leq n} X_k - \ln n \xrightarrow{D} Y$ where $Y$ has the Gumbel distribution: $P(Y \leq x) = \exp(-e^{-x})$.

Indeed, $P(\max_{1 \leq k \leq n} X_k - \ln n \leq x) = P(\max_{1 \leq k \leq n} X_k \leq x + \ln n) = P(X_1 \leq x + \ln n)^n = (1 - e^{-x\ln n})^n = (1 - e^{-x/n})^n \to e^{-e^{-x}}$

The following example illustrates that we cannot require convergence for all $x \in \mathbb{R}$.

**Example 9.2.** Suppose $X_n$ are uniform $U(0,1/n)$ with

$$F_n(x) = \begin{cases} 
1 & x > 1/n \\
 nx & x \in [0,1/n] \\
0 & x < 0 
\end{cases}$$

It is clear that $X_n \to 0$ with probability one, so we expect (and can prove, see Theorem 9.2 below) that $X_n \xrightarrow{D} 0$, i.e. that $F_n(x) \to F(x) = \begin{cases} 
1 & x \geq 0 \\
0 & x < 0 
\end{cases}$. And indeed, $F_n(x) = 0$ for $x < 0$ and $F_n(x) \to 1$ for $x > 0$. But note that $F_n(0) = 0$ does not converge to $F(0)$.
The following example illustrates that a popular interpretation of weak convergence as “approximating all probabilities” for $X_n$ by the asymptotic probabilities for $X$ has significant restrictions.

**Example 9.3.** Suppose $P(X_n = k) = 1/n$. Then $\frac{1}{n}X_n \xrightarrow{D} U(0,1)$. Indeed, $F_n(x) = [nx+1]/n \to x$. Note however that $P(\frac{1}{n}X_n \in V)$ may fail to converge to $\lambda(V)$ for some Borel sets $V \in \mathcal{B}$.

In view of Example 9.3, it is interesting to have a criterion where a stronger form of weak convergence holds.

**Theorem 9.1** (Scheffe’s theorem). Suppose $X_n$ has a density $f_n(x)$ with respect to a (possibly infinite, possibly discrete) measure $\nu(dx)$ on $\mathbb{R}$. If $f_n(x) \to f(x)$ pointwise and $f$ is a density of a random variable $X$, then

$$\sup_{U} |P(X_n \in U) - P(X \in U)| \to 0$$

**Proof.** Consider $g_n = f - f_n$. Then $g_n^+ \to 0$ and $0 \leq g_n^+ \leq f$ so by the dominated convergence theorem $\int g_n^+ \nu(dx) \to 0$. Now $\int |g_n| \nu(dx) = \int_{g_n > 0} g_n \nu(dx) - \int_{g_n \leq 0} g_n \nu(dx)$. Since $\int g_n \nu(dx) = 0$ we have $\int_{g_n \leq 0} g_n \nu(dx) = -\int_{g_n > 0} g_n \nu(dx)$ so

$$\int |g_n| \nu(dx) = 2 \int g_n^+ \nu(dx) \to 0$$

Thus $|P(X_n \in U) - P(X \in U)| \leq \int_U |g_n(x)| \nu(dx) \to 0$ for any $U \in \mathcal{B}$. \hfill $\square$

**Example 9.4.** Suppose $X_n$ is binomial $\text{Bin}(n,p = \lambda/n)$. Then $X_n \xrightarrow{D} Y$ where $Y$ is Poiss($\lambda$). Indeed, the density with respect to the counting measure $\nu$ converges pointwise

$$P(X_n = k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)\lambda^k}{k!(1-\lambda/n)^n} \to e^{-\lambda}\lambda^k/k!$$

Thus in this case $P(X_n \in A) \to P(Y \in A)$ for all $A$. For example,

$$P(X_n \text{ is even}) \to e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{2k!}$$

Similarly, as the number of degrees of freedom $d \to \infty$, the density of student $T_d$ distribution converges to the standard normal density.

**Theorem 9.2.** If $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{D} X$.

**Proof.** Let $x$ be a point of continuity of $F(x)$. Then

$$P(X_n \leq x) = P(X_n \leq x, |X_n - X| \leq \varepsilon) + P(X_n \leq x, |X_n - X| > \varepsilon)$$

So

$$P(X_n \leq x) \leq P(X \leq x + \varepsilon) + P(|X_n - X| > \varepsilon)$$

Similarly,

$$P(X \leq x - \varepsilon) \leq P(X_n \leq x) + P(|X_n - X| > \varepsilon)$$

So

$$F(x - \varepsilon) \leq \liminf P(X_n \leq x) \leq \limsup P(X_n \leq x) \leq F(x + \varepsilon)$$

Taking the limit $\varepsilon \to 0$,

$$F(x^-) \leq \liminf P(X_n \leq x) \leq \limsup P(X_n \leq x) \leq F(x)$$
Remark 9.3. If $X_n \xrightarrow{D} a$ for a deterministic random variable $a$ then $X_n \xrightarrow{P} a$. Indeed, $P(X_n \leq a - \varepsilon) \to 0$ and $P(X_n \leq a + \varepsilon) \to 1$ so

$$P(|X_n - a| > \varepsilon) \leq P(X_n \leq a - \varepsilon) + 1 - P(X_n \leq a + \varepsilon) \to 0$$

Theorem 9.4 (Slutsky’s Theorem). Suppose $(X_n, Y_n)$ are defined on the same probability space. If $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} 0$ then $X_n + Y_n \xrightarrow{D} X$.

Proof. Take $y' < y''$ two continuity points of the law of $X$ and $y' < x - \varepsilon < x < x + \varepsilon < y''$. Then

$$P(X_n \leq y') - P(|Y_n| > \varepsilon) \leq P(X_n + Y_n \leq x) \leq P(X_n \leq y'') + P(|Y_n| > \varepsilon)$$

so

$$F(x) \leq \lim \inf P(X_n + Y_n \leq x) \leq \lim \sup P(X_n + Y_n \leq x) \leq F(y'')$$

We now note that the set $\{x : F(x- - F(x) \geq 1/n\}$ has at most $n$ points, so the set of all discontinuities of $F$ is at most countable. Therefore, if $x$ is a continuity point of $F$ we can find continuity points $y' < x < y''$ that are arbitrarily close to $x$. Thus taking a sequence $y' \to x$ and $y'' \to x$ of such points we get

$$F(x) \leq \lim \inf P(X_n + Y_n \leq x) \leq \lim \sup P(X_n + Y_n \leq x) \leq F(x)$$

The following corollary is often useful. (Its proof requires tightness!).

Corollary 9.5. Suppose $(X_n, Y_n)$ are defined on the same probability space. If $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} c$ then $X_n Y_n \xrightarrow{D} cX$.

Proof. See Exercise 9.2.

2. Fundamental results

(Omitted in 2019)

Theorem 9.6 (Skorohod’s theorem). Suppose $X_n \xrightarrow{D} X$ i.e. $F_n \xrightarrow{D} F$. Then there exist a probability space $(\Omega, \mathcal{F}, P)$ and random variables $Y_n, Y$ with CDF $F_n, F$ on $(\Omega, \mathcal{F}, P)$ such that $Y_n \to Y$ for all $\omega \in \Omega$.

Proof. We choose $\Omega = (0, 1)$ with Lebesgue measure. Recalling the quantile function (4.3) we define

$$Y_n(\omega) = \inf \{x : F_n(x) \geq \omega\}$$

$$Y(\omega) = \inf \{x : F(x) \geq \omega\}$$

Recall that $Y(\omega) \leq x$ iff $F(x) \geq \omega$ so $Y(\omega) > x$ implies $F(x) < \omega$.

Given $\varepsilon > 0$ choose $Y(\omega) - \varepsilon < x < Y(\omega)$ such that $F(x- - F(x)$. Since $F_n(x) \to F(x)$, this implies that $F_n(x) < \omega$ for large $n$. Thus $Y_n(\omega) > x > Y(\omega) - \varepsilon$. Since $\varepsilon > 0$ this shows that $\lim \inf Y_n \geq Y$.

Now choose $\omega < \omega'$ and a continuity point $y$ of $F$ such that $Y(\omega') < y < Y(\omega') + \varepsilon$. The first inequality then implies that $\omega < \omega' \leq F(y)$, so for large $n$ we have $F_n(y) > \omega$. Thus $Y_n(\omega) \leq y < Y(\omega') + \varepsilon$. This shows that $\lim \sup Y_n(\omega) \leq Y(\omega)$ for all points $\omega$ of continuity of $Y$. Note that $Y$ is an increasing function so it can have at most countable number of discontinuities. At such points we re-define $Y_n(\omega)$ to be $Y(\omega)$. This changes $Y_n$ on the set of measure zero, so does not affect the result. □
Theorem 9.7 (Portmanteau Theorem). The following conditions are equivalent:

(i) \( X_n \xrightarrow{D} X \)

(ii) \( E(f(X_n)) \to E(f(X)) \) for every bounded continuous function \( f \)

(iii) \( E(f(X_n)) \to E(f(X)) \) for every bounded Lipschitz (uniformly continuous) function \( f \)

(iv) \( P(X_n \in U) \to P(X \in U) \) for every Borel set \( U \) such that \( P(X \in \delta U) = 0 \)

**Proof.** (Omitted in 2019) (1)⇒(2) Using Theorem 9.6 we have \( f(Y_n) \to f(Y) \) so by Lebesgue’s dominated convergence theorem (Theorem 6.9), the integrals converge. Note that this proof works for \( \mathbb{R} \) but not for \( \mathbb{R}^2 \), so it is of interest to have a direct proof that will not use Theorem 9.6. See [Billingsley, Theorem 29.1]

(2)⇒(3) is obvious

(3)⇒(1) Fix a point of continuity \( x_0 \) of \( F \) and let

\[
 f(x) = \begin{cases} 
 1 & x \leq x_0 \\
 \text{linear} & x_0 < x < x_0 + \varepsilon \\
 0 & x > x_0 + \varepsilon 
\end{cases}
\]

Then \( F_n(x_0) \leq E(f(X_n)) \to Ef(X) \leq F(x_0 + \varepsilon) \) so \( \limsup F_n(x_0) \leq F(x_0) \).

Next, take

\[
 f(x) = \begin{cases} 
 1 & x \leq x_0 - \varepsilon \\
 \text{linear} & x_0 - \varepsilon < x < x_0 \\
 0 & x \geq x_0
\end{cases}
\]

Then \( F_n(x_0) \geq E(f(X_n)) \to Ef(X) \geq F(x_0 - \varepsilon) \) Thus \( \liminf F_n(x_0) \geq F(x_0) \).

(3)⇒(4) The plan here is to prove that for any Borel set \( U \) with interior \( U^o \) and closure \( \bar{U} \) we have

\[
 \mu(U^o) \leq \liminf \mu_n(U^o) \leq \liminf \mu_n(U) \leq \limsup \mu_n(U) \leq \limsup \mu_n(\bar{U}) \leq \mu(\bar{U})
\]

To prove the last inequality, take continuous \( f \) with values between 0, 1 and such that \( f = 1 \) on \( \bar{U} \) and \( f = 0 \) outside of an \( \varepsilon \)-closure of \( U \). Then \( \mu_n(U) \leq \int f(x) \mu_n(dx) \to f(x) \mu(dx) \leq \mu(U^{[\varepsilon]}) \) Noting that \( U = \bigcap_{\varepsilon > 0} U^{[\varepsilon]} \) we get the last inequality.

The first inequality then follows by taking complements.

(4)⇒(1) is obvious

(1)⇒(2) [Second proof] Suppose \( f \) is continuously differentiable and \( f' = 0 \) outside of a finite interval \([ -K, K ] \). Then from Fubini’s theorem\(^1\) we get

\[
 Ef(X_n) = f(0) + \int_0^K f'(t) P(X_n > t)dt - \int_{-K}^0 f'(t) P(X_n \leq t)dt
\]

Since \( P(X_n \leq t) \to P(X \leq t) \) except for a countable (Lebesgue-measure zero) set of \( t \), by Lebesgue dominated convergence theorem we get \( Ef(X_n) \to Ef(X) \).

\(^1\) \( f(x) = f(0) + \int_{x>0} f'(t)dt - \int_{x<0} f'(t)dt \)
As an immediate corollary, we get an important result.

**Example 9.5.** In the setting of Example 9.3, we have a unbounded function then $f$.

**Proof.** Since $\text{Theorem 9.9 (Helly, Prokhorov)}$.

Define $9.2$. A sequence of probability measures $\mu_n$ on $\mathbb{R}$ is tight if for every $\varepsilon > 0$ there exists $K$ such that $\mu_n([-K,K]) > 1 - \varepsilon$.

It is clear that if $X_n \xrightarrow{D} X$ then $X_n$ is tight.

**Theorem 9.9 (Helly, Prokhorov).** If $\mu_n$ is a tight family of probability measures then there is a probability measure $\mu$ and a subsequence $n_k \to \infty$ such that $\mu_{n_k} \xrightarrow{D} \mu$.

**Proof.** Since $F_n(r)$ is a bounded sequence of numbers, there is a subsequence that converges. In fact, by using a diagonal method, there is a subsequence $n_k$ such that $F_{n_k}(r) \to G(r)$ for all $r \in \mathbb{Q}$.

To see this, enumerate all rational numbers $q_1, q_2, \ldots$. Since $[0,1]$ is compact, we can choose a sequence $n(k) = n_1(k)$ such that $F_{n_1(k)}(q_1)$ converges to, say, $G(q_1)$. Choose a subsequence $n_2(k)$ of $n_1(k)$ such that $F_{n_2(k)}(q_1)$ converges to, say, $G(q_1)$ and so on.

\[
\begin{array}{cccccc}
n_1(1) & n_1(2) & n_1(3) & \ldots & \ldots & F_{n_1(k)}(q_1) \to G(q_1) \\
n_2(1) & n_2(2) & n_2(3) & \ldots & \ldots & F_{n_2(k)}(q_2) \to G(q_2) \\
n_3(1) & n_3(2) & n_3(3) & \ldots & \ldots & F_{n_3(k)}(q_3) \to G(q_3) \\
\vdots & \ddots & \vdots & \ldots & \ldots & \vdots \\
n_j(1) & n_j(2) & n_j(3) & \ldots & \ldots & F_{n_j(k)}(q_j) \to G(q_j) \\
\vdots & \ddots & \vdots & \ldots & \ldots & \vdots 
\end{array}
\]

Then the diagonal subsequence $m_k := n_k(k)$ has the property that $F_{m_k}(q) \to G(q)$ for every $q \in \mathbb{Q}$.

Define

\[F(x) = \inf \{G(r) : r > x\}\]
Note that $F(x) = \lim_{r_k \downarrow x} G(r)$, so $F$ is non-decreasing and right-continuous.

By tightness, $F(x) < \varepsilon$ if $x < K$ and $F(x) > 1 - \varepsilon$ if $x > K$.

Now we verify the weak convergence. Let $x$ be a point of continuity of $F$. Choose $r_k \uparrow x$ and $r_k' \downarrow x$.

Then

$$F_n(r_k) \leq F_n(x) \leq F_n(r_k')$$

so for every $k$ we have

$$G(r_k) \leq \liminf F_n(x) \leq \limsup F_n(x) \leq G(r_k')$$

But $G(r_k') \to F(x)$ as $k \to \infty$. And for any $\varepsilon_k > 0$ converging to 0 we have $G(r_k) \geq F(r_k - \varepsilon_k) \to F(x)$ by continuity.

□

Example 9.6. Suppose $X_n$ are uniform on $(0, n)$. Then

$$F_n(x) = \begin{cases} 
0 & x < 0 \\
\frac{x}{n} & 0 \leq x \leq n \\
1 & x > n
\end{cases}$$

So $F_n(x) \to F_\infty(x)$ for all $x$. Clearly $F_\infty(x)$ is not a cumulative distribution function, and $X_n$ is not a tight sequence.

We will need the following corollary.

**Theorem 9.10.** If $\mu_n$ is a tight family of probability measures and if each subsequence converges to the same probability measure $\mu$ then $\mu_n \overset{D}\to \mu$.

**Proof.** Suppose $\mu_n$ fails to converge to $\mu$ with CDF $F$. Then there is a point of continuity $x$ of $F$ and an infinite sequence $n_k$ such that $|F_{n_k}(x) - F(x)| > \delta$ for all $k$. Since subsequence $\mu_{n_k}$ is tight, choose a convergent subsequence. By assumption, this sequence converges to $\mu$, so $F_{n_{k_j}}(x) \to F(x)$, which contradicts that $|F_{n_k}(x) - F(x)| > \delta$ for all $k$.

Recall Definition 6.3: Family $\{X_n\}$ is uniformly integrable if for every $\varepsilon > 0$ there is $K$ such that $\int_{|X_n| > K} |X_n| dP < \varepsilon$.

**Proposition 9.11.** If $\{X_n\}$ is uniformly integrable, then $\sup_n E|X_n| < \infty$

**Proof.** (This should have been an assigned exercise!) $E|X_n| = \int_{|X_n| \leq K} |X_n| dP + \int_{|X_n| > K} |X_n| dP < K\varepsilon + \varepsilon$.

□

**Theorem 9.12.** Suppose $X_n \overset{D}\to X$ and $\{X_n\}$ is uniformly integrable. Then $X$ is integrable and $E(X_n) \to E(X)$.

**First proof.** From Theorem 9.6 there exists a sequence $Y_n \to Y$ such that $E(Y_n) = E(X_n)$. [For this proof, we need to know that $|Y_n - Y|$ is uniformly integrable!] By Lebesgue’s dominated convergence theorem (See Remark 6.13), $E(Y_n) \to E(Y)$.

□
Second proof. The first step is to prove that $X$ is integrable, which we will omit\(^2\).

Given $\varepsilon > 0$ choose $K$ such that $\int_{|X| > K} |X|dP < \varepsilon$. Since $X$ is integrable, we can increase $K$ to ensure that we also have $\int_{|X| > K} |X|dP < \varepsilon$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise-linear bounded continuous function such that $f(x) = x$ for $x \in [-K, K]$ and $f(x) = 0$ for $x \notin [-K-1, K+1]$. (Draw the graph. Note that $f = f_K$ depends on $K$.) By Theorem 9.7, $E(f(X_n)) \to E(f(X))$. On the other hand, $X_n = f(X_n)$ for $|X_n| \leq K$ and $|f(x)| \leq |x|$ for all $x$, so

$$|E(X_n) - E(X)| \leq |E(X_n) - E(f(X_n))| + |E(X) - E(f(X))| + |Ef(X_n) - Ef(X)|$$

$$\leq 2 \int_{|X| \geq K} |X|dP + 2 \int_{|X| \geq K} |X|dP + |Ef(X_n) - Ef(X)|$$

$$\leq 4\varepsilon + |Ef(X_n) - Ef(X)|$$

Since $|Ef(X_n) - Ef(X)| \to 0$ this implies convergence.

The proof of the next result contains solution of (a generalization of) Exercise 6.6.

**Corollary 9.13.** Suppose $\sup_n E|X_n|^r < \infty$ for a natural $r$ and $\delta > 0$. If $X_n \xrightarrow{D} X$ then $E(|X|^r) < \infty$ and $E(X_n^r) \to E(X^r)$.

**Proof.** We verify that $|X_n|^r$ is uniformly integrable, compare Exercise 6.6.

$$\int_{|X_n| > t} |X_n|^r dP = \int_{|X_n| > t} |X_n|^r 1_{|X_n| > t} dP \leq \int_{|X_n| > t} |X_n|^r dP \leq \frac{1}{t^\delta} \sup_n E|X_n|^{r+\delta}$$

\[\square\]

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**Required Exercises**

**Exercise 9.1.** Suppose $\{X_k\}$ are independent uniform $U(0, 1)$ random variables. Show that

$$n \min_{1 \leq k \leq n} X_k \xrightarrow{D} Y$$

and determine the law of $Y$.

**Exercise 9.2.** Prove Corollary 9.5: if $X_n \xrightarrow{P} c$ for a constant $c$ and $Y_n \xrightarrow{D} Y$, show that $X_n Y_n \xrightarrow{D} cY$.

**Exercise 9.3.** Suppose $X_n \xrightarrow{D} X$. Show that the laws of $X_n$ are tight.

**Exercise 9.4.** Suppose $X_n \in \mathbb{Z}$ and $X_n \xrightarrow{D} X$. Show that $P(X \in \mathbb{Z}) = 1$ and that $P(X_n = k) \to P(X = k)$ for every $k \in \mathbb{Z}$.

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\(^2\)Choose bounded continuous $f_K$ as in the main part of the proof but apply it to $|X_n|$ so that $0 \leq f_K(|X_n|) \leq |X_n|$. Then $E(|X|^r 1_{|X| \leq K}) \leq Ef_K(|X|)$. But $Ef_K(|X|) = \lim_{K \to \infty} Ef_K(|X_n|)$. And $Ef_K(|X_n|) \leq E|X_n| \leq M$ by Proposition 9.11. So $E|X| = \lim_{K \to \infty} E(|X|^r 1_{|X| \leq K}) \leq M < \infty.$
Exercise 9.5. Suppose $X_n$ has density $f_n(x) = 1 + \cos(2\pi nx)$ on $[0, 1]$. Prove that $X_n \xrightarrow{D} X$ (and determine the law of $X$).

Exercise 9.6. Suppose $E(X_n^2) = 1$. Show that $F_n$ is tight.

Exercise 9.7. Suppose $E(X_n^2) = 1$. Show that $\{X_n\}$ is uniformly integrable.

Exercise 9.8. Show that $X$ is integrable if and only if for every $\varepsilon > 0$ there exists $K$ such that $\int_{|X|>K} |X| dP < \varepsilon$. (This is Corollary 6.12 on page 73)

Exercise 9.9. Suppose that $\sup_n E(|X_n| f(|X_n|)) < \infty$ for some non-decreasing function $f$ such that $\lim_{x \to \infty} f(x) = \infty$. Show that $\{X_n\}$ is uniformly integrable.

Exercise 9.10. The Lévy distance between two probability measures on $\mathbb{R}$ is defined as
\[ d(F, G) = \inf\{\varepsilon > 0 : G(x - \varepsilon) - \varepsilon \leq F(x) \leq G(x + \varepsilon) + \varepsilon \text{ for all } x \}\]
(i) Verify that this is a metric\footnote{Non-negative, triangle inequality, and $d(F, G) = 0$ only for $F = G$}.
(ii) Verify that $F_n \xrightarrow{D} F$ iff $d(F_n, F) \to 0$
(iii) Verify that for every probability measure $\mu$ on Borel sets of $\mathbb{R}$ there exist probability measures $\mu_n$ with finite support such that $\mu_n \xrightarrow{D} \mu$. Show further that the support can be taken from $\mathbb{Q}$, so that the space of distribution functions is separable in the Lévy metric.

Definition 9.3. We say that $(X_n, Y_n) \xrightarrow{D} (X, Y)$ if for every bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ we have $E(f(X_n, Y_n)) \to E(f(X, Y))$.

Exercise 9.11. Suppose $(X_n, Y_n)$ are independent and $X_n \xrightarrow{D} X, Y_n \xrightarrow{D} Y$. Prove that $(X_n, Y_n) \xrightarrow{D} \mu$ where $\mu = F_X \otimes F_Y$ is the product measure.

Exercise 9.12. Suppose $(X_n, Y_n) \xrightarrow{D} (X, Y)$. Prove that $X_n^2 + Y_n^2$ converges in distribution.
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