Chapter 11

The Central Limit Theorem

1. Sums of independent identically distributed random variables

Denote by $Z$ the "standard normal random variable" with density $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

**Lemma 11.1.** $Ee^{itZ} = e^{-t^2/2}$

**Proof.** We use the same calculation as for the moment generating function:

$$\int_{-\infty}^{\infty} \exp(itx - \frac{1}{2}x^2)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}(x-it)^2)dx = \sqrt{2\pi}$$

Note that $e^{-x^2/2}$ is an analytic function so $\int e^{-x^2/2}dz = 0$ over any closed path. So

$$\int_{-A}^{A} \exp-(x-it)^2/2dx - \int_{-A}^{A} e^{-x^2/2}dx + \int_{0}^{it} \exp(-(A-is)^2/2)ds - \int_{0}^{it} \exp(-(A-is)^2/2)ds = 0$$

**Theorem 11.2** (CLT for i.i.d.). Suppose $\{X_n\}$ is i.i.d. with mean $m$ and variance $0 < \sigma^2 < \infty$. Let $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n - nm}{\sigma\sqrt{n}} \xrightarrow{d} Z$$

This is one of the special cases of the Lindeberg theorem and the proof uses characteristic functions. Note that $\varphi_{S_n/\sqrt{n}}(t) = e^{-t^2/2}$ when $X_j$ are independent $N(0, 1)$.

In general, $\varphi_{S_n/\sqrt{n}}(t)$ is a complex number. For example, when $X_n$ are exponential with parameter $\lambda = 1$, the conclusion says that

$$\varphi_{S_n/\sqrt{n}}(t) = \frac{e^{-it\sqrt{n}}}{(1 - i\frac{t}{\sqrt{n}})^n} \rightarrow e^{-t^2/2}$$
which is not so obvious to see. On the other hand, characteristic function in Exercise 10.5 on page 120 is real and the limit can be found using calculus:

\[ \varphi_{s_n/\sqrt{n}}(t) = \cos^n(t/\sqrt{n}) \rightarrow e^{-t^2/2}. \]

Here is a simple inequality that will suffice for the proof in the general case.

**Lemma 11.3.** If \( z_1, \ldots, z_m \) and \( w_1, \ldots, w_m \) are complex numbers of modulus at most 1 then

\[
|z_1 \ldots z_m - w_1 \ldots w_m| \leq \sum_{k=1}^{m} |z_k - w_k|
\]

**Proof.** Write the left hand side of (11.1) as a telescoping sum:

\[
z_1 \ldots z_m - w_1 \ldots w_m = \sum_{k=1}^{m} z_1 \ldots z_{k-1} (z_k - w_k) w_{k+1} \ldots w_m
\]

(11.1)

**Example 11.1.** We show how to complete the proof for the exponential distribution.

\[
\left| \frac{e^{-it\sqrt{n}}}{1 - i t/\sqrt{n}} - e^{-t^2/2} \right| = \left| \left( \frac{e^{-it\sqrt{n}}}{1 - i t/\sqrt{n}} \right)^n - \left( e^{-t^2/(2n)} \right)^n \right| \leq n \left| \frac{e^{-it\sqrt{n}}}{1 - i t/\sqrt{n}} - e^{-t^2/(2n)} \right|
\]

\[
= n \left| 1 - itn + t^2/(2n) + it^3/(6n\sqrt{n}) - \ldots - 1 + t^2/(2n) - t^4/(6n^2) + \ldots \right|
\]

\[
= n \left| \left( 1 - \frac{it}{\sqrt{n}} - \frac{t^2}{2n} - \frac{it^3}{6n\sqrt{n}} + \ldots \right) \left( 1 + \frac{t}{\sqrt{n}} - \frac{t^2}{n} + \ldots \right) - 1 + \frac{t^2}{2n} - \frac{t^4}{6n^2} + \ldots \right|
\]

\[
= n \left| 1 - \frac{it}{\sqrt{n}} - \frac{t^2}{2n} + \frac{it^3}{6n\sqrt{n}} - \ldots - 1 + \frac{t^2}{2n} - \frac{t^4}{6n^2} + \ldots \right| \leq n C(t) \frac{n}{\sqrt{n}} \rightarrow 0.
\]

**Proof of Theorem 11.2.** Without loss of generality we may assume \( m = 0 \) and \( \sigma = 1 \). We have \( \varphi_{s_n/\sqrt{n}}(t) = \varphi_X(t/\sqrt{n})^n \). For a fixed \( t \in \mathbb{R} \) choose \( n \) large enough so that \( 1 - \frac{t^2}{2n} > -1 \). For such \( n \), we can apply (11.1) with \( z_k = \varphi_X(t/\sqrt{n}) \) and \( w_k = 1 - \frac{t^2}{2n} \). We get

\[
\left| \varphi_{s_n/\sqrt{n}}(t) - \left( 1 - \frac{t^2}{2n} \right)^n \right| \leq n \left| \varphi_X(t/\sqrt{n}) - 1 - \frac{t^2}{2n} \right| \leq t^2 E \min \left\{ \frac{|t||X|^3}{\sqrt{n}} , X^2 \right\}
\]

Noting that \( \lim_{n \to \infty} \min \{ |t||X|^3/\sqrt{n}, X^2 \} = 0 \), by dominated convergence theorem (the integrand is dominated by the integrable function \( X^2 \)) we have \( E \min \left\{ \frac{|t||X|^3}{\sqrt{n}} , X^2 \right\} \to 0 \) as \( n \to \infty \). So

\[
\lim_{n \to \infty} \left| \varphi_{s_n/\sqrt{n}}(t) - \left( 1 - \frac{t^2}{2n} \right)^n \right| = 0.
\]

It remains to notice that \( (1 - \frac{t^2}{2n})^n \to e^{-t^2/2} \). □

**Remark 11.4.** If \( X_n \overset{D}{\to} Z \) then the cumulative distribution functions converge uniformly: \( \sup_n |P(X_n \leq x) - P(Z \leq x)| \to 0 \).
Example 11.2 (Normal approximation to Binomial). If $X_n$ is $Bin(n,p)$ and $p$ is fixed then $P\left(\frac{1}{\sqrt{n}}X_n < p + x/\sqrt{n}\right) \to P(Z \leq x\sqrt{p(1-p)})$ as $n \to \infty$.

Example 11.3 (Normal approximation to Poisson). If $X_\lambda$ is Poiss and $p$ is fixed then $(X_\lambda - \lambda)/\sqrt{\lambda} \overset{D}{\to} Z$ as $\lambda \to \infty$. (Strictly speaking, the CLT gives only convergence of $(X_{n\lambda} - \lambda n)/\sqrt{n\lambda} \overset{D}{\to} Z$ as $n \to \infty$.)

2. General form of a limit theorem

The general problem of convergence in distribution can be stated as follows: Given a sequence $Z_n$ of random variables, find normalizing constants $a_n, b_n$ and a limiting distribution/random variable $Z$ such that $(Z_n - b_n)/a_n \to Z$.

In Example 9.1, $Z_n$ is a maximum, $a_n = 1, b_n = \log n$.

In Theorem 11.2, $Z_n$ is the sum, the normalizing constants are $b_n = E(S_n)$ and $a_n = \sqrt{Var(S_n)}$, and we will make the same choice for sums of independent random variables in the next section. However, finding an appropriate normalization for CLT may be not obvious or easy, see Section 5.

One may wonder how much flexibility do we have in the choice of the normalizing constants $a_n, b_n$.

Theorem 11.5 (Convergence of types). Suppose $X_n \overset{D}{\to} X$ and $a_nX_n + b_n \overset{D}{\to} Y$ for some $a_n > 0, b_n \in \mathbb{R}$, and both $X,Y$ are non-degenerate. Then $a_n \to a > 0$ and $b_n \to b$ and in particular $Y$ has the same law as $aX + b$.

So if $(Z_n - b_n)/a_n \to Z$ and $(Z_n - b'_n)/a'_n \to Z'$ then $(Z_n - b'_n)/a'_n = \frac{a_n}{a_n'} ((Z_n - b_n)/a_n) + (b_n - b'_n)/a'_n$, which means that $a_n/a'_n \to a > 0$ and $(b_n - b'_n)/a'_n \to b$. So $a'_n = a_n/a, b'_n = b_n - \frac{b}{a}a_n$ and $Z' = aZ + b$.

(Omitted in 2019)

Proof. To be written...

It is clear that independence alone is not sufficient for the CLT.

3. Lindeberg’s theorem

The setting is of sums of triangular arrays: For each $n$ we have a family of independent random variables

$$X_{n,1}, \ldots, X_{n,r_n}$$

and we set $S_n = X_{n,1} + \cdots + X_{n,r_n}$.

For Theorem 11.2, the triangular array can be $X_{n,k} = \frac{X_k - m}{\sigma \sqrt{n}}$. Or one can take $X_{n,k} = \frac{X_k - m}{\sigma}$...

Through this section we assume that random variables are square-integrable with mean zero, and we use the notation

$$(11.2) \quad E(X_{n,k}) = 0, \quad \sigma^2_{nk} = E(X_{n,k}^2), \quad s_n^2 = \sum_{k=1}^{r_n} \sigma^2_{nk}$$
Definion 11.1 (The Lindeberg condition). We say that the Lindeberg condition holds if

\[
\lim_{n \to \infty} \frac{1}{s_n^2} \sum_{k=1}^{r_n} \int_{|X_{nk}| > \varepsilon s_n} X_{nk}^2 dP = 0 \text{ for all } \varepsilon > 0
\]

(Note that strict inequality \( \int_{|X_{nk}| > \varepsilon s_n} X_{nk}^2 dP \) can be replaced by \( \int_{|X_{nk}| \geq \varepsilon s_n} X_{nk}^2 dP \) and the resulting condition is the same.)

Remark 11.6. Under the Lindeberg condition, we have

\[
\lim_{n \to \infty} \max_{k \leq r_n} \frac{\sigma_{nk}^2}{s_n^2} = 0
\]

Indeed,

\[
\sigma_{nk}^2 = \int_{|X_{nk}| \leq \varepsilon s_n} X_{nk}^2 dP + \int_{|X_{nk}| > \varepsilon s_n} X_{nk}^2 dP \leq \varepsilon s_n^2 + \int_{|X_{nk}| > \varepsilon s_n} X_{nk}^2 dP
\]

So

\[
\max_{k \leq r_n} \frac{\sigma_{nk}^2}{s_n^2} \leq \varepsilon + \frac{1}{s_n^2} \max_{k \leq r_n} \int_{|X_{nk}| > \varepsilon s_n} X_{nk}^2 dP \leq \varepsilon + \frac{1}{s_n^2} \sum_{k=1}^{r_n} \int_{|X_{nk}| > \varepsilon s_n} X_{nk}^2 dP
\]

Theorem 11.7 (Lindeberg CLT). Suppose that for each \( n \) the sequence \( X_{n1} \ldots X_{nr_n} \) is independent with mean zero. If the Lindeberg condition holds for all \( \varepsilon > 0 \) then \( S_n/s_n \xrightarrow{D} Z \).

Example 11.4 (Proof of Theorem 11.2). In the setting of Theorem 11.2, we have \( X_{n,k} = \frac{X_k - m}{\sigma} \) and \( s_n = \sqrt{n} \). The Lindeberg condition is

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \int_{|X_k - m| > \varepsilon \sqrt{n}} \frac{(X_k - m)^2}{\sigma^2} dP = \lim_{n \to \infty} \frac{1}{\sigma^2} \int_{|X_1 - m| > \varepsilon \sqrt{n}} (X_1 - m)^2 = 0
\]

by Lebesgue dominated convergence theorem, say. (Or by Corollary 6.12 on page 73.)

Proof. Without loss of generality we may assume that \( s_n^2 = 1 \) so that \( \sum_{k=1}^{r_n} \sigma_{nk}^2 = 1 \). Denote \( \varphi_{nk} = E(e^{it X_{nk}}) \). From (10.13) we have

\[
|\varphi_{nk}(t) - (1 - \frac{1}{2} t^2 \sigma_{nk}^2)| \leq E\left( \min\{|t X_{nk}|^2, |t X_{nk}|^3\} \right)
\]

\[
\leq \int_{|X_{nk}| < \varepsilon} |t X_{nk}|^3 dP + \int_{|X_{nk}| \geq \varepsilon} |t X_{nk}|^2 dP \leq t^3 \varepsilon \sigma_{nk}^2 + t^2 \int_{|X_{nk}| \geq \varepsilon} X_{nk}^2 dP
\]

Using (11.1), we see that

\[
|\varphi_{S_n}(t) - \prod_{k=1}^{n} (1 - \frac{1}{2} t^2 \sigma_{nk}^2)| \leq \epsilon t^3 \sum_{k=1}^{n} \sigma_{nk}^2 + t^2 \sum_{k=1}^{n} \int_{|X_{nk}| > \varepsilon} X_{nk}^2 dP
\]

This shows that

\[
|\varphi_{S_n}(t) - \prod_{k=1}^{n} (1 - \frac{1}{2} t^2 \sigma_{nk}^2)| = 0
\]

It remains to verify that \( \lim_{n \to \infty} e^{-t^2/2} - \prod_{k=1}^{n} (1 - \frac{1}{2} t^2 \sigma_{nk}^2) = 0. \)
To do so, we apply the previous proof to the triangular array \( \sigma_{n,k} Z_k \) of independent normal random variables. Note that

\[
\phi \sum Z_{nk}(t) = \prod_{k=1}^{r_n} e^{-t^2 \sigma_{nk}^2 / 2} = e^{-t^2 / 2}
\]

We only need to verify the Lindeberg condition for \( \{Z_{nk}\} \):

\[
\int_{Z_{nk} > \varepsilon} Z_{nk}^2 dP = \sigma_{nk}^2 \int_{|x| > \varepsilon / \sigma_{nk}} x^2 f(x) dx
\]

So

\[
\sum_{k=1}^{r_n} \int_{Z_{nk} > \varepsilon} Z_{nk}^2 dP \leq \sum_{k=1}^{r_n} \sigma_{nk}^2 \int_{|x| > \varepsilon / \sigma_{nk}} x^2 f(x) dx \leq \max_{1 \leq k \leq r_n} \int_{|x| > \varepsilon / \sigma_{nk}} x^2 f(x) dx
\]

The right hand side goes to zero as \( n \to \infty \), because by \( \max_{1 \leq k \leq r_n} \sigma_{nk} \to 0 \) by (11.4).

4. Lyapunov’s theorem

Theorem 11.8. Suppose that for each \( n \) the sequence \( X_{n1} \ldots X_{nr_n} \) is independent with mean zero. If the Lyapunov’s condition

\[
\lim_{n \to \infty} \frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} E|X_{nk}|^{2+\delta} = 0
\]

holds for some \( \delta > 0 \), then \( S_n / s_n \overset{D}{\to} Z \)

Proof. We use the following bound to verify Lindeberg’s condition:

\[
\frac{1}{s_n^{2}} \sum_{k=1}^{r_n} \int_{|X_{nk}| > \varepsilon s_n} X_{nk}^2 dP \leq \frac{1}{\varepsilon^2 s_n^{2+\delta}} \sum_{k=1}^{r_n} \int_{|X_{nk}| > \varepsilon s_n} |X_{nk}|^{2+\delta} dP \leq \frac{1}{\varepsilon^2 s_n^{2+\delta}} \sum_{k=1}^{n} E|X_{nk}|^{2+\delta}
\]

Corollary 11.9. Suppose \( X_k \) are independent with mean zero, variance \( \sigma^2 \) and that \( \sup_k E|X_k|^{2+\delta} < \infty \). Then \( S_n / \sqrt{n} \overset{D}{\to} \sigma Z \)

Proof. Let \( C = \sup_k E|X_k|^{2+\delta} \). Then \( s_n = \sqrt{n} \) and \( \frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} E(|X_k|^{2+\delta}) \leq C / n^{\delta / 2} \to 0 \), so Lyapunov’s condition is satisfied.

Corollary 11.10. Suppose \( X_k \) are independent, uniformly bounded, and have mean zero. If \( \sum_n \text{Var}(X_n) = \infty \), then \( S_n / \sqrt{\text{Var}(S_n)} \overset{D}{\to} N(0,1) \).

Proof. Suppose \( |X_n| \leq C \) for a constant \( C \). Then

\[
\frac{1}{s_n^{3}} \sum_{k=1}^{n} E|X_n|^3 \leq C \frac{s_n^2}{s_n^3} = \frac{C}{s_n} \to 0
\]
5. Normal approximation without Lindeberg condition

One basic idea is truncation: \( X_n = X_n I_{|X_n| \leq a_n} + X_n I_{|X_n| > a_n} \). One wants to show that \( \frac{1}{s_n} \sum X_k I_{|X_k| \leq a_n} \to Z \) and that \( \frac{1}{s_n} \sum X_k I_{|X_k| > a_n} \to 0 \). Then \( S_n/s_n \) is asymptotically normal by Slutski’s theorem.

Example 11.5. Let \( X_1, X_2, \ldots \) be independent random variables with the distribution \((k \geq 1)\)

\[
\begin{align*}
    \Pr(X_k = \pm 1) &= \frac{1}{4}, \\
    \Pr(X_k = k^k) &= \frac{1}{4^k}, \\
    \Pr(X_k = 0) &= \frac{1}{2} - \frac{1}{4^k}.
\end{align*}
\]

Then \( \sigma^2_k = \frac{1}{2} + \left(\frac{k}{4}\right)^k \) and \( s_n \geq n^{n/4^n} \). But \( S_n/s_n \to 0 \) and in fact we have \( S_n/s_n \to Z/\sqrt{2} \). To see this, note that \( Y_k = X_k I_{|X_k| \leq 1} \) are independent with mean 0, variance \( \frac{1}{2} \) and \( P(Y_k \neq X_k) = 1/4^k \) so by the first Borel Cantelli Lemma (Theorem 3.8) \( |\frac{1}{\sqrt{n}} \sum_{k=1}^n (Y_k - X_k)| \leq \frac{U_n}{\sqrt{n}} \to 0 \) with probability one.

It is sometimes convenient to use Corollary 9.5 (Exercise 9.2) combined with the law of large numbers. This is how one needs to proceed in Exercise 11.2.

Example 11.6. Suppose \( X_1, X_2, \ldots, \) are i.i.d. with mean 0 and variance \( \sigma^2 > 0 \). Then

\[
\frac{\sum_{k=1}^n X_k}{\sqrt{\sum_{k=1}^n X_k^2}}
\]

converges in distribution to \( N(0, 1) \). To see this, write

\[
\frac{\sum_{k=1}^n X_k}{\sqrt{\sum_{k=1}^n X_k^2}} = \frac{\sigma}{\sqrt{\frac{1}{n} \sum_{k=1}^n X_k^2}} \times \frac{\sum_{k=1}^n X_k}{\sigma \sqrt{n}}
\]

and note that the first factor converges to 1 with probability one.

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Required Exercises

Exercise 11.1. Suppose \( a_{nk} \) is an array of numbers such that \( \sum_{k=1}^n a_{nk}^2 = 1 \) and \( \max_{1 \leq k \leq n} |a_{nk}| \to 0 \). Let \( X_j \) be i.i.d. with mean zero and variance 1. Show that \( \sum_{k=1}^n a_{nk} X_k \to N(0, 1) \).

Exercise 11.2. Suppose that \( X_1, X_2, \ldots \) are i.i.d., \( E(X_1) = 1, E(X_1^2) = \sigma^2 < \infty \). Let \( \bar{X}_n = \frac{1}{n} \sum_{j=1}^n X_j \). Show that for all \( k > 0 \)

\[
\sqrt{n} \left( \bar{X}_n^k - 1 \right) \to N(0, k \sigma)
\]

as \( n \to \infty \).

Exercise 11.3. Suppose \( X_1, X_2, \ldots \) are independent, \( X_k = \pm 1 \) with probability \( \frac{1}{2}(1 - k^{-2}) \) and \( X_k = \pm k \) with probability \( \frac{1}{2} k^{-2} \). Let \( S_n = \sum_{k=1}^n X_k \)

(i) Show that \( S_n/\sqrt{n} \to N(0, 1) \)
(ii) Is the Lindeberg condition satisfied?

**Exercise 11.4.** Suppose \(X_1, X_2, \ldots\) are independent random variables with distribution \(\Pr(X_k = 1) = p_k\) and \(\Pr(X_k = 0) = 1 - p_k\). Prove that if \(\sum Var(X_k) = \infty\) then
\[
\frac{\sum_{k=1}^{n}(X_k - p_k)}{\sqrt{\sum_{k=1}^{n}p_k(1 - p_k)}} \xrightarrow{D} N(0, 1).
\]

**Exercise 11.5.** Suppose \(X_k\) are independent and have density \(\frac{1}{|x|^2}\) for \(|x| > 1\). Show that \(S_n \sqrt{n \log n} \xrightarrow{D} N(0, 1)\).

*Hint:* Verify that Lyapunov’s condition (11.7) holds with \(\delta = 1\) for truncated random variables.

Several different truncations can be used, but technical details differ:

- \(Y_k = X_k I_{|X_k| \leq \sqrt{k}}\) is a solution in [Billingsley]. To show that \(\frac{1}{\sqrt{n \log n}} \sum_{k=1}^{n}(X_k - Y_k) \xrightarrow{P} 0\) use \(L_1\)-convergence.
- Triangular array \(Y_{nk} = X_k I_{|X_k| \leq \sqrt{n}}\) is simpler computationally
- Truncation \(Y_k = X_k I_{|X_k| \leq \sqrt{k \log k}}\) leads to “asymptotically equivalent” sequences.

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**Some previous prelim problems**

**Exercise 11.6** (May 2018). (To be added latter)

**Exercise 11.7** (Aug 2017). Let \(\{X_n\}_{n \in \mathbb{N}}\) be a collection of independent random variables with
\[
\Pr(X_n = \pm n^2) = \frac{1}{2n^\beta} \quad \text{and} \quad \Pr(X_n = 0) = 1 - \frac{1}{n^\beta}, \quad n \in \mathbb{N},
\]
where \(\beta \in (0, 1)\) is fixed for all \(n \in \mathbb{N}\). Consider \(S_n := X_1 + \cdots + X_n\). Show that
\[
\frac{S_n}{n^\gamma} \xrightarrow{D} \mathcal{N}(0, \sigma^2)
\]
for some \(\sigma > 0, \gamma > 0\). Identify \(\sigma\) and \(\gamma\) as functions of \(\beta\). You may use the formula
\[
\sum_{k=1}^{n} k^\theta \sim \frac{n^{\theta+1}}{\theta+1}
\]
for \(\theta > 0\), and recall that by \(a_n \sim b_n\) we mean \(\lim_{n \to \infty} a_n/b_n = 1\).

**Exercise 11.8** (May 2017). Let \(\{X_n\}_{n \in \mathbb{N}}\) be independent random variables with \(\Pr(X_n = 1) = 1/n = 1 - \Pr(X_n = 0)\). Let \(S_n := X_1 + \cdots + X_n\) be the partial sum.

(i) Show that
\[
\lim_{n \to \infty} \frac{\mathbb{E}S_n}{\log n} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{Var(S_n)}{\log n} = 1.
\]

(ii) Prove that
\[
\frac{S_n - \log n}{\sqrt{\log n}} \xrightarrow{D} \mathcal{N}(0, 1)
\]
as \( n \to \infty \). Explain which central limit theorem you use. State and verify all the conditions clearly.

Hint: recall the relation \( \lim_{n \to \infty} \sum_{k=1}^{n} 1/k \log n = 1 \).

**Exercise 11.9** (May 2016). (a) State Lindeberg–Feller central limit theorem. (b) Use Lindeberg–Feller central limit theorem to prove the following. Consider a triangular array of random variables \( \{ Y_{n,k} \}_{n \in \mathbb{N}, k=1,\ldots,n} \) such that for each \( n \), \( \mathbb{E} Y_{n,k} = 0, k = 1, \ldots, n \), and \( \{ Y_{n,k} \}_{k=1,\ldots,n} \) are independent. In addition, with \( \sigma_n := \left( \sum_{k=1}^{n} \mathbb{E} Y_{n,k}^2 \right)^{1/2} \), assume that \( \lim_{n \to \infty} \frac{1}{\sigma_n^4} \sum_{k=1}^{n} \mathbb{E} Y_{n,k}^4 = 0 \).

Show that \( \frac{Y_{n,1} + \cdots + Y_{n,n}}{\sigma_n} \xrightarrow{D} \mathcal{N}(0,1) \).

**Exercise 11.10** (Aug 2015). Let \( \{ U_n \}_{n \in \mathbb{N}} \) be a collection of i.i.d. random variables with \( \mathbb{E} U_n = 0 \) and \( \mathbb{E} U_n^2 = \sigma^2 \in (0, \infty) \). Consider random variables \( \{ X_n \}_{n \in \mathbb{N}} \) defined by \( X_n = U_n + U_{2n}, n \in \mathbb{N} \), and the partial sum \( S_n = X_1 + \cdots + X_n \). Find appropriate constants \( \{ a_n, b_n \}_{n \in \mathbb{N}} \) such that \( \frac{S_n - b_n}{a_n} \xrightarrow{D} \mathcal{N}(0,1) \).

**Exercise 11.11** (May 2015). Let \( \{ U_n \}_{n \in \mathbb{N}} \) be a collection of i.i.d. random variables distributed uniformly on interval \((0,1)\). Consider a triangular array of random variables \( \{ X_{n,k} \}_{k=1,\ldots,n,n \in \mathbb{N}} \) defined as

\[
X_{n,k} = 1_{\{\sqrt{n}U_k \leq 1\}} - \frac{1}{\sqrt{n}}.
\]

Find constants \( \{ a_n, b_n \}_{n \in \mathbb{N}} \) such that

\[
\frac{X_{n,1} + \cdots + X_{n,n} - b_n}{a_n} \xrightarrow{D} \mathcal{N}(0,1).
\]

**Exercise 11.12** (Aug 2014). Let \( X_1, X_2, \ldots \) be independent and identically distributed random variables with

\[
P(X_i = 1) = P(X_i = -1) = 1/2.
\]

Prove that

\[
\frac{\sqrt{3}}{\sqrt{n^3}} \sum_{k=1}^{n} kX_k \xrightarrow{D} \mathcal{N}(0,1)
\]

(You may use formulas \( \sum_{j=1}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1) \) and \( \sum_{j=1}^{n} j^3 = \frac{1}{4}n^2(n+1)^2 \) without proof.)

**Exercise 11.13** (May 2014). Let \( \{ X_{nk} : k = 1, \ldots, n, n \in \mathbb{N} \} \) be a family of independent random variables satisfying

\[
P \left( X_{nk} = \frac{k}{\sqrt{n}} \right) = P \left( X_{nk} = -\frac{k}{\sqrt{n}} \right) = P(X_{nk} = 0) = 1/3
\]

Let \( S_n = X_{n1} + \cdots + X_{nn} \). Prove that \( S_n/s_n \) converges in distribution to a standard normal random variable for a suitable sequence of real numbers \( s_n \).
Some useful identities:
\[
\sum_{k=1}^{n} k = \frac{1}{2} n(n + 1)
\]
\[
\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n + 1)(2n + 1)
\]
\[
\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2(n + 1)^2
\]

Exercise 11.14 (Aug 2013). Suppose \(X_1, Y_1, X_2, Y_2, \ldots,\) are independent identically distributed with mean zero and variance 1. For integer \(n,\) let
\[
U_n = \frac{1}{n} \left( \sum_{j=1}^{n} X_j \right)^2 + \frac{1}{n} \left( \sum_{j=1}^{n} Y_j \right)^2.
\]
Prove that \(\lim_{n \to \infty} P(U_n \leq u) = 1 - e^{-u/2}\) for \(u > 0.\)

Exercise 11.15 (May 2013). Suppose \(X_{n,1}, X_{n,2}, \ldots,\) are independent random variables centered at expectations (mean 0) and set \(s_n^2 = \sum_{k=1}^{n} E\left((X_{n,k})^2\right).\) Assume for all \(k\) that \(|X_{n,k}| \leq M_n\) with probability 1 and that \(M_n/s_n \to 0.\) Let \(Y_{n,i} = 3X_{n,i} + X_{n,i+1}.\) Show that
\[
\frac{Y_{n,1} + Y_{n,2} + \ldots + Y_{n,n}}{s_n}
\]
converges in distribution and find the limiting distribution.
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