Prep Questions II.

(1) Suppose that $p, q, r \geq 1$ are such that $1/p + 1/q + 1/r = 1$ and $X, Y, Z$ are random variables such that $E|X|^p < \infty$, $E|Y|^q < \infty$, $E|Z|^r < \infty$. Use Hölder’s inequality to show that

$$|E(XYZ)| \leq \|X\|_p\|Y\|_q\|Z\|_r$$

(2) Show that for $X \geq 0$,

$$E e^X = 1 + \int_0^\infty e^t P(X > t) dt$$

(3) If $X_n \xrightarrow{P} 1$ and $\sup_n \ E(X_n^2) = M < \infty$ then $E(X_n) \rightarrow 1$ (This is essentially Corollary 6.14, simplified to shorten its proof).

(4) Suppose that random variables $X_1, X_2, \ldots$ have mean zero and variance 1. Prove that $\frac{1}{n} X_n \rightarrow 0$ with probability one.

(5) Suppose $X_1, X_2, \ldots$ are independent identically distributed and integrable. Prove that $\frac{1}{n} X_n \rightarrow 0$ with probability one. Is independence used in your proof?

(6) Suppose $X_1, X_2, \ldots$ are independent identically distributed random variables and $p > 0$ is a fixed real number. Prove that the following are equivalent:

(a) $\frac{1}{\sqrt{n}} X_n \rightarrow 0$ with probability one

(b) $E(|X_n|^p) < \infty$.

Be sure to indicate where in the proof you use the assumption of independence, and where in the proof you use the assumption of the same distribution.

Hint: Use the Borel Cantelli Lemmas and tail integration.

(7) Suppose $Z_1, Z_2, \ldots, Z_n, \ldots$ are random variables such that $E(Z_n) = n$ and $Var(Z_n) = n$.

(a) Prove that $\frac{Z_n}{n} \xrightarrow{P} 1$ in probability.

(b) Suppose that in addition that $\{Z_n\}$ is an increasing sequence, $Z_n \leq Z_{n+1}$ for $n \in \mathbb{N}$. Prove that $\frac{Z_n}{n} \rightarrow 1$ with probability one. Hint: One of the proofs of SLLN works!

(8) Suppose $X_1, X_2, \ldots$ are independent identically distributed with mean zero, variance 1, the cubic moment $E(X_j^3) = c$ and the 4-th moment $E(X_j^4) = q$. Derive formulas for $E(\sum_{j=1}^n X_j)^3$ and $E(\sum_{j=1}^n X_j)^4$ as a function of $n, c, q$.

(9) Suppose $X_1, X_2, \ldots, X_n, \ldots$ are independent identically distributed exponential random variables on $[0, \infty)$, i.e. with the same CDF $F(x) = 1 - e^{-x}$ for $x \geq 0$. Show that the sequence $M_n = \min_{1 \leq k \leq n} X_k$ converges to 0 in probability.

(10) If $X_n \leq Y_n \leq Z_n$ for all $n$, and $X_n \rightarrow X$, $Z_n \rightarrow X$ in probability, show that $Y_n \rightarrow X$ in probability.

(11) Suppose $X_1, X_2, \ldots$ are independent square-integrable random variables with $E(X_j) = 0$ and $E(X_j^2) = 1$. Let $S_n = X_1 + \cdots + X_n$. Prove that $\frac{1}{n} E(|S_n|) \rightarrow 0$. Hint: square-integrable random variables have second moments!

(12) Use Markov’s inequality to deduce that if $E[\exp X] < \infty$ then there exists $C > 0$ such that $P(X > t) \leq Ce^{-t}$ for all $t > 0$.

(13) Define $d(X, Y) = E(\frac{|X - Y|}{1 + |X - Y|})$. Prove that $X_n \xrightarrow{P} X$ iff $d(X_n, X) \rightarrow 0$. (Note that this requires two proofs!)
(14) Suppose that random variables $X_1, X_2, \ldots$ are i.i.d. with $E(X_j) = 0$ and $E(X_j^4) < \infty$. Let $S_n = X_1 + \cdots + X_n$. If $\theta > 3/4$, prove that $\frac{1}{n^\theta} S_n \to 0$ with probability one.

(15) Suppose $X_1, X_2, \ldots$, are i.i.d. with mean $m$ and variance $\sigma^2 > 0$. Let $\bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$ be the sample mean and $S_n^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2$ be the sample variance. Show that with probability one $S_n \to \sigma$.

(16) Suppose $X_1, X_2, \ldots$, are i.i.d. with mean $m$ and variance $\sigma^2 > 0$. Show that with probability one

$$L = \lim_{n \to \infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^2}$$

exists, and determine $L$ as a function of $m, \sigma$.

(17) Suppose that random variables $X_1, X_2, \ldots$ are independent identically distributed, with mean $m$, variance $\sigma^2$, and with finite 4-th moments. Let $S_n = \sum_{k=1}^n X_k X_{k+1}$. Show that $\frac{1}{n} S_n$ converges almost surely (and find the limit).

(18) Suppose $X_1, X_2, \ldots$ are independent identically distributed uniform $U(1, e)$. Let $Z_n = X_1 \ldots X_n$. Show that $\sqrt{Z_n}$ converges almost surely (and find the limit).

(19) State and prove Kolmogorov’s maximal inequality

$$P(\max_{k \leq n} |S_k| > t) \leq \frac{Var(S_n)}{t^2}$$

(Note that $Y_1, Y_2, \ldots$ are dependent!)

(20) Suppose $X_1, X_2, \ldots$ are independent, square-integrable, with mean zero. Define $Y_n = X_1 X_2 \ldots X_n$ and $S_n = Y_1 + \cdots + Y_n$. Adapt the proof of Kolmogorov’s maximal inequality to prove that

$$P(\max_{k \leq n} |S_k| > t) \leq \frac{Var(S_n)}{t^2}$$

(21) Suppose that random variables $\{X_k\}$ are independent uniform $U(-k, k)$ for $k \in \mathbb{N}$. Show that the series $\sum_n \frac{1}{n^2 + 2n + 3\theta} X_n$ converges with probability one for $\theta > \theta_0$, where $\theta_0$ is to be determined by you.

(22) Suppose that random variables $\{X_k\}$ are independent exponential with parameters $\lambda_k = k$ i.e. $P(X_k \leq x) = 1 - e^{-\lambda_k x}$ for $x > 0$ and $k \in \mathbb{N}$. Prove that the series

$$\sum_n (n^2 + n + 1)\theta (X_n - \frac{1}{n})$$

converges with probability one if $\theta < \theta_0$, where $\theta_0$ is to be determined by you.

(23) Let $X_1, X_2, \ldots$ be independent random variables with the distribution $(k \geq 1)$

$$\Pr(X_k = \pm 1) = 1/4,$$
$$\Pr(X_k = k^k) = 1/4^k,$$
$$\Pr(X_k = 0) = 1/2 - 1/4^k.$$ Use Kolmogorov’s three-series theorem to prove that the series $\sum \frac{1}{k} X_k$ converges with probability one.

(24) Prove that if $X_m \overset{P}{\to} X$ then $X_n \overset{D}{\to} X$.

(25) If $X_n \overset{D}{\to} 1$ then $X_n \overset{P}{\to} 1$. 


(26) Suppose \( \{X_k\} \) are independent uniform \( U(0,1) \) random variables. Show that

\[
\min_{1 \leq k \leq n} X_k \xrightarrow{\mathcal{D}} Y
\]

and determine the law of \( Y \).

(27) Suppose \( X_n \xrightarrow{P} c \) for a constant \( c \) and \( Y_n \xrightarrow{\mathcal{D}} Y \). Use the definitions to show that \( X_n Y_n \xrightarrow{\mathcal{D}} cY \).

(28) Suppose \( X_n \) has density \( f_n(x) = 1 + \cos(2\pi nx) \) on \( [0,1] \). Prove that \( X_n \xrightarrow{\mathcal{D}} X \) (and determine the law of \( X \)).

(29) Suppose random variables \( Z_n \) with laws \( \mu_n \) are such that \( E(Z_n^2) = 1 \). Show that \( \{\mu_n\} \) is tight.