Prep Questions II.

1. Suppose that \( p, q, r \geq 1 \) are such that \( 1/p + 1/q + 1/r = 1 \) and \( X, Y, Z \) are random variables such that \( E|X|^p < \infty, E|Y|^q < \infty, E|Z|^r < \infty \). Use Hölder’s inequality to show that
\[
|E(XYZ)| \leq \|X\|_p \|Y\|_q \|Z\|_r
\]

2. Show that for \( X \geq 0 \),
\[
Ee^X = 1 + \int_0^\infty e^t P(X > t)dt
\]

3. If \( X_n \xrightarrow{p} 1 \) and \( \sup_n E(X_n^2) = M < \infty \) then \( E(X_n) \to 0 \). (This is essentially Corollary 6.14, simplified to shorten its proof).

4. Suppose that random variables \( X_1, X_2, \ldots \) have mean zero and variance 1. Prove that \( \frac{1}{n} X_n \to 0 \) with probability one.

5. Suppose \( X_1, X_2, \ldots \) are independent identically distributed random variables and \( p > 0 \) is a fixed real number. Prove that the following are equivalent:
   (a) \( \frac{1}{\sqrt{n}} X_n \to 0 \) with probability one
   (b) \( E(|X|_p) < \infty \).

   Be sure to indicate where in the proof you use the assumption of independence, and where in the proof you use the assumption of the same distribution.

   \textit{Hint:} Use the Borel Cantelli Lemmas and tail integration.

6. Suppose \( Z_1, Z_2, \ldots, Z_n, \ldots \) are random variables such that \( E(Z_n) = n \) and \( Var(Z_n) = n \).
   (a) Prove that \( \frac{Z_n}{n} \xrightarrow{P} 1 \) in probability.
   (b) Suppose that in addition that \( \{Z_n\} \) is an increasing sequence, \( Z_n \leq Z_{n+1} \) for \( n \in \mathbb{N} \). Prove that \( \frac{Z_n}{n} \to 1 \) with probability one. \( \textit{Hint:} \) One of the proofs of SLLN works!

7. Suppose \( X_1, X_2, \ldots \) are independent identically distributed with mean zero, variance 1, the cubic moment \( E(X_i^3) = c \) and the 4-th moment \( E(X_i^4) = q \). Derive formulas for \( E(\sum_{j=1}^n X_j)^3 \) and \( E(\sum_{j=1}^n X_j)^4 \) as a function of \( n, c, q \).

8. Suppose that random variables \( X_1, X_2, \ldots \) are \( i.i.d. \) with \( E(X_j) = 0 \) and \( E(X_j^4) < \infty \). Let \( S_n = X_1 + \cdots + X_n \). If \( \theta > 3/4 \), prove that \( \frac{1}{n} S_n \to 0 \) with probability one.

9. Suppose \( X_1, X_2, \ldots \) are \( i.i.d. \) with mean \( m \) and variance \( \sigma^2 > 0 \). Let \( \bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k \) be the sample mean and \( S_n^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2 \) be the sample variance. Show that with probability one \( S_n \to \sigma \).

10. Suppose \( X_1, X_2, \ldots \) are \( i.i.d. \) with mean \( m \) and variance \( \sigma^2 > 0 \). Show that with probability one
\[
L = \lim_{n \to \infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^2}
\]
exists, and determine \( L \) as a function of \( m, \sigma \).

11. Suppose that random variables \( X_1, X_2, \ldots \) are independent identically distributed, with mean \( m \), variance \( \sigma^2 \), and with finite 4-th moments. Let \( S_n = \sum_{k=1}^n X_k X_{k+1} \). Show that \( \frac{1}{n} S_n \) converges almost surely (and find the limit).

12. Suppose \( X_1, X_2, \ldots \) are independent identically distributed uniform \( U(1, e) \). Let \( Z_n = X_1 \cdots X_n \). Show that \( \sqrt{n} Z_n \) converges almost surely (and find the limit).

13. State and prove Kolmogorov’s maximal inequality

14. Suppose \( X_1, X_2, \ldots \) are independent, square-integrable, with mean zero. Define \( Y_n = X_1 X_2 \cdots X_n \) and \( S_n = Y_1 + \cdots + Y_n \). Adapt the proof of Kolmogorov’s maximal inequality
to prove that
\[
P(\max_{k \leq n} |S_k| > t) \leq \frac{Var(S_n)}{t^2}
\]
(Note that \(Y_1, Y_2, \ldots\) are dependent!)

(15) Suppose that random variables \(\{X_k\}\) are independent uniform \(U(-k, k)\) for \(k \in \mathbb{N}\). Show that the series \(\sum_n \frac{1}{(n^2 + 2n + 3)^n} X_n\) converges with probability one for \(\theta > \theta_0\), where \(\theta_0\) is to be determined by you.

(16) Suppose that random variables \(\{X_k\}\) are independent exponential with parameters \(\lambda_k = k\) (i.e. \(P(X_k \leq x) = 1 - e^{-\lambda_k x}\) for \(x > 0\)) for \(k \in \mathbb{N}\). Prove that the series
\[
\sum_n (n^2 + n + 1)^\theta (X_n - \frac{1}{n})
\]
converges with probability one if \(\theta < \theta_0\), where \(\theta_0\) is to be determined by you.

(17) Let \(X_1, X_2, \ldots\) be independent random variables with the distribution \((k \geq 1)\)
\[
\Pr(X_k = \pm 1) = 1/4,
\Pr(X_k = k^k) = 1/4^k,
\Pr(X_k = 0) = 1/2 - 1/4^k.
\]
Use Kolmogorov’s three-series theorem to prove that the series \(\sum \frac{1}{k} X_k\) converges with probability one.

(18) Prove that if \(X_m \xrightarrow{P} X\) then \(X_n \xrightarrow{D} X\).

(19) If \(X_n \xrightarrow{D} 1\) then \(X_n \xrightarrow{P} 1\).

(20) Suppose \(\{X_k\}\) are independent uniform \(U(0, 1)\) random variables. Show that
\[
n \min_{1 \leq k \leq n} X_k \xrightarrow{D} Y
\]
and determine the law of \(Y\).

(21) Suppose \(X_n \xrightarrow{P} c\) for a constant \(c\) and \(Y_n \xrightarrow{D} Y\). Use the definitions to show that \(X_n Y_n \xrightarrow{D} cY\).

(22) Suppose \(X_n\) has density \(f_n(x) = 1 + \cos(2\pi nx)\) on \([0, 1]\). Prove that \(X_n \xrightarrow{D} X\) (and determine the law of \(X\)).

(23) Suppose random variables \(Z_n\) with laws \(\mu_n\) are such that \(E(Z_n^2) = 1\). Show that \(\{\mu_n\}\) is tight.

(24) If a family \(\{X_n\}\) of random variables is uniformly integrable, prove that there exists \(M < \infty\) such that \(E|X_n| \leq M\).

(25) If a family \(\{X_n\}\) of random variables is square integrable and there exists \(M < \infty\) such that \(EX_n^2 \leq M\), prove that \(\{X_n\}\) is uniformly integrable.