Chapter 2

Probability measures

1. Existence

**Theorem 2.1** (Caratheodory). A (countably additive) probability measure on a field has an extension to the generated $\sigma$-field

**Proof of Theorem 2.1.** Let $\mathcal{F}_0$ be a field of subsets of $\Omega$ and let $P_0$ be a probability measure on $\mathcal{F}_0$. Put $\mathcal{F} = \sigma(\mathcal{F}_0)$.

For each subset $A$ of $\Omega$, define the outer measure

$$(2.1) \quad P^*(A) = \inf \left\{ \sum_{n=1}^{\infty} P_0(A_n) : A_n \in \mathcal{F}_0, \bigcup_{n=1}^{\infty} A_n \supset A \right\}$$

**Question 2.1.** Can $P^*(A) = \infty$?

Let’s first check that $P^*$ is a genuine extension of $P_0$ to a set function defines on all subsets of $\Omega$.

**Proposition 2.2.** $P^*$ and $P$ agree on $\mathcal{F}_0$.

**Proof.** (Omitted in 2018 - see full set of notes)

In general, $P^*$ is not additive, at least not on $2^\Omega$, but it still has a number of nice properties.

**Proposition 2.3.** The outer probability has the following properties:

(i) $P^*(\emptyset) = 0$;

(ii) $P^*(A) \geq 0$

(iii) $A \subset B$ implies $P^*(A) \leq P^*(B)$

(iv) $P^*(\bigcup_n A_n) \leq \sum_n P^*(A_n)$
Proof. (Omitted in 2018 -see full set of notes)

Next, consider the class $\mathcal{M}$ of subsets $A$ of $\Omega$ with the property that

\begin{equation}
(2.2) \quad P^*(A \cap E) + P^*(A^c \cap E) = P^*(E) \quad \text{for all } E \subset \Omega
\end{equation}

Note that by subadditivity of $P^*$, identity (2.2) is equivalent to inequality

\begin{equation}
(2.3) \quad P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E) \quad \text{for all } E \subset \Omega
\end{equation}

Lemma 2.4. $\mathcal{M}$ is a $\sigma$-field. Set function $P : \mathcal{M} \to \mathbb{R}$ defined by $P(A) = P^*(A)$ is a probability measure.

We can now complete the proof of Theorem. Since $P$ and $P^*$ coincide on $\mathcal{M}$ and $P^*$ and $P_0$ coincide on $\mathcal{F}_0$, we already know that $P$ and $P_0$ coincide on $\mathcal{F}_0$. Since $\mathcal{F}_0 \subset \mathcal{M}$, therefore it is also countably additive on a smaller $\sigma$-field $\mathcal{F}$ generated by the field $\mathcal{F}_0$.

Remark 2.1. $P_*(A) = 1 - P^*(A)$ is called the inner measure. [Billingsley] gives other expressions for the outer and inner measures which are of importance in the theory of stochastic processes.

2. Uniqueness

This section is based on [Billingsley, Section 3].

Theorem 2.5. A (countably additive) probability measure on a field has a unique extension to the generated $\sigma$-field.

In view of Theorem 2.1, we only need to prove uniqueness.
### 2.1. Dynkin’s $\pi$-$\lambda$ Theorem.

**Definition 2.1.** A class $\mathcal{P}$ of subsets of $\Omega$ is a $\pi$-system if

(\pi) \ A, B \in \mathcal{P} \ implies \ A \cap B \in \mathcal{P}.

Examples of $\pi$-systems are

(i) \ \{\emptyset\}, which generates sigma-field ...

(ii) Family of intervals $(-\infty, a]$ with $a \in \mathbb{R}$, which generates Borel sigma-field $\mathcal{B}_{\mathbb{R}}$

(iii) Family $(-\infty, a] \times (-\infty, b]$, which generates Borel sigma field $\mathcal{B}_{\mathbb{R}^2}$

(iv) Family of sets $B_1 \times B_2 \times \cdots \times B_d \times \mathbb{R}^\infty$ with $B_j \in \mathcal{B}_{\mathbb{R}}$ which generates the Borel sigma field $\mathcal{B}_{\mathbb{R}^\infty}$.

**Definition 2.2.** A class $\mathcal{L}$ of subsets of $\Omega$ is a $\lambda$-system if

(\lambda_1) \ \Omega \in \mathcal{L}.

(\lambda_2) \ A \in \mathcal{L} \ implies \ A^c \in \mathcal{L}.

(\lambda_3) \ If \ A_1, A_2, \in \mathcal{L} \ are \ disjoint \ then \ \bigcup_n A_n \in \mathcal{L}.

**Remark 2.2.** From (\lambda_1) and (\lambda_2) we see that $\emptyset \in \mathcal{L}$. So if $A, B \in \mathcal{L}$ are disjoint then by (\lambda_3) we get $A \cup B = A \cup B \cup \emptyset \cup \emptyset \cup \cdots \in \mathcal{L}$.

Of course, every field is a $\pi$-system, and every $\sigma$-field is a $\lambda$-system.

**Lemma 2.6.** A class that is both a $\pi$-system and a $\lambda$-system is a $\sigma$-field.

(Omitted in 2018 -see full set of notes)

**Theorem 2.7** (Dynkin’s $\pi$-$\lambda$ Theorem). Suppose a $\lambda$-system $\mathcal{L}$ includes a $\pi$-system $\mathcal{P}$. Then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

(Omitted in 2018 -see full set of notes)

**Corollary 2.8.** Let $\mathcal{P}$ be a $\pi$-system and denote $\mathcal{F} = \sigma(\mathcal{P})$. Suppose $P_1, P_2$ are two probability measures on $\mathcal{F}$ that agree on $\mathcal{P}$. Then $P_1 = P_2$ (on $\mathcal{F}$.)

**Proof.** Let $\mathcal{L}$ be the family of all sets in $\mathcal{F}$ on which $P_1$ and $P_2$ agree. Then $\mathcal{L}$ is a $\lambda$-system. By Theorem 2.7 $\mathcal{F} \subset \mathcal{L}$. \qed

### 3. Probability measures on $\mathbb{R}$

This is based on [Billingsley, Section 12] and [Durrett, Section 1.2].

**Definition 2.3.** $F: \mathbb{R} \to \mathbb{R}$ is a cumulative distribution function, if

(i) $F$ is non-decreasing: $x < y$ implies $F(x) \leq F(y)$

(ii) $\lim_{x \to \infty} F(x) = 0$ and $\lim_{x \to -\infty} F(x) = 1$. 
(iii) \( F \) is right-continuous, \( \lim_{x \to x_0^+} F(x) = F(x_0) \)

Suppose that \( P \) is a probability measure on the Borel subsets of \( \mathbb{R} \). Consider a function \( F : \mathbb{R} \to \mathbb{R} \) defined by \( F(x) = P((-\infty, x]) \). Then \( F \) is a cumulative distribution function. (You should be able to supply the proof!)

The following is a combination of Lebesgue’s Theorem 1.3, with Caratheodory’s Theorem 2.1 and uniqueness Theorem 2.5.

**Proposition 2.9.** Every cumulative distribution function \( F \) corresponds to a unique probability measure \( P \) on the Borel sigma-field set of \( \mathbb{R} \), such that \( F(x) = P((-\infty, x]) \).

**Proof.** Intervals of the form \((-\infty, a]\) form a \( \pi \)-system, and generate the Borel \( \sigma \)-field. So uniqueness follows from Theorem 2.5.

Consider the field \( B_0 \) of finite disjoint unions of intervals \((a, b] \) where \(-\infty \leq a < b \leq \infty \).

For finite \( a < b \), define \( P((a, b]) = F(b) - F(a) \). Also define \( P((-\infty, a]) = F(a) \) and \( P((a, \infty)) = 1 - F(a) \).

Extend \( P \) by additivity to \( B_0 \). As in Week 1, Theorem 1.2, one needs to show that this definition is consistent, that \( P \) is finitely-additive, and that \( P \) is countably-additive on \( B_0 \). Once we prove this, we invoke Theorem 2.1.

(Omitted in 2018 - see full set of notes)

3.1. Examples.

3.1.1. Uniform distributions.

**Example 2.1** (Uniform I). Uniform distribution on the set of real numbers \( \{x_1 < x_2 < \cdots < x_n\} \) is (see Examples 1.4 and 1.5) \( P = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j} \) and corresponds to \( F(x) = \#\{j : x_j \leq x\}/n \).

**Example 2.2** (Uniform II). Uniform distribution on the interval \((a, b)\) is the probability measure \( P \) which corresponds to

\[
F(x) = \begin{cases} 
0 & x < 0 \\
   x & \text{for } 0 \leq x \leq 1 \\
 1 & x > 1
\end{cases}
\]

Recall the construction of the Cantor set: split \([0, 1]\) into \([0, 1/3] \cup (1/3, 2/3) \cup [2/3, 1]\) and remove the middle part. Continue recursively the same procedure with each of the closed intervals retained.
Example 2.3 (Uniform III). Uniform distribution on the Cantor set corresponds to $F$ that is constant on all deleted intervals,

$$
F(x) = \begin{cases} 
0 & x < 0 \\
\vdots \\
1/4 & 1/9 \leq x < 2/9 \\
\vdots \\
1/2 & 1/3 \leq x < 2/3 \\
\vdots \\
3/4 & 7/9 \leq x < 8/9 \\
\vdots \\
1 & x \leq 1 
\end{cases}
$$

The interval removed in $d$-th step is $(\sum_{k=1}^{d} x_k/3^k, \sum_{k=1}^{d} x_k/3^k+1/3^d)$ with $x_d = 1$ and $x_1, \ldots, x_{k-1} \in \{0,2\}$. For example, for $d = 1$ it is $(1/3, 1/3 + 1/3)$. For $d = 2$ the intervals are $(1/3^2, 1/3^2 + 1/3^2)$ and $(2/3 + 1/3^2, 2/3 + 1/3^2 + 1/3^2)$. On each removed interval, $F(x) = \sum_{k=1}^{d-1} x_k/2^{k+1} + 1/2^d$ is constant.

![Graph of $F(x)$](image.png)

3.1.2. Important (absolutely) continuous distributions. Continuous distributions arise from $F(x) = \int_{-\infty}^{x} f(y)dy$, where the so called density function $f \geq 0$ and $\int_{-\infty}^{\infty} f(y)dy = 1$. Example 2.2 is absolutely continuous with $f(y) = 1_{[a,b]}$.

Example 2.4 (Exponential distribution). Take $f(x) = 1_{x>0} \lambda e^{-\lambda x}$, where $\lambda > 0$. This gives

$$
F(x) = \begin{cases} 
0 & x < 0 \\
1 - e^{-\lambda x} & x \geq 0 
\end{cases}
$$
Example 2.5 (Standard normal distribution). Take \( f(x) = \exp(-x^2/2) / \sqrt{2\pi} \). Notation: \( N(0,1) \).

3.1.3. Other examples.

Example 2.6 (mixed type). It is clear that

\[
F(x) = \begin{cases} 0 & x < 0 \\ x/9 & 0 \leq x < 1 \\ x/3 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}
\]

is a cumulative distribution function which cannot be written as an integral of the density\(^1\).

4. Probability measures on \( \mathbb{R}^k \)

For simplicity consider only \( k = 2,3 \).

4.1. Probability measures on \( \mathbb{R}^2 \). The \( \pi \) system that generates Borel sets of \( \mathbb{R}^2 \) consists of sets \(( -\infty, x] \times (-\infty, y] \). Thus every probability measure \( P \) on Borel sets of \( \mathbb{R}^2 \) is determined uniquely by its values on such sets, \( F(x,y) = P((-\infty, x] \times (-\infty, y]) \). Function \( F(x,y) \) is a joint cumulative distribution function.

The probability measure must assign nonnegative numbers to all rectangles \( A = (a_1, b_1] \times (a_2, b_2] \). It is clear (draw a picture) that

\[
(-\infty, b_1] \times (-\infty, b_2] = (-\infty, a_1] \times (-\infty, b_2] \cup (-\infty, b_1] \times (-\infty, a_2] \cup A
\]

Thus

\[
(2.4) \quad F(b_1, b_2) = P(A) + P((-\infty, a_1] \times (-\infty, b_2] \cup (-\infty, b_1]) = P(A) + F(a_1, b_2) + F(a_2, b_1) - P((-\infty, a_1] \times (-\infty, b_2] \cap (-\infty, b_1]) = P(A) + F(a_1, b_2) + F(a_2, b_1) - F(a_1, a_2)
\]

Thus

\[
(2.5) \quad P(A) = \Delta_A(F) := F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(a_2, b_1)
\]

This shows that we must have \( \Delta_A F \geq 0 \).

It is also clear that we have the following properties:

- \( F \) is "right-continuous": if \( a_n, b_n > 0 \) converge to 0 then \( F(x + a_n, y + b_n) \to F(x, y) \).
- \( \lim_{x,y \to \infty} F(x, y) = 1 \)
- \( \lim_{y \to -\infty} F(x, y) = \lim_{x \to -\infty} F(x, y) = 0 \)
- \( G(x) = \lim_{y \to -\infty} F(x, y) \) and \( H(y) = \lim_{x \to -\infty} F(x, y) \) exist and define non-decreasing functions, called the marginal cumulative distribution functions

\(^1\)Probability measures of mixed type arise in actuarial models, where the loss of an insured person might have a density but the insurance payoff may be capped, or be a fraction of the of loss that changes when the loss exceeds some predefined thresholds.
The field \( \mathcal{B}_0 \) generated by the sets \((-\infty, b_1] \times (-\infty, b_2]\) consists of finite unions of disjoint sets that arise as intersections of such sets or their complements, see Exercise 1.17.

This gives sets \((-\infty, b_1] \times (-\infty, b_2]\), their complements, finite rectangles \( A \), sets of the form \((-\infty, b_1] \times (a_2, b_2] \) and \((a_1, b_1] \times (-\infty, b_2].\)

We define \( P((a_1, \infty) \times (a_2, \infty)) = 1 - F(a_1, a_2), \) \( P((-\infty, b_1] \times (-\infty, b_2]) = F(b_1, b_2) \) and \( P((-\infty, b_1] \times (a_2, b_2]) = \lim_{a_1 \to -\infty} \Delta A F. \) We extend the definition by additivity to \( \mathcal{B}_0. \)

Next we check that the assumptions of Exercise 1.12 are again satisfied, so we can conclude that \( P \) has a unique countably additive extension to the Borel \( \sigma \)-field.

It suffices to find a suitable compact set for each of the four types of the "generalized" rectangles. If \( A = (a_1, \infty) \times (a_2, \infty) \) we take \( K = [a_1 + \delta, B_1] \times [a_2 + \delta, B_2] \) and \( B = (a_1 + \delta, B_1] \times (a_2 + \delta, B_2]. \)

Given \( \varepsilon > 0 \) choose \( \delta \) such that \( F(a_1 + \delta, a_2 + \delta) < F(a_1, a_2) + \varepsilon B_1, B_2 \) such that \( F(B_1, B_2) > 1 - \varepsilon. \)

\[
P(B) = F(B_1, B_2) + F(a_1 + \delta, a_2 + \delta) - F(a_1 + \delta, B_2) - F(a_2 + \delta, B_1)
\]

4.2. Probability measures on \( \mathbb{R}^3 \). The \( \pi \) system that generates Borel sets of \( \mathbb{R}^3 \) consists of sets \((-\infty, x] \times (-\infty, y] \times (-\infty, z]. \) Thus every probability measure is determined uniquely by its values on such sets, \( F(x, y, z). \)

We need to assign values of the measure to all rectangles \( A = (a_1, b_1] \times (a_2, b_2] \times (a_3, b_3]. \)

It is clear that

\[
(2.6) \quad (-\infty, b_1] \times (-\infty, b_2] \times (-\infty, b_3]
= A \cup (-\infty, a_1] \times (-\infty, b_2] \times (-\infty, b_3] \cup (-\infty, b_1] \times (-\infty, a_2] \times (-\infty, b_3] \cup (-\infty, b_1] \times (-\infty, b_2] \times (-\infty, a_3]
\]

Noting that \( A \) is disjoint with the remaining set, by the inclusion-exclusion formula (1.1), we get

\[
(2.7) \quad F(b_1, b_2, b_3) = P(A) + F(a_1, b_2, b_3) + F(b_1, a_2, b_3) + F(b_1, b_2, a_3)
- F(a_1, a_2, b_3) - F(a_1, b_2, a_3) - F(b_1, a_2, a_3) + F(a_1, a_2, a_3)
\]

So

\[
(2.8) \quad P(A) = \Delta A (F) :=
F(b_1, b_2, b_3) + F(a_1, a_2, b_3) + F(a_1, b_2, a_3) + F(b_1, a_2, a_3) - F(a_1, a_2, b_3) - F(b_1, a_2, a_3) - F(b_1, b_2, a_3) - F(a_1, a_2, a_3)
\]

4.3. Probability measures on \( \mathbb{R}^\infty \). Recall that \( \mathbb{R}^\infty \) is the set of all infinite real sequences, with metric (0.4). Probability measures on \( \mathbb{R}^\infty \) are determined uniquely by the families of joint finite-dimensional distributions that arise from a special \( \pi \)-system of cylindirical sets, i.e. sets of the form

\[
(-\infty, a_1] \times (-\infty, a_2] \times \ldots \times (-\infty, a_n] \times \mathbb{R} \times \mathbb{R} \times \ldots
\]

A special case of such a measure is constructed in Theorem 4.5.

(Omitted in 2018 - see full set of notes)
Required Exercises

**Exercise 2.1** (Different representations of the same measure on $\mathcal{F}$). Let $\lambda$ be the Lebesgue measure on the Borel $\sigma$-field of subsets of $\Omega = [0, 1]$. Consider $\pi$-system $\mathcal{P} = \{[0, 1/n) : n \in \mathbb{N}\}$ and let $\mathcal{F} = \sigma(\mathcal{P})$. Show that there exists a discrete probability measure $P = \sum_{n=1}^{\infty} p_n \delta_{\omega_n}$ on $2^\Omega$ (see Example 1.5) such that $\lambda$ restricted to $\mathcal{F}$ coincides with $P$ restricted to $\mathcal{F}$. (In formal notation, $\lambda|_\mathcal{F} = P|_\mathcal{F}$.) Is $P$ unique?

**Exercise 2.2** (Statistics). It is illustrative to produce empirical histograms at various sample sizes for the uniform distribution on the Cantor set from Example 2.3. Somewhat surprisingly, this is easy to simulate: take $\sum_{k=1}^{\infty} \varepsilon_k/3^k$ where $\varepsilon_k$ represents a “toss of a fair coin” with values 0 or 1. This exercise asks you to reproduce histograms from [Proschan-Shaw].

**Exercise 2.3** (measure-preserving maps). Let $f : [0, 1] \to [0, 1]$ be the fractional part of $2x$. That is,

$$f(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2x - 1 & \text{if } x > 1/2 \end{cases}$$

Show that for every Borel subset $A$ of $[0, 1]$ the Lebesgue measure of $f^{-1}(A)$ equals to the Lebesgue measure of $A$. (Compare Exercise 2.10.)

Additional Exercises

**Exercise 2.4.** Let $\Omega = (0, 1] \times (0, 1]$ and let $\mathcal{F}$ be the class of sets of the form $A_1 \times (0, 1]$ with $A_1 \in \mathcal{B}$ the Borel $\sigma$-field in $(0, 1]$ and $(P(A_1 \times (0, 1])) = \lambda(A_1)$ (the Lebesgue measure). Then $(\Omega, \mathcal{F}, P)$ is a probability space. For the diagonal $D = \{(x, x) : 0 < x \leq 1\}$, find $P^*(D)$ and $P^*(D^c)$.

**Exercise 2.5.** Inspect the proofs of Theorems 2.1 and 2.5. Find all places where additivity or countable additivity is used.

**Exercise 2.6** (Compare Exercise 1.18). For $\Omega = (0, 1]$ with the field $\mathcal{B}_0$ generated by intervals $I = (a, b]$, consider $\lambda_0(I) = |I|$, extended by additivity to $\mathcal{B}_0$. Let $Q$ be the set of all rational numbers in $(0, 1]$. Use the definition of $\lambda^*$ (not subadditivity) to show that $\lambda^*(Q) = 0$.

**Exercise 2.7.** The family $\mathcal{P}$ of open intervals $(-1/n, 1/n)$ with $n \in \mathbb{N}$ is a $\pi$-system in $\Omega = (-1, 1)$. Describe what sets are in the $\sigma$-field $\sigma(\mathcal{P})$. In particular, is set $\{0\}$ in $\sigma(\mathcal{P})$?

**Exercise 2.8.** Let $\mathcal{A}$ be the smallest field generated by a $\pi$-system $\mathcal{P}$ (see Exercise 1.17). Use the inclusion-exclusion formula from Exercise 1.6 to show that finitely additive probability measures that agree on $\mathcal{P}$ must also agree on $\mathcal{A}$.
Exercise 2.9. Suppose $\mathcal{L}$ is a $\lambda$-system. Show that $A, B \in \mathcal{L}$ and $A \subset B$ implies that $B \setminus A \in \mathcal{L}$.  

*Hint:* Show that $(B \setminus A)^c \in \mathcal{L}$.

**Exercise 2.10.** Consider $\Omega = (0,1)$ with Lebesgue measure. Use the Dynkin’s $\pi$-$\lambda$ Theorem to prove that for all Borel sub-sets $B$ of $(0,1/2)$ and all $x \in (0,1/2)$, the Lebesgue measure of $B + x$ is the same as the Lebesgue measure of $B$. 
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Index

$L - p$-norm, 50
$\lambda$-system, 23
$\pi$-system, 22
$\sigma$-field, 14
$\sigma$-field generated by $X$, 39
distribution of a random variable, 40
Bernoulli random variables, 43
Binomial distribution, 15, 56
Borel $\sigma$-field, 39
Borel sigma-field, 14
Cantelli’s inequality, 52
cardinality, 9
Cauchy distribution, 87
Cauchy-Schwarz inequality, 49
centered, 51
Central Limit Theorem, 91
characteristic function, 83
characteristic function – continuity theorem, 87
Characteristic functions – uniqueness, 86
Characteristic functions – inversion formula, 86
Chebyshev’s inequality, 49
complex numbers, 82
conjugate exponents, 50
continuity condition, 12
convergence in distribution, 45, 73
converges in distribution, 97
converges in probability, 43
converges pointwise, 7
converges uniformly, 7
converges with probability 1, 44
convex function, 49
countable additivity, 12
covariance matrix, 100
cumulative distribution function, 24, 40
cylindrical sets, 29
cylindrical sets, 28
DeMorgan’s law, 8
density function, 26
diadic interval, 105
discrete random variable, 56
discrete random variables, 42
equal in distribution, 40
events, 11, 15
expected value, 47
Exponential distribution, 57
exponential distribution, 26
Fatou’s lemma, 55
field, 11
finite dimensional distributions, 29
finitely-additive probability measure, 12
Fubini’s Theorem, 62
Geometric distribution, 57
Hölder’s inequality, 50, 58
inclusion-exclusion, 16
independent $\sigma$-fields, 33
independent events, 33
independent identically distributed, 42
indicator functions, 8
induced measure, 40
infinite number of tosses of a coin, 105
integrable, 54
intersection, 8
Jensen’s inequality, 49
joint cumulative distribution function, 27
joint distribution of random variables, 40
Kolmogorov’s maximal inequality, 67
Kolmogorov’s one series theorem, 68
Kolmogorov’s three series theorem, 68
Kolmogorov’s two series theorem, 68
Kolmogorov’s zero-one law, 67
Kronecker’s Lemma, 69
Lévy distance, 79
Lebesgue’s dominated convergence theorem, 55
Lebesgue’s dominated convergence theorem – used, 56, 66, 75, 88
Levy’s theorem, 70
Index

Lindeberg condition, 93
Lyapunov’s condition, 94
Lyapunov’s inequality, 49

Marginal cumulative distribution functions, 27
Markov’s inequality, 49
Maximal inequality, Etemadi’s, 70
Maximal inequality, Kolmogorov’s, 67
Measurable function, 39
Measurable rectangle, 61
Minkowski’s inequality, 50
Minkowski’s inequality, 58
Moment generating function, 60
Monotone Convergence Theorem, 55
Multivariate normal distribution, 99

Negative binomial distribution, 16
Normal distribution, 27

Poisson distribution, 16, 57
Polya’s distribution, 16
Portmanteau Theorem, 75
Power set, 7
Probability, 11
Probability measure, 12
Probability space, 11, 15
Product measure, 62

Quantile function, 41, 75
Random element, 39
Random variable, 40
Random vector, 40

Sample space, 11
Scheffe’s theorem, 73
Section, 61
Semi-algebra, 13
Semi-ring, 13
Sigma-field generated by $A$, 14
Simple random variable, 47
Skorohod’s theorem, 75
Slutsky’s Theorem, 74
Standard normal density, 57
Stochastic process with continuous trajectories, 40
Stochastically bounded, 46
Symmetric distribution, 71

Tail $\sigma$-field, 34
Tail integration formula, 63
tight, 46
tight probability measure, 17
Tonelli’s theorem, 62
Truncation of r.v., 45

Uncorrelated, 51
Uniform continuous, 25
Uniform density, 57
Uniform discrete, 25
Uniform singular, 25
Uniformly integrable, 56, 77

Union, 8
Variance, 48
Zero-one law, 34, 67