Three.VI Projection

Linear Algebra
Jim Hefferon

http://joshua.smcvt.edu/linearalgebra
Orthogonal Projection Into a Line
Project a vector into a line

This shows a figure walking out on the line to a point $\vec{p}$ such that the tip of $\vec{v}$ is directly above them, where “above” does not mean parallel to the $y$-axis but instead means orthogonal to the line.
Project a vector into a line

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Since the line is the span of some vector \( \ell = \{ c \cdot \vec{s} \mid c \in \mathbb{R} \} \), we have a coefficient \( c_{\vec{p}} \) with the property that \( \vec{v} - c_{\vec{p}} \vec{s} \) is orthogonal to \( c_{\vec{p}} \vec{s} \).
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Since the line is the span of some vector $\ell = \{ c \cdot \vec{s} \mid c \in \mathbb{R} \}$, we have a coefficient $c_{\vec{p}}$ with the property that $\vec{v} - c_{\vec{p}} \vec{s}$ is orthogonal to $c_{\vec{p}} \vec{s}$.

To solve for this coefficient, observe that because $\vec{v} - c_{\vec{p}} \vec{s}$ is orthogonal to a scalar multiple of $\vec{s}$, it must be orthogonal to $\vec{s}$ itself. Then $(\vec{v} - c_{\vec{p}} \vec{s}) \cdot \vec{s} = 0$ gives that $c_{\vec{p}} = \vec{v} \cdot \vec{s}/\vec{s} \cdot \vec{s}$. 
We have decomposed $\vec{v}$ into two parts $\vec{v} = (c_p \vec{s}) + (\vec{v} - c_p \vec{s})$.

Intuitively, some of $\vec{v}$ lies with the line and that gives the first part $c_p \vec{s}$. The part of $\vec{v}$ that lies with a line orthogonal to $\ell$ is $\vec{v} - c_p \vec{s}$. What’s compelling about pairing these two parts is that they don’t interact, in that the projection of one into the line spanned by the other is the zero vector.
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*Note:* We have not given a definition of ‘angle’ in spaces other than $\mathbb{R}^n$’s, so we will stick here to those spaces. Extending the definitions to other spaces is perfectly possible but we don’t need them here.
1.1 Definition  The orthogonal projection of $\vec{v}$ into the line spanned by a nonzero $\vec{s}$ is this vector.

$$\text{proj}_{[\vec{s}]}(\vec{v}) = \frac{\vec{v} \cdot \vec{s}}{\vec{s} \cdot \vec{s}} \cdot \vec{s}$$

Example  The projection of this $\mathbb{R}^3$ vector into the line

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{L} = \{ c \cdot \vec{s} = c \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R} \}$$

is this vector.

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$
Because $\vec{v}$ is nearly orthogonal to the line $L$, only a small part of $\vec{v}$ lies with the direction of that line, so the projected-to red vector $\text{proj}_{[\vec{s}]}(\vec{v})$ is quite short: ($|\vec{v}| = \sqrt{6} \approx 2.45$ while $|\text{proj}_{[\vec{s}]}(\vec{v})| = \sqrt{\frac{1}{6}} \approx 0.41$).
Gram-Schmidt Orthogonalization
Mutually orthogonal vectors

The prior subsection suggests that projecting a vector \( \vec{v} \) into the line spanned by \( \vec{s} \) decomposes \( \vec{v} \) into two parts, a part with the line and a part orthogonal to that.

\[
\vec{v} = \text{proj}_{[\vec{s}]}(\vec{v}) + (\vec{v} - \text{proj}_{[\vec{s}]}(\vec{v}))
\]

Because these are orthogonal they are in some sense non-interacting. Here we will develop that.
Mutually orthogonal vectors

The prior subsection suggests that projecting a vector $\vec{v}$ into the line spanned by $\vec{s}$ decomposes $\vec{v}$ into two parts, a part with the line and a part orthogonal to that.

Because these are orthogonal they are in some sense non-interacting. Here we will develop that.

2.1 Definition  Vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ are mutually orthogonal when any two are orthogonal: if $i \neq j$ then the dot product $\vec{v}_i \cdot \vec{v}_j$ is zero.

Example  The vectors of the standard basis $\mathcal{E}_3 \subset \mathbb{R}^3$ are mutually orthogonal.

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$
Example  These two vectors in $\mathbb{R}^2$ are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
The next result makes ‘non-interacting’ precise.

2.2 *Theorem* If the vectors in a set \( \{ \vec{v}_1, \ldots, \vec{v}_k \} \subset \mathbb{R}^n \) are mutually orthogonal and nonzero then that set is linearly independent.

2.3 *Corollary* In a \( k \)-dimensional vector space, if the vectors in a size \( k \) set are mutually orthogonal and nonzero then that set is a basis for the space.

2.5 *Definition* An orthogonal basis for a vector space is a basis of mutually orthogonal vectors.
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2.2 Theorem If the vectors in a set \( \{ \vec{v}_1, \ldots, \vec{v}_k \} \subset \mathbb{R}^n \) are mutually orthogonal and nonzero then that set is linearly independent.

Proof Consider \( \vec{0} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k \). For \( i \in \{1, \ldots, k\} \), taking the dot product of \( \vec{v}_i \) with both sides of the equation
\[
\vec{v}_i \cdot (c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k) = \vec{v}_i \cdot \vec{0},
\]
which gives \( c_i \cdot (\vec{v}_i \cdot \vec{v}_i) = 0 \), shows that \( c_i = 0 \) since \( \vec{v}_i \neq \vec{0} \).

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If the vectors in a set \( \{ \vec{v}_1, \ldots, \vec{v}_k \} \subset \mathbb{R}^n \) are mutually orthogonal and nonzero then that set is linearly independent.

**Proof**  
Consider \( \vec{0} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k \). For \( i \in \{1, \ldots, k\} \), taking the dot product of \( \vec{v}_i \) with both sides of the equation \( \vec{v}_i \cdot (c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k) = \vec{v}_i \cdot \vec{0} \), which gives \( c_i \cdot (\vec{v}_i \cdot \vec{v}_i) = 0 \), shows that \( c_i = 0 \) since \( \vec{v}_i \neq \vec{0} \).

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2.3 **Corollary**  
In a \( k \) dimensional vector space, if the vectors in a size \( k \) set are mutually orthogonal and nonzero then that set is a basis for the space.
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2.3 Corollary In a \( k \) dimensional vector space, if the vectors in a size \( k \) set are mutually orthogonal and nonzero then that set is a basis for the space.

Proof Any linearly independent size \( k \) subset of a \( k \) dimensional space is a basis.

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2.5 Definition An orthogonal basis for a vector space is a basis of mutually orthogonal vectors.
2.7 *Theorem* 

If $\langle \vec{\beta}_1, \ldots, \vec{\beta}_k \rangle$ is a basis for a subspace of $\mathbb{R}^n$ then the vectors

\[
\vec{\kappa}_1 = \vec{\beta}_1 \\
\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{\vec{\kappa}_1}(\vec{\beta}_2) \\
\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{\vec{\kappa}_1}(\vec{\beta}_3) - \text{proj}_{\vec{\kappa}_2}(\vec{\beta}_3) \\
\vdots \\
\vec{\kappa}_k = \vec{\beta}_k - \text{proj}_{\vec{\kappa}_1}(\vec{\beta}_k) - \cdots - \text{proj}_{\vec{\kappa}_{k-1}}(\vec{\beta}_k)
\]

form an orthogonal basis for the same subspace.

The book has the proof. We will instead illustrate.
Example This basis for $\mathbb{R}^2$

$$B = \langle \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ \end{pmatrix} \rangle$$

does not have orthogonal vectors. To derive from it a basis $K = \langle \vec{\kappa}_1, \vec{\kappa}_2 \rangle$ that is orthogonal, start by taking the first vector unchanged.

$$\vec{\kappa}_1 = \vec{\beta}_1 = \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}$$

For $\vec{\kappa}_2$ take the part of $\vec{\beta}_2$ that does not lie with $\vec{\kappa}_1$.

$$\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) = \begin{pmatrix} 1 \\ 3 \\ \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 3 \\ \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \\ \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix} = \begin{pmatrix} -2/5 \\ 1/5 \end{pmatrix}$$

Note that $\vec{\kappa}_1$ and $\vec{\kappa}_2$ are indeed orthogonal.
Example This is a basis for $\mathbb{R}^3$.

\[
B = \langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \rangle
\]

Start the orthogonal basis with $\mathbf{\bar{\beta}}_1$.

$\mathbf{\bar{\kappa}}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$
Example This is a basis for $\mathbb{R}^3$.

$$B = \langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \rangle$$

Start the orthogonal basis with $\vec{\beta}_1$.

$$\vec{\kappa}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

As in the prior slide, the next step is $\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$.

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}$$
The third step is \( \vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \).

\[
\begin{pmatrix}
0 \\
3 \\
-1
\end{pmatrix}
- \begin{pmatrix}
0 \\
3 \\
-1
\end{pmatrix} \cdot \begin{pmatrix}
1 \\
1 \\
2
\end{pmatrix}
\cdot \begin{pmatrix}
1 \\
1 \\
2
\end{pmatrix}
- \begin{pmatrix}
0 \\
3 \\
-1
\end{pmatrix} \cdot \begin{pmatrix}
-3/2 \\
3/2 \\
0
\end{pmatrix}
\cdot \begin{pmatrix}
-3/2 \\
3/2 \\
0
\end{pmatrix}
\cdot \begin{pmatrix}
-3/2 \\
3/2 \\
0
\end{pmatrix}
= \begin{pmatrix}
4/3 \\
4/3 \\
-4/3
\end{pmatrix}
\]
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\[
\begin{bmatrix}
0 \\
3 \\
-1
\end{bmatrix} - \frac{\begin{bmatrix}
0 \\
3 \\
-1
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}}{\begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}} \cdot \begin{bmatrix}
1 \\
2
\end{bmatrix} - \frac{\begin{bmatrix}
0 \\
3 \\
-1
\end{bmatrix} \cdot \begin{bmatrix}
-3/2 \\
3/2 \\
0
\end{bmatrix}}{\begin{bmatrix}
-3/2 \\
3/2 \\
0
\end{bmatrix} \cdot \begin{bmatrix}
-3/2 \\
3/2 \\
0
\end{bmatrix}} \cdot \begin{bmatrix}
3/2 \\
0
\end{bmatrix} = \begin{bmatrix}
4/3 \\
4/3 \\
-4/3
\end{bmatrix}
\]

The members of \( B \) are at odd angles but the members of \( K \) are mutually orthogonal.

\[
B = \langle \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}, \begin{bmatrix}
-1 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
0 \\
3 \\
-1
\end{bmatrix} \rangle
\]
The third step is $\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$.

$$
\begin{bmatrix}
0 \\
3 \\
-1
\end{bmatrix}
- \frac{
\begin{bmatrix}
0 \\
3 \\
-1
\end{bmatrix} \cdot 
\begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}
}{
\begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix} \cdot 
\begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}
}
- \frac{
\begin{bmatrix}
0 \\
3 \\
-1
\end{bmatrix} \cdot 
\begin{bmatrix}
-3/2 \\
3/2 \\
0
\end{bmatrix}
}{
\begin{bmatrix}
-3/2 \\
3/2 \\
0
\end{bmatrix} \cdot 
\begin{bmatrix}
-3/2 \\
3/2 \\
0
\end{bmatrix}
}
\cdot 
\begin{bmatrix}
-3/2 \\
3/2 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
4/3 \\
4/3 \\
-4/3
\end{bmatrix}
$$

The members of $\mathcal{B}$ are at odd angles but the members of $\mathcal{K}$ are mutually orthogonal.

$$\mathcal{K} = \langle 
\begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}, 
\begin{bmatrix}
-3/2 \\
3/2 \\
0
\end{bmatrix}, 
\begin{bmatrix}
4/3 \\
4/3 \\
-4/3
\end{bmatrix}
\rangle$$
The third step is $\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$.

\[
\begin{pmatrix}
0 \\ 3 \\ -1
\end{pmatrix}
- \frac{ \begin{pmatrix}
0 \\ 3 \\ -1
\end{pmatrix} \cdot \begin{pmatrix}
1 \\ 2 \\ 0
\end{pmatrix} } { \begin{pmatrix}
1 \\ 2 \\ 0
\end{pmatrix} \cdot \begin{pmatrix}
1 \\ 2 \\ 0
\end{pmatrix} }
\begin{pmatrix}
1 \\ 2 \\ 0
\end{pmatrix}
- \frac{ \begin{pmatrix}
0 \\ 3 \\ -1
\end{pmatrix} \cdot \begin{pmatrix}
-3/2 \\ 3/2 \\ 0
\end{pmatrix} } { \begin{pmatrix}
-3/2 \\ 3/2 \\ 0
\end{pmatrix} \cdot \begin{pmatrix}
-3/2 \\ 3/2 \\ 0
\end{pmatrix} }
\begin{pmatrix}
-3/2 \\ 3/2 \\ 0
\end{pmatrix}
= \begin{pmatrix}
4/3 \\ 4/3 \\ -4/3
\end{pmatrix}
\]

The members of $B$ are at odd angles but the members of $K$ are mutually orthogonal.

$K = \langle \begin{pmatrix}
1 \\ 1 \\ 2
\end{pmatrix}, \begin{pmatrix}
-3/2 \\ 3/2 \\ 0
\end{pmatrix}, \begin{pmatrix}
4/3 \\ 4/3 \\ -4/3
\end{pmatrix}\rangle$

We could go on to make this basis even more like $E_3$ by normalizing all of its members to have length $1$, making an orthnormal basis.