Three.II Homomorphisms

Linear Algebra
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http://joshua.smcvt.edu/linearalgebra
Definition
1.1 Definition  
A function between vector spaces $h: V \rightarrow W$ that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a homomorphism or linear map.
Example Of these two maps $h, g: \mathbb{R}^2 \to \mathbb{R}$, the first is a homomorphism
while the second is not.

$$
\begin{pmatrix}
 x \\
 y 
\end{pmatrix} \mapsto h 2x - 3y \quad \begin{pmatrix}
 x \\
 y 
\end{pmatrix} \mapsto g 2x - 3y + 1
$$
Example Of these two maps $h, g : \mathbb{R}^2 \to \mathbb{R}$, the first is a homomorphism while the second is not.

\[
\begin{pmatrix}
  x \\
  y 
\end{pmatrix} \xrightarrow{h} 2x - 3y \quad \begin{pmatrix}
  x \\
  y 
\end{pmatrix} \xrightarrow{g} 2x - 3y + 1
\]

The map $h$ respects addition

\[
h\left(\begin{pmatrix}
  x_1 \\
  y_1 
\end{pmatrix} + \begin{pmatrix}
  x_2 \\
  y_2 
\end{pmatrix}\right) = h\left(\begin{pmatrix}
  x_1 + x_2 \\
  y_1 + y_2 
\end{pmatrix}\right) = 2(x_1 + x_2) - 3(y_1 + y_2)
\]

\[
= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h\left(\begin{pmatrix}
  x_1 \\
  y_1 
\end{pmatrix}\right) + h\left(\begin{pmatrix}
  x_2 \\
  y_2 
\end{pmatrix}\right)
\]

and scalar multiplication.

\[
r \cdot h\left(\begin{pmatrix}
  x \\
  y 
\end{pmatrix}\right) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h(r \cdot \begin{pmatrix}
  x \\
  y 
\end{pmatrix})
\]
**Example** Of these two maps $h, g: \mathbb{R}^2 \to \mathbb{R}$, the first is a homomorphism while the second is not.

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto h \ 2x - 3y \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto g \ 2x - 3y + 1$

The map $h$ respects addition

$$h\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = h\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) = 2(x_1 + x_2) - 3(y_1 + y_2)$$

$$= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + h\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$$

and scalar multiplication.

$$r \cdot h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h(r \cdot \begin{pmatrix} x \\ y \end{pmatrix})$$

In contrast, $g$ does not respect addition.

$$g\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}\right) = -17 \quad g\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right) + g\left(\begin{pmatrix} 5 \\ 6 \end{pmatrix}\right) = -16$$
We proved these two while studying isomorphisms.

1.6 Lemma A homomorphism sends the zero vector to the zero vector.

1.7 Lemma The following are equivalent for any map $f: V \to W$ between vector spaces.

(1) $f$ is a homomorphism

(2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$

(3) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \ldots, c_n \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_n \in V$

To verify that a map is a homomorphism, we most often use (2).
We proved these two while studying isomorphisms.

1.6 **Lemma** A homomorphism sends the zero vector to the zero vector.

1.7 **Lemma** The following are equivalent for any map \( f: V \rightarrow W \) between vector spaces.

   1. \( f \) is a homomorphism
   2. \( f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2) \) for any \( c_1, c_2 \in \mathbb{R} \) and \( \vec{v}_1, \vec{v}_2 \in V \)
   3. \( f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n) \) for any \( c_1, \ldots, c_n \in \mathbb{R} \) and \( \vec{v}_1, \ldots, \vec{v}_n \in V \)

To verify that a map is a homomorphism, we most often use (2).

**Example** Between any two vector spaces the zero map \( Z: V \rightarrow W \) given by \( Z(\vec{v}) = \vec{0}_W \) is a linear map. Using (2):

\[
Z(c_1 \vec{v}_1 + c_2 \vec{v}_2) = \vec{0}_W = \vec{0}_W + \vec{0}_W = c_1 Z(\vec{v}_1) + c_2 Z(\vec{v}_2).
\]
**Example** The *inclusion map* $\iota: \mathbb{R}^2 \to \mathbb{R}^3$

$$\iota\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism.

$$\iota(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) = \iota\left(\begin{pmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 x_1 \\ c_1 y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2 x_2 \\ c_2 y_2 \\ 0 \end{pmatrix}$$

$$= c_1 \cdot \iota\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + c_2 \cdot \iota\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$$
**Example** The derivative is a transformation on polynomial spaces. For instance, consider \( \frac{d}{dx} : \mathcal{P}_2 \to \mathcal{P}_1 \) given by

\[
\frac{d}{dx} (ax^2 + bx + c) = 2ax + b
\]

(examples are \( \frac{d}{dx} (3x^2 - 2x + 4) = 6x - 2 \) and \( \frac{d}{dx} (x^2 + 1) = 2x \)).

It is a homomorphism.

\[
\frac{d}{dx} \left( r_1 (a_1 x^2 + b_1 x + c_1) + r_2 (a_2 x^2 + b_2 x + c_2) \right)
\]

\[
= \frac{d}{dx} \left( (r_1 a_1 + r_2 a_2) x^2 + (r_1 b_1 + r_2 b_2) x + (r_1 c_1 + r_2 c_2) \right)
\]

\[
= 2(r_1 a_1 + r_2 a_2) x + (r_1 b_1 + r_2 b_2)
\]

\[
= (2r_1 a_1 x + r_1 b_1) + (2r_2 a_2 x + r_2 b_2)
\]

\[
= r_1 \cdot \frac{d}{dx} (a_1 x^2 + b_1 x + c_1) + r_2 \cdot \frac{d}{dx} (a_2 x^2 + b_2 x + c_2)
\]
Example The *trace* of a square matrix is the sum down the upper-left to lower-right diagonal. Thus \( \text{Tr}: \mathcal{M}_{2 \times 2} \to \mathbb{R} \) is this.

\[
\text{Tr} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a + d
\]

It is linear.

\[
\text{Tr} ( r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) = \text{Tr} \left( \begin{pmatrix} r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\ r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2 \end{pmatrix} \right)
\]

\[
= (r_1 a_1 + r_2 a_2) + (r_1 d_1 + r_2 d_2)
\]

\[
= r_1 (a_1 + d_1) + r_2 (a_2 + d_2)
\]

\[
= r_1 \cdot \text{Tr} \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right) + r_2 \cdot \text{Tr} \left( \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right)
\]
1.9 **Theorem** A homomorphism is determined by its action on a basis: if $V$ is a vector space with basis $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$, if $W$ is a vector space, and if $\vec{w}_1, \ldots, \vec{w}_n \in W$ (these codomain elements need not be distinct) then there exists a homomorphism from $V$ to $W$ sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.
1.9 **Theorem** A homomorphism is determined by its action on a basis: if \( V \) is a vector space with basis \( \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle \), if \( W \) is a vector space, and if \( \vec{w}_1, \ldots, \vec{w}_n \in W \) (these codomain elements need not be distinct) then there exists a homomorphism from \( V \) to \( W \) sending each \( \vec{\beta}_i \) to \( \vec{w}_i \), and that homomorphism is unique.

**Example** The book has the proof. Here is an illustration. Consider a map \( h: \mathbb{R}^2 \to \mathbb{R}^2 \) with this action on a basis.

\[
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} 2 \\ 0 \end{pmatrix}
\]

The effect of the map on any vector \( \vec{v} \) at all is determined by those two facts. Represent that vector \( \vec{v} \) with respect to the basis.

\[
\begin{pmatrix} -1 \\ 5 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 6 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

Compute \( h(\vec{v}) \) using the definition of homomorphism.

\[
h(\vec{v}) = h(5 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 6 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = 5 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 6 \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -7 \\ 15 \end{pmatrix}
\]
**Example** Consider $f: \mathbb{R}^3 \to \mathbb{R}^3$ with this effect on the standard basis.

$$f(\vec{e}_1) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad f(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad f(\vec{e}_3) = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

Because this is the standard basis, the effect of the map on any vector $\vec{v} \in \mathbb{R}^3$ is especially easy to compute. For instance,

$$\text{Rep}_{\vec{e}_3, \vec{e}_3} \left( \begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix} \right) = \begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}$$

and so we have this.

$$f(\begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}) = -5 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 10 \cdot \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ -20 \\ 5 \end{pmatrix}$$
**Example** Consider \( f: \mathbb{R}^3 \to \mathbb{R}^3 \) with this effect on the standard basis.

\[
\begin{align*}
  f(\vec{e}_1) &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & f(\vec{e}_2) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & f(\vec{e}_3) &= \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}
\end{align*}
\]

Because this is the standard basis, the effect of the map on any vector \( \vec{v} \in \mathbb{R}^3 \) is especially easy to compute. For instance,

\[
\text{Rep}_{\varepsilon_3, \varepsilon_3} \left( \begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix} \right) = \begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}
\]

and so we have this.

\[
\begin{align*}
  f(\begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}) &= -5 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 10 \cdot \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ -20 \\ 5 \end{pmatrix}
\end{align*}
\]

**1.10 Definition** Let \( V \) and \( W \) be vector spaces and let \( B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle \) be a basis for \( V \). A function defined on that basis \( f: B \to W \) is *extended linearly* to a function \( \hat{f}: V \to W \) if for all \( \vec{v} \in V \) such that \( \vec{v} = c_1 \vec{\beta}_1 + \cdots + c_n \vec{\beta}_n \), the action of the map is \( \hat{f}(\vec{v}) = c_1 \cdot f(\vec{\beta}_1) + \cdots + c_n \cdot f(\vec{\beta}_n) \).
**Example** Consider the action $t_\Theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of rotating all vectors in the plane through an angle $\Theta$. These drawings show that this map satisfies the addition and scalar multiplication conditions.

We will develop the formula for $t_\Theta$. 
Fix a basis for the domain $\mathbb{R}^2$; the standard basis $\mathcal{B}_2$ is convenient. We want the basis vectors mapped as here.

\[
\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}
\]
Fix a basis for the domain $\mathbb{R}^2$; the standard basis $\mathcal{E}_2$ is convenient. We want the basis vectors mapped as here.

$$\begin{pmatrix} -\sin \Theta \\ \cos \Theta \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

Extend linearly.

$$t_\theta(\begin{pmatrix} x \\ y \end{pmatrix}) = t_\theta(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

$$= x \cdot t_\theta(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + y \cdot t_\theta(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

$$= x \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + y \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

$$= \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix}$$
**Example** One basis of the space of quadratic polynomials \( P_2 \) is \( B = \langle x^2, x, 1 \rangle \). Define the *evaluation map* \( \text{eval}_3 : P_2 \to \mathbb{R} \) by specifying its action on that basis

\[
\begin{align*}
    x^2 & \mapsto 9 \\
    x & \mapsto 3 \\
    1 & \mapsto 1
\end{align*}
\]

and then extending linearly.

\[
\text{eval}_3(ax^2 + bx + c) = a \cdot \text{eval}_3(x^2) + b \cdot \text{eval}_3(x) + c \cdot \text{eval}_3(1) \\
= 9a + 3b + c
\]

For instance, \( \text{eval}_3(x^2 + 2x + 3) = 9 + 6 + 3 = 18 \).
Example  One basis of the space of quadratic polynomials $\mathcal{P}_2$ is $B = \langle x^2, x, 1 \rangle$. Define the evaluation map $\text{eval}_3 : \mathcal{P}_2 \to \mathbb{R}$ by specifying its action on that basis

$$ x^2 \xrightarrow{\text{eval}_3} 9 \quad x \xrightarrow{\text{eval}_3} 3 \quad 1 \xrightarrow{\text{eval}_3} 1 $$

and then extending linearly.

$$ \text{eval}_3(ax^2 + bx + c) = a \cdot \text{eval}_3(x^2) + b \cdot \text{eval}_3(x) + c \cdot \text{eval}_3(1) $$

$$ = 9a + 3b + c $$

For instance, $\text{eval}_3(x^2 + 2x + 3) = 9 + 6 + 3 = 18$.

On the basis elements, we can describe the action of this map as: plugging the value 3 in for $x$. That remains true when we extend linearly, so $\text{eval}_3(p(x)) = p(3)$. 

1.12 Definition  A linear map from a space into itself $t: V \rightarrow V$ is a linear transformation.
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Example For any vector space $V$ the identity map $id : V \to V$ given by $\vec{v} \mapsto \vec{v}$ is a linear transformation. The check is easy.
1.12 **Definition**  A linear map from a space into itself \( t: V \rightarrow V \) is a *linear transformation*.

**Example** For any vector space \( V \) the *identity* map \( \text{id}: V \rightarrow V \) given by \( \vec{v} \mapsto \vec{v} \) is a linear transformation. The check is easy.

**Example** In \( \mathbb{R}^3 \) the function \( f_{yz} \) that reflects vectors over the \( yz \)-plane

\[
\begin{pmatrix}
\chi \\
y \\
z
\end{pmatrix}
\mapsto
\begin{pmatrix}
-x \\
y \\
z
\end{pmatrix}
\]

is a linear transformation.

\[
f_{yz}(r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = f_{yz}(\begin{pmatrix} r_1 x_1 + r_2 x_2 \\ r_1 y_1 + r_2 y_2 \\ r_1 z_1 + r_2 z_2 \end{pmatrix}) = \begin{pmatrix} -(r_1 x_1 + r_2 x_2) \\ r_1 y_1 + r_2 y_2 \\ r_1 z_1 + r_2 z_2 \end{pmatrix}
\]

\[
= r_1 \begin{pmatrix} -x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} -x_2 \\ y_2 \\ z_2 \end{pmatrix} = r_1 f_{yz}(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + r_2 f_{yz}(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix})
\]
1.17 Lemma For vector spaces $V$ and $W$, the set of linear functions from $V$ to $W$ is itself a vector space, a subspace of the space of all functions from $V$ to $W$.

We denote the space of linear maps from $V$ to $W$ by $\mathcal{L}(V, W)$.

The book contains the proof.

Example We can combine the two homomorphisms $f, g: \mathcal{P}_1 \rightarrow \mathbb{R}^2$

$$f(a_0 + a_1 x) = \begin{pmatrix} a_0 + a_1 \\ 0 \end{pmatrix} \quad g(a_0 + a_1 x) = \begin{pmatrix} 4a_1 \\ a_1 \end{pmatrix}$$

into a function $2f + 3g: \mathcal{P}_1 \rightarrow \mathbb{R}^2$ whose action is this.

$$(2f + 3g)(a_0 + a_1 x) = \begin{pmatrix} 2a_0 + 14a_1 \\ 3a_1 \end{pmatrix}$$

The point of the lemma is that $2f + 3g$ is also a homomorphism; the check is routine. The collection of homomorphisms from $\mathcal{P}_1$ to $\mathbb{R}^2$ is closed under linear combinations of those homomorphisms—it is a vector space.
**Example** Consider $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$. A member of $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ is a linear map. A linear map is determined by its action on a basis of the domain space. Fix these bases.

$$B_{\mathbb{R}} = \mathcal{E}_1 = \langle 1 \rangle \quad B_{\mathbb{R}^2} = \mathcal{E}_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

Thus the functions that are elements of $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ are determined by $c_1$ and $c_2$ here.

$$1 \xrightarrow{t} c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We could write each such map as $h = h_{c_1, c_2}$. There are two parameters and thus $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ is a dimension 2 space.
Range space and null space
2.1 Lemma  Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

The book has the proof; we instead consider an example.
Example Let \( f: \mathbb{R}^2 \to M_{2\times 2} \) be

\[
\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a & a + b \\ 2b & b \end{pmatrix}
\]

(the check that it is a homomorphism is routine). One subspace of the domain is the \( x \) axis.

\[ S = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{R} \right\} \]

The image under \( f \) of the \( x \) axis is a subspace of of the codomain \( M_{2\times 2} \).

\[
f(S) = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}
\]
Example Let \( f: \mathbb{R}^2 \to M_{2 \times 2} \) be

\[
\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a & a + b \\ 2b & b \end{pmatrix}
\]

(the check that it is a homomorphism is routine). One subspace of the domain is the \( x \) axis.

\[ S = \{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{R} \} \]

The image under \( f \) of the \( x \) axis is a subspace of the codomain \( M_{2 \times 2} \).

\[ f(S) = \{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \} \]

Another subspace of \( \mathbb{R}^2 \) is \( \mathbb{R}^2 \) itself. The image of \( \mathbb{R}^2 \) under \( f \) is this subspace of \( M_{2 \times 2} \).

\[ f(\mathbb{R}^2) = \{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \cdot c_2 \mid c_1, c_2 \in \mathbb{R} \} \]
Example For any angle $\theta$, the function $t_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors counterclockwise through an angle $\theta$ is a homomorphism.

In the domain $\mathbb{R}^2$ each line through the origin is a subspace. The image of that line under this map is another line through the origin, a subspace of the codomain $\mathbb{R}^2$. 
2.2 Definition The range space of a homomorphism \( h: V \rightarrow W \) is

\[
\mathcal{R}(h) = \{ h(\vec{v}) \mid \vec{v} \in V \}
\]

sometimes denoted \( h(V) \). The dimension of the range space is the map’s rank.
Range space

2.2 Definition The range space of a homomorphism $h: V \rightarrow W$ is

$$\mathcal{R}(h) = \{h(\vec{v}) | \vec{v} \in V\}$$

sometimes denoted $h(V)$. The dimension of the range space is the map’s rank.

Example This map from $M_{2 \times 2}$ to $\mathbb{R}^2$ is linear.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{h} \begin{pmatrix} a + b \\ 2a + 2b \end{pmatrix}$$

The range space is a line through the origin.

$$\{ \begin{pmatrix} t \\ 2t \end{pmatrix} | t \in \mathbb{R} \}$$

Every member of that set is the image of a $2 \times 2$ matrix.

$$\begin{pmatrix} t \\ 2t \end{pmatrix} = h\left( \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \right)$$

The map’s rank is 1.
Example The derivative map \( d/dx : \mathcal{P}_4 \rightarrow \mathcal{P}_4 \) is linear. Its range is \( \mathcal{R}(d/dx) = \mathcal{P}_3 \). (Verifying that every member of \( \mathcal{P}_3 \) is the derivative of some member of \( \mathcal{P}_4 \) is easy.) The rank of this derivative function is the dimension of \( \mathcal{P}_3 \), namely 4.
**Example** The derivative map \( d/dx: P_4 \rightarrow P_4 \) is linear. Its range is \( \mathcal{R}(d/dx) = P_3 \). (Verifying that every member of \( P_3 \) is the derivative of some member of \( P_4 \) is easy.) The rank of this derivative function is the dimension of \( P_3 \), namely 4.

**Example** Projection \( \pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \)

\[
\begin{pmatrix}
\chi \\
y \\
z
\end{pmatrix} \mapsto \begin{pmatrix}
\chi \\
y
\end{pmatrix}
\]

is a linear map; the check is routine. The range space is \( \mathcal{R}(\pi) = \mathbb{R}^2 \) because given a vector \( \vec{w} \in \mathbb{R}^2 \)

\[
\vec{w} = \begin{pmatrix}
a \\
b
\end{pmatrix}
\]

we can find a \( \vec{v} \in \mathbb{R}^3 \) that maps to it, specifically any \( \vec{v} \) with a first component \( a \) and second component \( b \). Thus the rank of \( \pi \) is 2.

In the book’s next section, on computing linear maps, we will do more examples of determining the range space.
Example The derivative map $d/dx: \mathcal{P}_4 \rightarrow \mathcal{P}_4$ is linear. Its range is $\mathcal{R}(d/dx) = \mathcal{P}_3$. (Verifying that every member of $\mathcal{P}_3$ is the derivative of some member of $\mathcal{P}_4$ is easy.) The rank of this derivative function is the dimension of $\mathcal{P}_3$, namely 4.

Example Projection $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$

is a linear map; the check is routine. The range space is $\mathcal{R}(\pi) = \mathbb{R}^2$ because given a vector $\vec{w} \in \mathbb{R}^2$

$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$

we can find a $\vec{v} \in \mathbb{R}^3$ that maps to it, specifically any $\vec{v}$ with a first component $a$ and second component $b$. Thus the rank of $\pi$ is 2.

In the book’s next section, on computing linear maps, we will do more examples of determining the range space.
Many-to-one

In moving from isomorphisms to homomorphisms we dropped the requirement that the maps be onto and one-to-one. But any homomorphism $h: V \rightarrow W$ is onto its range space $\mathcal{R}(h)$, so dropping the onto condition has, in a way, no effect on the range. It doesn’t allow any essentially new maps.
Many-to-one

In moving from isomorphisms to homomorphisms we dropped the requirement that the maps be onto and one-to-one. But any homomorphism \( h : V \rightarrow W \) is onto its range space \( \mathcal{R}(h) \), so dropping the onto condition has, in a way, no effect on the range. It doesn’t allow any essentially new maps.

In contrast, consider the effect of dropping the one-to-one condition. With that, an output vector \( \vec{w} \in W \) may have many associated inputs, many \( \vec{v} \in V \) such that \( h(\vec{v}) = \vec{w} \).

Recall that for any function \( h : V \rightarrow W \), the set of elements of \( V \) that map to \( \vec{w} \in W \) is the \textit{inverse image} \( h^{-1}(\vec{w}) = \{ \vec{v} \in V \mid h(\vec{v}) = \vec{w} \} \).

The structure of the inverse image sets will give us insight into the definition of homomorphism.
Example Projection \( \pi: \mathbb{R}^2 \to \mathbb{R} \) onto the \( x \) axis is linear.

\[
\pi\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = x
\]
**Example**  Projection $\pi: \mathbb{R}^2 \to \mathbb{R}$ onto the $x$ axis is linear.

$$\pi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x$$

Here are some elements of $\pi^{-1}(2)$. Think of these as “2 vectors.”

Think of elements of $\pi^{-1}(3)$ as “3 vectors.”

These elements of $\pi^{-1}(5)$ are “5 vectors.”
These drawings give us a way to make the definition of homomorphism more concrete. Consider preservation of addition.

\[ \pi(\vec{u}) + \pi(\vec{v}) = \pi(\vec{u} + \vec{v}) \]

If \( \vec{u} \) is such that \( \pi(\vec{u}) = 2 \), and \( \vec{v} \) is such that \( \pi(\vec{v}) = 3 \), then \( \vec{u} + \vec{v} \) will be such that the sum \( \pi(\vec{u} + \vec{v}) = 5 \).
These drawings give us a way to make the definition of homomorphism more concrete. Consider preservation of addition.

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If \( \vec{u} \) is such that \( \pi(\vec{u}) = 2 \), and \( \vec{v} \) is such that \( \pi(\vec{v}) = 3 \), then \( \vec{u} + \vec{v} \) will be such that the sum \( \pi(\vec{u} + \vec{v}) = 5 \). That is, a “2 vector” plus a “3 vector” is a “5 vector.” Red plus blue makes magenta.

A similar interpretation holds for preservation of scalar multiplication: the image of an “\( r \cdot 2 \) vector” is \( r \) times 2.
**Example** This function \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) is linear.

\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ 2x + 2y \end{pmatrix}
\]

Here are elements of \( h^{-1}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\). (Only one inverse image element is shown as a vector, most are indicated with dots.)

Here are some elements of \( h^{-1}\left(\begin{pmatrix} 1.5 \\ 3 \end{pmatrix}\right)\) and \( h^{-1}\left(\begin{pmatrix} 2.5 \\ 5 \end{pmatrix}\right)\).
The way that the range space vectors add

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 5 \end{pmatrix}$$

is reflected in the domain: red plus blue makes magenta.
The way that the range space vectors add

\[
\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 5 \end{pmatrix}
\]

is reflected in the domain: red plus blue makes magenta.

That is, preservation of addition is: \( h(\vec{v}_1) + h(\vec{v}_2) = h(\vec{v}_1 + \vec{v}_2) \).
Homomorphisms organize the domain

So the intuition is that a linear map organizes its domain into inverse images,

such that those sets reflect the structure of the range.
Example  Projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism.

$$
\begin{pmatrix}
\chi \\
y \\
z
\end{pmatrix} \mapsto
\begin{pmatrix}
\chi \\
y
\end{pmatrix}
$$

Here we draw the range $\mathbb{R}^2$ as the $xy$-plane inside of $\mathbb{R}^3$.

In the range the parallelogram shows a vector addition $\vec{w}_1 + \vec{w}_2 = \vec{w}_3$. 
**Example**  Projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism.

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The diagram shows some of the points in each inverse image $\pi^{-1}(\vec{w}_1)$, $\pi^{-1}(\vec{w}_2)$, and $\pi^{-1}(\vec{w}_3)$. 
Example  Projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism.

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]

Here we draw the range $\mathbb{R}^2$ as the $xy$-plane inside of $\mathbb{R}^3$.

In the range the parallelogram shows a vector addition $\vec{w}_1 + \vec{w}_2 = \vec{w}_3$.

The diagram shows some of the points in each inverse image $\pi^{-1}(\vec{w}_1)$, $\pi^{-1}(\vec{w}_2)$, and $\pi^{-1}(\vec{w}_3)$. The sum of a vector $\vec{v}_1 \in \pi^{-1}(\vec{w}_1)$ and a vector $\vec{v}_2 \in \pi^{-1}(\vec{w}_2)$ equals a vector $\vec{v}_3 \in \pi^{-1}(\vec{w}_3)$. A $\vec{w}_1$ vector plus a $\vec{w}_2$ vector equals a $\vec{w}_3$ vector.
This interpretation of the definition of homomorphism also holds when the spaces are not ones that we can sketch.

**Example** Let \( h: P_2 \to \mathbb{R}^2 \) be

\[
ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}
\]

and consider these three members of the range such that \( \vec{w}_1 + \vec{w}_2 = \vec{w}_3 \)

\[
\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
This interpretation of the definition of homomorphism also holds when the spaces are not ones that we can sketch.

**Example** Let $h: \mathcal{P}_2 \to \mathbb{R}^2$ be

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}$$

and consider these three members of the range such that $\vec{w}_1 + \vec{w}_2 = \vec{w}_3$

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The inverse image of $\vec{w}_1$ is $h^{-1}(\vec{w}_1) = \{a_1x^2 + 1x + c_1 \mid a_1, c_1 \in \mathbb{R}^2\}$. Members of this set are “$\vec{w}_1$ vectors.”
This interpretation of the definition of homomorphism also holds when the spaces are not ones that we can sketch.

*Example* Let \( h: \mathcal{P}_2 \to \mathbb{R}^2 \) be

\[
ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}
\]

and consider these three members of the range such that \( \vec{w}_1 + \vec{w}_2 = \vec{w}_3 \)

\[
\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

The inverse image of \( \vec{w}_1 \) is \( h^{-1}(\vec{w}_1) = \{a_1 x^2 + 1x + c_1 \mid a_1, c_1 \in \mathbb{R}^2\} \). Members of this set are “\( \vec{w}_1 \) vectors.” The inverse image of \( \vec{w}_2 \) is \( h^{-1}(\vec{w}_2) = \{a_2 x^2 - 1x + c_2 \mid a_2, c_2 \in \mathbb{R}\} \); these are “\( \vec{w}_2 \) vectors.” The “\( \vec{w}_3 \) vectors” are members of \( h^{-1}(\vec{w}_3) = \{a_3 x^2 + 0x + c_3 \mid a_3, c_3 \in \mathbb{R}^2\} \).
This interpretation of the definition of homomorphism also holds when the spaces are not ones that we can sketch.

Example Let \( h: \mathcal{P}_2 \to \mathbb{R}^2 \) be

\[
ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}
\]

and consider these three members of the range such that \( \vec{w}_1 + \vec{w}_2 = \vec{w}_3 \)

\[
\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

The inverse image of \( \vec{w}_1 \) is \( h^{-1}(\vec{w}_1) = \{ a_1x^2 + 1x + c_1 \mid a_1, c_1 \in \mathbb{R}^2 \} \). Members of this set are “\( \vec{w}_1 \) vectors.” The inverse image of \( \vec{w}_2 \) is \( h^{-1}(\vec{w}_2) = \{ a_2x^2 - 1x + c_2 \mid a_2, c_2 \in \mathbb{R} \} \); these are “\( \vec{w}_2 \) vectors.” The “\( \vec{w}_3 \) vectors” are members of \( h^{-1}(\vec{w}_3) = \{ a_3x^2 + 0x + c_3 \mid a_3, c_3 \in \mathbb{R}^2 \} \).

Any \( \vec{v}_1 \in h^{-1}(\vec{w}_1) \) plus any \( \vec{v}_2 \in h^{-1}(\vec{w}_2) \) equals a \( \vec{v}_3 \in h^{-1}(\vec{w}_3) \): a quadratic with an \( x \) coefficient of 1 plus a quadratic with an \( x \) coefficient of \(-1\) equals a quadratic with an \( x \) coefficient of 0.
Null space

In each of those examples, the homomorphism \( h: V \to W \) shows how to view the domain \( V \) as organized into the inverse images \( h^{-1}(\vec{w}) \).

In the examples these inverse images are all the same, but shifted. So if we describe one of them then we understand how the domain is divided. Vector spaces have a distinguished element, \( \vec{0} \). So we next consider the inverse image \( h^{-1}(\vec{0}) \).
2.10 **Lemma** For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

   The book has the verification.
2.11 Definition The **null space** or **kernel** of a linear map $h: V \to W$ is the inverse image of $\vec{0}_W$.

\[ \mathcal{N}(h) = h^{-1}(\vec{0}_W) = \{ \vec{v} \in V \mid h(\vec{v}) = \vec{0}_W \} \]

The dimension of the null space is the map’s **nullity**.
2.11 Definition The null space or kernel of a linear map $h: V \to W$ is the inverse image of $\vec{0}_W$.

$$\mathcal{N}(h) = h^{-1}(\vec{0}_W) = \{ \vec{v} \in V \mid h(\vec{v}) = \vec{0}_W \}$$

The dimension of the null space is the map’s nullity.

**Note** Strictly, the trivial subspace of the codomain is not $\vec{0}_W$, it is $\{\vec{0}_W\}$, and so we may think to write the nullspace as $h^{-1}(\{\vec{0}_W\})$. But we have defined the two sets $h^{-1}(\vec{w})$ and $h^{-1}(\{\vec{w}\})$ to be equal and the first is easier to write.
**Example** Consider the derivative \( \frac{d}{dx} : \mathcal{P}_2 \to \mathcal{P}_1 \). This is the nullspace; note that it is a subset of the domain

\[
\mathcal{N}(\frac{d}{dx}) = \{ ax^2 + bx + c \mid 2ax + b = 0 \}
\]

(the ‘0’ there is the zero polynomial \( 0x + 0 \)). Now, \( 2ax + b = 0 \) if and only if they have the same constant coefficient \( b = 0 \), the same \( x \) coefficient of \( a = 0 \), and the same coefficient of \( x^2 \) (which gives no restriction). So this is the nullspace, and the nullity is 1.

\[
\mathcal{N}(\frac{d}{dx}) = \{ ax^2 + bx + c \mid a = 0, b = 0, c \in \mathbb{R} \} = \{ c \mid c \in \mathbb{R} \}
\]

**Example** The function \( h: \mathbb{R}^2 \to \mathbb{R}^1 \) given by

\[
\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b
\]

has this null space and so its nullity is 1.

\[
\mathcal{N}(h) = \{ \begin{pmatrix} a \\ b \end{pmatrix} \mid 2a + b = 0 \} = \{ \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} b \mid b \in \mathbb{R} \}
\]
Example  The homomorphism \( f: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}^2 \)

\[
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} \mapsto \begin{pmatrix}
a + b \\
c + d \\
\end{pmatrix}
\]

has this null space

\[
\mathcal{N}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = 0 \text{ and } c + d = 0 \right\}
\]

\[
= \left\{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \right\}
\]

and a nullity of 2.

Example  The dilation function \( d_3: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[
\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}
\]

has \( \mathcal{N}(d_3) = \{ \vec{0} \} \). A trivial space has an empty basis so \( d_3 \)'s nullity is 0.


**Rank plus nullity**

Recall the example map $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ 2x + 2y \end{pmatrix}
$$

whose range space $\mathbb{R}(h)$ is the line $y = 2x$ and whose domain is organized into lines, $\mathcal{N}(h)$ is the line $y = -x$. There, an entire line’s worth of domain vectors collapses to the single range point.

In moving from domain to range, this maps drops a dimension. We can account for it by thinking that each output point absorbs a one-dimensional set.
2.14 Theorem A linear map's rank plus its nullity equals the dimension of its domain.

The book contains the proof.

Example Consider this map \( h: \mathbb{R}^3 \rightarrow \mathbb{R} \).

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} \mapsto \frac{x}{2} + \frac{y}{5} + z
\]

The null space is this plane.

\[
\mathcal{N}(h) = h^{-1}(0) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \frac{x}{2} + \frac{y}{5} + z = 0 \right\}
\]

Other inverse image sets are also planes.

\[
h^{-1}(1) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \frac{x}{2} + \frac{y}{5} + z = 1 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid z = 1 - \frac{x}{2} - \frac{y}{5} \right\}
\]
This shows the inverse images $h^{-1}(0)$ and $h^{-1}(1)$ lined up on the $z$ axis.

So $h$ breaks the domain into stacked planes—any two inverse images $h^{-1}(r_1)$ and $h^{-1}(r_2)$ are collections of domain vectors whose endpoints form a plane. The only difference between these 2-dimensional subsets is where they sit in the stack, shown here as where they intersect the $z$ axis.

That is, $h$ partitions the 3-dimensional domain into 2-dimensional sets, leaving 1 dimension of freedom, which matches the dimension of the map’s range.
Example  Projection \( \pi: \mathbb{R}^3 \to \mathbb{R}^2 \)

\[
\begin{pmatrix}
  a \\
  b \\
  c
\end{pmatrix} \mapsto \begin{pmatrix}
  a \\
  b
\end{pmatrix}
\]

takes a 3-dimensional domain to a 2-dimensional range. Its null space is the z-axis, so its nullity is 1.
Example Projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$

\[
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix} \mapsto \begin{pmatrix}
    a \\
    b
\end{pmatrix}
\]

takes a 3-dimensional domain to a 2-dimensional range. Its null space is the z-axis, so its nullity is 1.

This example shows the idea of the proof particularly clearly. Take the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space.
Example  Projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$

takes a 3-dimensional domain to a 2-dimensional range. Its null space is the z-axis, so its nullity is 1.

This example shows the idea of the proof particularly clearly. Take the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space. Expand that to the basis $E_3$ for the entire domain.
Example Projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$

\[
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}
\]

takes a 3-dimensional domain to a 2-dimensional range. Its null space is the z-axis, so its nullity is 1.

This example shows the idea of the proof particularly clearly. Take the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space. Expand that to the basis $E_3$ for the entire domain. On an input vector the action of $\pi$ is

\[
c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3 \mapsto c_1 \vec{e}_1 + c_2 \vec{e}_2 + \vec{0}
\]

and so the domain is organized by $\pi$ into inverse images that are vertical lines, one-dimensional sets like the null space.
Example  The derivative function \( \frac{d}{dx} : \mathcal{P}_2 \to \mathcal{P}_1 \)

\[
ax^2 + bx + c \mapsto 2a \cdot x + b
\]

has this range space

\[
\mathcal{R}(\frac{d}{dx}) = \{d \cdot x + e | \ d, e \in \mathbb{R}\} = \mathcal{P}_1
\]

(the linear polynomial \( dx + e \in \mathcal{P}_1 \) is the image of any antiderivative \((d/2)x^2 + ex + C\), where \( C \in \mathbb{R}\)). This is its null space.

\[
\mathcal{N}(\frac{d}{dx}) = \{0x^2 + 0x + c | \ c \in \mathbb{R}\} = \{c | \ c \in \mathbb{R}\}
\]

The rank is 2 while the nullity is 1, and they add to the domain’s dimension 3.
**Example** The dilation function $d_3 : \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$

has range space $\mathbb{R}^2$ and a trivial nullspace $\mathcal{N}(d_3) = \{ \vec{0} \}$. So its rank is 2 and its nullity is 0.
Example  The dilation function $d_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$

has range space $\mathbb{R}^2$ and a trivial nullspace $\mathcal{N}(d_3) = \{\vec{0}\}$. So its rank is 2 and its nullity is 0.

The book’s next section is on computing linear maps, and we will compute more null spaces there.
2.18 Lemma Under a linear map, the image of a linearly dependent set is linearly dependent.

Proof Suppose that
\[ c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n = \mathbf{0}, \]
with some \( c_i \) nonzero. Apply \( h \) to both sides:
\[ h(c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n) = c_1 h(\mathbf{v}_1) + \cdots + c_n h(\mathbf{v}_n) \]
and
\[ h(\mathbf{0}) = \mathbf{0}. \]
Thus we have
\[ c_1 h(\mathbf{v}_1) + \cdots + c_n h(\mathbf{v}_n) = \mathbf{0} \]
with some \( c_i \) nonzero. \( \Box \)

Example The trace function \( \text{Tr} : M_{2 \times 2} \to \mathbb{R} \)
\[ (a \ b) \mapsto a + d \]
is linear. This set of matrices is dependent. \( S = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \} \)
The three matrices map to 1, 0, and 2 respectively. The set \( \{1, 0, 2\} \subseteq \mathbb{R} \)
is linearly dependent.
2.18 Lemma Under a linear map, the image of a linearly dependent set is linearly dependent.

Proof Suppose that \( c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n = \vec{0}_V \) with some \( c_i \) nonzero. Apply \( h \) to both sides: \( h(c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n) = c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) \) and \( h(\vec{0}_V) = \vec{0}_W \). Thus we have \( c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) = \vec{0}_W \) with some \( c_i \) nonzero. QED
Lemma 2.18 Under a linear map, the image of a linearly dependent set is linearly dependent.

Proof Suppose that \( c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n = \vec{0}_V \) with some \( c_i \) nonzero. Apply \( h \) to both sides: \( h(c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n) = c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) \) and \( h(\vec{0}_V) = \vec{0}_W \). Thus we have \( c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) = \vec{0}_W \) with some \( c_i \) nonzero. QED

Example The trace function \( \text{Tr}: \mathcal{M}_{2 \times 2} \to \mathbb{R} \)

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d
\]
is linear. This set of matrices is dependent.

\[
S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \right\}
\]
The three matrices map to 1, 0, and 2 respectively. The set \( \{1, 0, 2\} \subseteq \mathbb{R} \) is linearly dependent.
A one-to-one homomorphism is an isomorphism

2.20 Theorem  Where $V$ is an $n$-dimensional vector space, these are equivalent statements about a linear map $h: V \rightarrow W$.

1. $h$ is one-to-one
2. $h$ has an inverse from its range to its domain that is a linear map
3. $\mathcal{N}(h) = \{\vec{0}\}$, that is, $\text{nullity}(h) = 0$
4. $\text{rank}(h) = n$
5. If $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for $V$ then $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n) \rangle$ is a basis for $\mathcal{R}(h)$

The book has the proof.