MATH 6012 Exam-1-2019  Answer: Key

1. Find the cosine of the angle between vectors \( \vec{u} + \vec{v} \) and \( \vec{u} - \vec{v} \) if \( \vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \) and \( \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \).

Answer: \( \vec{s} = \vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \) and \( \vec{d} = \vec{u} - \vec{v} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \) so

\[
\cos \theta = \frac{\vec{s} \cdot \vec{d}}{\| \vec{s} \| \times \| \vec{d} \|} = \frac{-2}{\sqrt{10} \sqrt{2}} = -\frac{\sqrt{5}}{5}
\]

The angle is obtuse, with \( \theta \approx 2.03444 \) radians, i.e., about 116.565°.

2. Use the definition to show that functions \( g_1(x) = 1, g_2(x) = x, g_3(x) = x(e^x + e^{-x}) \) are linearly independent.

Answer: Suppose

\[ c_1 g_1 + c_2 g_2 + c_3 g_3 = 0 \text{ for all real } x. \]  

(\(^*)\)

Our goal is to show that this implies \( c_1 = c_2 = c_3 = 0. \)

Routine solution: Denote by \( f(x) \) the left hand side of \((*)\). Then \( f(0) = c_1 = 0, f'(0) = c_2 + 2c_3 = 0, f''(0) = 0, f'''(0) = 6c_3 = 0. \) This gives a system of 4 equations for 3 unknown coefficients \( c_1, c_2, c_3: \)

\[
\begin{align*}
    c_1 &= 0 \\
    c_2 + 2c_3 &= 0 \\
    0 &= 0 \\
    6c_3 &= 0
\end{align*}
\]

Clearly, all \( c_j = 0. \)

In summary, we showed that if \((*)\) holds then we must have \( c_0 = c_1 = c_2 = c_3 = 0, \) i.e. the functions are linearly independent.

There are numerous other solutions. An ad-hoc method: Evaluating \((*)\) expression at \( x = 0 \) we get \( c_1 = 0, \) as \( g_2(0) = g_3(0) = 0. \)

So \((*)\) becomes \( c_2 g_2 + c_3 g_3 = 0, \) i.e.

\[ c_2 x + c_3 x(e^x + e^{-x}) = 0 \]

Dividing by \( x, \) we get

\[ c_2 + c_3(x(e^x + e^{-x})) = 0 \]  

(**)

Differentiating \((**\) at \( x = 1 \) we get \( c_3(e - \frac{1}{e}) = 0, \) so \( c_3 = 0. \) Inserting this back into \((**)\) we see that \( c_2 = 0, \) too.
3. Matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{bmatrix}$ is row equivalent to matrix $B = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Use this information to find a basis (and the dimension) for the null space $\text{Null}(A) = \{ \vec{x} : A\vec{x} = \vec{0} \}$.

**Answer:** Equation $A\vec{x} = \vec{0}$ is equivalent to $B\vec{x} = \vec{0}$, and $B$ is in echelon form, so we can read out the solution. Basic variables are $x_1, x_2$. Free variables are $x_3 = u, x_4 = s, x_5 = t$. We get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2u - 3s - 4t \\ u \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So the solution set is the span of 3 linearly independent vectors in $\mathbb{R}^5$:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The dimension of the $\text{Null}(A)$ is 3.

4. Consider the following basis $B = \langle 1, 1 - t, (1 - t)^2 \rangle$ of the vector space $P_2$ of quadratic polynomials. (You do not need to check that this is a basis of $P_2$)

(a) Which polynomial $p$ has coordinates $\text{Rep}_B(p) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$? Simplify your answer.

**Answer:** From the definition of coordinates, we have $p(t) = 3 + 2(1 - t) + (1 - t)^2 = 6 - 4t + t^2$

(b) What are the coordinates of the monomial $t^2$ in basis $B$?

**Answer:** $t^2 = (t - 1 + 1)^2 = (t - 1)^2 + 2(t - 1) + 1 = (t - 1)^2 - 2(1 - t) + 1$ so the coordinates are $[t^2]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
5. Suppose \( V = \text{span}\{1, \cos x, \sin x\} \) and \( W = \text{span}\{1, x, \cos x, \sin x\} \). Let \( S : V \rightarrow W \) be a mapping which to a function \( f(x) \) assigns its definite integral, the function \( g(x) = \int_0^x f(t)dt \). Without checking you can assume that \( S \) is a linear mapping and that the above sets of functions are linearly independent, so they form respective bases of the spaces \( V \) and \( W \).

(a) Find the matrix representation of \( S \) with respect to the above bases. **Answer:**

\[
\begin{align*}
\int_0^x 1dt &= x, \\
\int_0^x \cos tdt &= \sin x, \\
\int_0^x \sin tdt &= 1 - \cos x
\end{align*}
\]

So the columns of the matrix representation are the expansions of these functions in the second basis, i.e.

\[
\begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
-1 \\
0
\end{bmatrix}
\]

(b) Is \( S \) one-to-one? Justify your answer. **Answer:** #1: Yes, the columns of \( A \) are linearly independent - this is easier seen after swapping the last two! **Answer:** #2: If \( \int_0^x f(t)dt = \int_0^x g(t)dt \) and \( f, g \) are continuous then by differentiation we get \( f = g \). So yes, it is one-to-one

(c) Is \( S \) onto? Justify your answer. **Answer:** #1: No, the dimensions do not match. The dimension of range of \( S \) can be at most 3. **Answer:** #2: If \( f(x) = a + b \cos x + c \sin x \) then \( S(f)(x) = ax + b \sin x + c - c \cos x \) so the range of range of \( S \) is span of functions \( x, \sin x, \cos x - 1 \) and is three dimensional, not the four dimensional space \( W \).
6. Find the inverse of \( A = \begin{bmatrix} 1 & b & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \) for arbitrary \( b \in \mathbb{R} \).

Answer: \( A^{-1} = \begin{bmatrix} 1 & -b & 2b - 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \)