Linear Algebra MATH 2076 Worksheet  Key

This is an in-class worksheet on coordinates.

1. Consider the following basis \( B = \{1, 1 + t, (1 + t)^2\} \) of the vector space \( \mathbb{P}_2 \) of quadratic polynomials.

(a) Which polynomial \( p \) has coordinates \([p]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\)? Simplify your answer.

\[
p(t) = 3 + 2(1 + t) + (1 + t)^2 = 6 + 4t + t^2
\]

(b) What are the coordinates of the monomial \( t^2 \) in basis \( B \)?

\[
t^2 = (t + 1 - 1)^2 = (t + 1)^2 - 2(t + 1) + 1 \text{ so the coordinates are } [t^2]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}
\]

2. Consider the subspace \( H \) of \( \mathbb{R}^4 \) spanned by the vectors \( \vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} \). Assume (without checking) that \( B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \) is linearly independent so that \( B \) is a basis of \( H \).

(a) Find the coordinates of vector \( \vec{v} = \begin{bmatrix} 5 \\ 7 \\ 11 \\ 11 \end{bmatrix} \) in basis \( B \).

We need to solve the system of equations \( c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 = \vec{v} \) for the unknown coordinates \( c_1, c_2, c_3 \) of vector \( \vec{v} \) in basis \( B \). That is

\[
c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \\ 11 \end{bmatrix}
\]

The augmented matrix of this system is

\[
\begin{bmatrix}
1 & 1 & 1 & 5 \\
1 & 2 & 2 & 7 \\
1 & 2 & 3 & 11 \\
1 & 2 & 3 & 11
\end{bmatrix}
\]

After row reduction we get

\[
\begin{bmatrix}
1 & 1 & 1 & 5 \\
0 & 1 & 1 & 2 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Solving the resulting system \( c_1 + c_2 + c_3 = 5, c_2 + c_3 = 2, c_3 = 4 \) we get \( c_1 = 3, c_2 = -2, c_3 = 4 \).

So the coordinates of \( \vec{v} \) in basis \( B \) are \( \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \).
3. Consider the vector space $V$ of all symmetric 2 by 2 matrices with the basis $B = \{ A_1, A_2, A_3 \}$ where $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. (These matrices come from the take-home quiz where you verified their linear independence.)

(a) Find the coordinates of the identity matrix $I$ in this basis.

(b) Which matrix $A$ has coordinates $[A]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$?

(c) Which $2 \times 2$ matrix is not in $V$?

4. Prove that polynomials $p_1(t) = t^3$, $p_2(t) = (1-t)^3$, $p_3(t) = (1+t)^3$ are linearly independent. (This can be done in many ways - for this worksheet, use the coordinates!)

There are many ways of solving this question: one can compute the derivatives, one can express the problem in the standard coordinates, or one can choose enough values of $t$. Here is one of the solutions by the latter method:

Consider $f(t) = C_1 t^3 + C_2 (1-t)^3 + C_3 (1+t)^3$ and suppose that $f(t) = 0$ for all $t$. Then $f(0) = 0$ and $f(1) = 0$ and $f(-1) = 0$ so we get the following system of equations:

\[
\begin{align*}
C_2 + C_3 &= 0 \\
C_1 + 8C_3 &= 0 \\
-C_1 - 8C_2 &= 0
\end{align*}
\]

The matrix of this system is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 8 \\ -1 & -8 & 0 \end{bmatrix}$. Now

\[
\det A = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 8 \\ 0 & -8 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 \\ -8 & 8 \end{bmatrix} = -16 \neq 0
\]

So $A$ is invertible. We now use the invertibility as follows:

Write $\vec{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$. The system of equations (1-3) in vector notation is $A \vec{C} = \vec{0}$. Since $A$ is invertible, the equation $A\vec{C} = \vec{0}$ has the unique solution $\vec{C} = A^{-1}\vec{0} = \vec{0}$. So $\vec{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This shows that $C_1 = C_2 = C_3 = 0$ is the only choice of the coefficients for the linear combination to make $f(t) = 0$ for all $t$. 