Linear Algebra MATH 2076 Quiz-2

A Answer: Key

Be sure to show your work. No credit for inspired answers!

1. Matrix \( A = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ 2 & -2 & 2 & 3 & -2 \\ 1 & -1 & 2 & 3 & -2 \\ 1 & -1 & 1 & 2 & -1 \end{bmatrix} \) is row equivalent to \( B = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \). Describe all solutions of equation \( Ax = 0 \) in parametric vector form (or as a span).

**Answer:** This problem is similar to problems 8, 11 in Section 1.5.

The free variables are \( x_2 \), \( x_5 \) and the general solution is \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} x_5 \). So the answer is \( \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \).

2. Find the value(s) of \( h \) for which the vectors \( v_1 = \begin{bmatrix} 3 \\ -6 \\ 0 \\ 1 \end{bmatrix} \), \( v_2 = \begin{bmatrix} -6 \\ 4 \\ 0 \\ -3 \end{bmatrix} \), and \( v_3 = \begin{bmatrix} 9 \\ h \end{bmatrix} \) are linearly dependent. Justify your answer. **Answer:** This problem is similar to problems 11, 12 in Section 1.7.

Recall the definition: vectors \( v_1, v_2, v_3 \) are linearly independent if one can find a nontrivial linear combination \( c_1v_1 + c_2v_2 + c_3v_3 \) that is equal to \( 0 \).

The system \( c_1 \begin{bmatrix} 3 \\ -6 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ 4 \\ 0 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 9 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) has nontrivial solution when the matrix
\[
\begin{bmatrix} 3 & -6 & 9 \\ -6 & 4 & h \\ 0 & 0 & 0 \\ 1 & -3 & -3 \end{bmatrix}
\]
row-reduces to an echelon form with at least one non-pivot column. Here are the first steps in the Gaussian elimination:
\[
\begin{bmatrix} 3 & -6 & 9 \\ -6 & 4 & h \\ 0 & 0 & 0 \\ 1 & -3 & -3 \end{bmatrix} \quad \leadsto \quad \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \leadsto \quad \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

The matrix \( \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & h + 18 \end{bmatrix} \) is in echelon form and has a pivot in each column, except when \( h + 18 = 0 \), as then the last column has no pivot!
The vectors are linearly dependent when \( h = -18 \).
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1. Matrix \( A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 6 & 1 \\ 3 & 2 & 5 & 8 & 2 \\ 4 & -3 & 1 & 5 & 3 \end{bmatrix} \) is row equivalent to \( B = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \). Describe all solutions of equation \( Ax = 0 \) in parametric vector form (or as a span).

Answer: This problems is similar to problems 8, 11 in Section 1.5.

Equation \( Ax = 0 \) w has the same solution set as equation \( Bx = 0 \). The free variables are \( x_3, x_4 \) and the general solution
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.
\]
So the the solution set is span \( \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \).

2. Find the value(s) of \( h \) for which the vectors \( v_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 4 \end{bmatrix}, \ v_2 = \begin{bmatrix} 4 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \) and \( v_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ h \end{bmatrix} \) are linearly dependent. Justify your answer. Answer: This problems is similar to problems 11, 12 in Section 1.7.

Recall the definition: vectors \( v_1, v_2, v_3 \) are linearly independent if one can find a nontrivial linear combination \( c_1 v_1 + c_2 v_2 + c_3 v_3 \) that is equal to \( 0 \).

The system
\[
c_1 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -6 \\ 0 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 0 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
has nontrivial solution when the matrix
\[
\begin{bmatrix} 2 & 4 & -2 \\ -2 & -6 & 2 \\ 0 & 0 & 0 \\ 4 & 7 & h \end{bmatrix}
\]
row-reduces to an echelon form with at least one non-pivot column. Here are the first steps in the Gaussian elimination:
\[
\begin{bmatrix} 2 & 4 & -2 \\ -2 & -6 & 2 \\ 0 & 0 & 0 \\ 4 & 7 & h \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & -1 & h+4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]
The matrix \( \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \) is in echelon form and has a pivot in each column, except when \( h+4 = 0 \), as then the last column has no pivot!

The vectors are linearly dependent when \( h = -4 \).