MATH 2076 Quiz-1 Answer: Key

Be sure to show your work. No credit for inspired answers!

1. Use Gaussian elimination and the echelon form to determine the values of \( h \) such that the matrix
\[
\begin{bmatrix}
1 & -1 & 4 \\
-2 & 3 & h \\
0 & 1 & 8 + h
\end{bmatrix}
\]
is the augmented matrix of a consistent linear system. (Be sure to write the echelon form!)

**Answer:** This problem is similar to problems 19-22 in Section 1.1. The echelon form of the augmented matrix is
\[
\begin{bmatrix}
1 & -1 & 4 \\
0 & 1 & 8 + h
\end{bmatrix}
\]
with pivots in each row. So the system is consistent for all \( h \).

(In fact, the reduced echelon form is
\[
\begin{bmatrix}
1 & 0 & 12 + h \\
0 & 1 & 8 + h
\end{bmatrix}
\]
. So for every real \( h \), the system has unique solution: \( x = 12 + h \) and \( y = 8 + h \). But the question did not ask for the solution...)

**Answer:** all \( h \)

2. Use Gaussian elimination to determine if vector \( \vec{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \) is a linear combination of the vectors formed from the columns of matrix
\[
A = \begin{bmatrix}
1 & 0 & 5 \\
-2 & 1 & -6 \\
0 & 2 & 8
\end{bmatrix}
\]

**Answer:** This problem is similar to problems 13,14 in Section 1.3. Recall that \( \vec{b} \) is in the span of the columns of
\[
A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]
\]
if we can find \( x_1, x_2, x_3 \in \mathbb{R} \) such that
\[
x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{b}
\]

With \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \), this is the same as
\[
A\vec{x} = \vec{b}
\]
for some \( \vec{x} \in \mathbb{R}^3 \).

To see whether the linear system \( A\vec{x} = \vec{b} \) has a solution we row-reduce the augmented matrix
\[
\begin{bmatrix}
1 & 0 & 5 & 2 \\
-2 & 1 & -6 & -1 \\
0 & 2 & 8 & 6
\end{bmatrix}
\]

The equivalent system has (infinitely many) solutions,
\[
x_1 = 2 - 5x_3, \ x_2 = 3 - 4x_3, \ x_3 \in \mathbb{R} \text{ is free}
\]

For example \( x_1 = 2, \ x_2 = 3, \ x_3 = 0 \) is one such solution, and it is easy to check that it indeed gives the right answer:
\[
2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}
\]

So \( \vec{b} \) is indeed in the span of the columns of \( A \), and one such linear combination of the columns is written in equation (1).
MATH 2076 Quiz-1B  Answer: Key

Be sure to show your work. **No credit for inspired answers!**

1. Use Gaussian elimination and the echelon form to determine the values of $h$ such that the matrix
\[
\begin{bmatrix}
1 & -1 & h \\
-2 & 3 & 3 \\
0 & 1 & 6 + h
\end{bmatrix}
\]
is the augmented matrix of a consistent linear system. (Be sure to write the echelon form!)

**Answer:**  This problem is similar to problems 19-22 is Section 1.1.

The echelon form of the augmented matrix is
\[
\begin{bmatrix}
1 & -1 & 3 \\
0 & 1 & 6 + h
\end{bmatrix}
\]
with pivots in each row. So the system is consistent for all $h$.

(In fact, the reduced echelon form is
\[
\begin{bmatrix}
1 & 0 & 9 + h \\
0 & 1 & 6 + h
\end{bmatrix}
\]. So for every real $h$, the system has unique solution: $x = 9 + h$ and $y = 6 + h$. But the question did not ask for the solution...)

**Answer:**  all $h$

2. Use Gaussian elimination to determine if vector $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of the vectors formed from the columns of matrix $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$.

**Answer:**  This problem similar to problems 13,14 is Section 1.3.

Recall that $\vec{b}$ is in the span of the columns of $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$ if we can find $x_1, x_2, x_3 \in \mathbb{R}$ such that

\[ x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{b} \]

With $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, this is the same as $A\vec{x} = \vec{b}$ for some $\vec{x} \in \mathbb{R}^3$.

To see whether the linear system $A\vec{x} = \vec{b}$ has a solution we row-reduce the augmented matrix
\[
\begin{bmatrix}
1 & 0 & 5 & 2 \\
-2 & 1 & -6 & -1 \\
0 & 2 & 8 & 6
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 5 & 2 \\
0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The equivalent system has (infinitely many) solutions,

\[ x_1 = 2 - 5x_3, \ x_2 = 3 - 4x_3, \ x_3 \in \mathbb{R} \text{ is free} \]

For example $x_1 = 2, \ x_2 = 3, \ x_3 = 0$ is one such solution, and it is easy to check that it indeed gives the right answer:

\[ 2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \] (2)

So $\vec{b}$ is indeed in the span of the columns of $A$, and one such linear combination of the columns is written in equation (2).