MATH 2076 Final-Exam-2017s

Be sure to show your work. No credit for inspired answers! No calculators/computers/cell phones.

1. Find a basis for the null space of matrix \( A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 2 \end{bmatrix} \)

2. Find the value(s) of parameter \( a \) for which polynomials
\[ p_1(x) = x^2 - x + 2, \quad p_2(x) = 4x^2 - 6x + 7, \quad p_3(x) = x^2 - x + a \]

are linearly dependent. Justify your answer!

3. Suppose \( B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \) is a basis of a 3-dimensional subspace \( V \) of \( \mathbb{R}^4 \).

(a) Which vector in \( V \) has coordinates \( \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} \) in basis \( B \)?

(b) One can check that vector \( \vec{y} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \end{bmatrix} \) is in \( V \). What are the coordinates of vector \( \vec{y} \) in basis \( B \)?

4. (a) Find the matrix representation for the orthogonal projection \( \text{Pr} : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) onto the plane \( P = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\} \)

**Warning:** these are orthogonal but not orthonormal vectors!

(b) Find the distance of vector \( \vec{y} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \end{bmatrix} \) from the plane \( P \). (Note: The answer for the distance involves \( \sqrt{3} \) but no fractions.)

5. You are given information that matrix \( A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \) has characteristic polynomial \( \det(A - \lambda I) = (4 - \lambda)(\lambda - 1)^2 \).

(a) Find a basis for the eigenspace corresponding to the eigenvalue \( \lambda = 1 \) of matrix \( A \).

(b) Find an orthogonal basis for that eigenspace.

6. Diagonalize the matrix \( A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \). Use the diagonalization to find the formula for \( A^n \).

7. Find the angle between vectors \( \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 6 \\ 2 \end{bmatrix} \) and \( \vec{v} = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 1 \end{bmatrix} \). Hint: \[ \theta = \frac{\pi}{3} \]

\[ \begin{array}{cccccc} \theta \cos \theta & 0 & \pi/6 & \pi/4 & \pi/3 & \pi/2 \\ \cos \theta & 1 & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 \\ \sin \theta & 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} & -1 \end{array} \]

8. For \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \), let \( Q(\vec{x}) = 3x_1^2 + 4x_1x_2 + x_2^2 \).

(a) Write this quadratic from in the matrix form. That is, write \( Q(\vec{x}) = \vec{x}^T A \vec{x} \) with symmetric matrix \( A \).

(b) Find the eigenvalues of \( A \) and use them to decide if the equation \( 3x_1^2 + 4x_1x_2 + x_2^2 = 100 \) describes an ellipse, or a hyperbola.