MATH 2076 Exam-2 Answer: Key

Be sure to show your work. No credit for inspired answers! No calculators/computers/cell phones.

1. Compute the determinant of matrix
\[
\begin{bmatrix}
0 & 0 & \sqrt{2} & e \\
0 & \sqrt{3} & 0 & e \\
0 & 0 & \sqrt{2} & 0 \\
\pi & \sqrt{3} & \sqrt{2} & e
\end{bmatrix}
\]
(e is the basis of the natural logarithm; \(\pi\) is the area of the unit disk.)

Answer: The determinant is \(-e\pi\sqrt{6}\).

2. Compute the determinant of matrix
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & a & 3 \\
1 & 2 & 3 & a
\end{bmatrix}
\]
Hint: Gaussian elimination works well here.

Answer: \(\det A = (a - 2)^2 - 1 = a^2 - 4a + 3\)

3. Let \(A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 4 & 3 & 5 \end{bmatrix}\). Answer: \(A \mapsto \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}

(a) Find a basis for the null space of matrix \(A\). Answer: \(x_1 = -s/2 + t/2, x_2 = -s/2 - 3/2t\) so a basis is
\[
\begin{bmatrix}
-1/2 \\
-1/2 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
1/2 \\
-3/2 \\
0 \\
1
\end{bmatrix}
\]

(b) Find a basis for the column space of matrix \(A\). Answer: \(\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\)

(c) Find a basis for the row space of matrix \(A\). Answer: \(\begin{bmatrix} 1 & 0 & 1/2 & -1/2, 1 & 1/2 & 3/2 \end{bmatrix}\)

4. Consider the vector space \(V\) of all symmetric 2 by 2 matrices with the basis \(B = \{A_1, A_2, A_3\}\) where \(A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\), \(A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\), \(A_3 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\).

(a) Find the coordinates of the matrix \(A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\) in this basis.

(b) Which matrix \(C\) has coordinates \([C]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\)?

Answer: \(A_2 - A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\), \(A_3 - A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\) and \(A_3 - A_1 = I\) so \(A = 2A_1 - A_3 + (A_2 - A_1) = A_1 + A_2 - A_3\).

\([A]_B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\)

\(B = A_1 + 2A_2 + 3A_3 = \begin{bmatrix} 9 & 6 \\ 6 & 11 \end{bmatrix}\)
5. Prove that polynomials $p_1(t) = t^2, p_2(t) = (1-t)^2, p_3(t) = (1+t)^2$ are linearly independent.

**Answer:** There are many ways of solving this question: one can compute the derivatives, one can express the problem in the standard coordinates, or one can choose enough values of $t$. Here is one of the solutions by the latter method:

Consider 

$$f(t) = C_1 t^3 + C_2 (1-t)^3 + C_3 (1+t)^3$$

and suppose that $f(t) = 0$ for all $t$. Then $f(0) = 0$ and $f(1) = 0$ and $f(-1) = 0$ so we get the following system of equations:

\[
\begin{align*}
C_2 + C_3 &= 0 \\
C_1 + 8C_3 &= 0 \\
-C_1 - 8C_2 &= 0
\end{align*}
\]

The matrix of this system is 

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 8 \\ -1 & -8 & 0 \end{bmatrix}.$$ 

Now 

$$\det A = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 8 \\ 0 & -8 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 \\ -8 & 8 \end{bmatrix} = -16 \neq 0$$

So $A$ is invertible. We now use the invertibility as follows:

Write $\vec{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$. The system of equations (1-3) in vector notation is $A\vec{C} = \vec{0}$. Since $A$ is invertible, the equation $A\vec{C} = \vec{0}$ has the unique solution $\vec{C} = A^{-1} \vec{0} = \vec{0}$. So $\vec{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This shows that $C_1 = C_2 = C_3 = 0$ is the only choice of the coefficients for the linear combination to make $f(t) = 0$ for all $t$. 