Differential Equations MATH 2073 Test-1-2019a *Key*

**Instructions.** Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve $y' + y = 5$, $y(0) = 1$.
   This is a linear and separable equation: \[
   \frac{dy}{y - 5} = -dt
   \]
   $\ln |y - 5| = -t + c$ or $y = 5 + Ce^{-t}$. The equation for constant is $5 + C = 1$, so $y = 5 - 4e^{-t}$

2. Solve the exact equation $2 + 5y + (3 + 5x)y' = 0$. Give explicit answer.
   \[
   \frac{\partial v}{\partial x} = 2 + 5y \Rightarrow 2x + 5xy + h(y) = 3 + 5x \Rightarrow h(y) = 3y
   \]
   Answer: Implicit: $2x + 5xy + 3y = C$ Explicit: $y = \frac{C - 2x}{3 + 5x}$

3. Find the solution of the initial value problem $y'' - 2y' + 2y = 2t$, $y(0) = 1$, $y'(0) = 0$.
   This is linear equation with constant coefficients. Characteristic equation $r^2 - 2r + 2$ has two roots: $r = 1 \pm i$. The general solution is $y = C_1e^t \cos t + C_2e^t \sin t$. Particular solution $y_p = A + Bt$ gives $y_p = 1 + t$.
   The general solution is $y = C_1e^t \cos t + C_2e^t \sin t + 1 + t$. Now we can compute $C_1, C_2$ from the initial condition.
   Answer: $\left\{ y = 1 + t - e^t \sin t \right\}$

4. Find the solution of the equation $y'' + 4y = 10e^t$.
   The homogeneous equation has characteristic equation $r^2 + 4r + 1 = 0$ with roots $r = \pm 2i$ that give $y_1 = \cos 2t, y_2 = \sin 2t$. Using method of undetermined parameters, we seek the particular solution $y_p = Ae^t$.
   Answer: $\left\{ y(t) = C_1 \cos 2t + C_2 \sin 2t + 2e^t \right\}$

5. Find the general solution of nonhomogeneous equation $y'' + 4y' + 4y = 25 \cos t$.
   The general solution of homogeneous equation is $y = C_1e^{-2t} + C_2te^{-2t}$. By undetermined coefficients method we seek particular solution of the form $y_p = A \cos Bt$. We get $A = 4, B = 3$ so the general solution is $y = C_1e^{-2t} + C_2te^{-2t} + 3 \cos t + 4 \sin t$.

6. Use the reduction-of-order substitution $v(t) = y'(t)$ to find the general solution of the (non-homogeneous) equation $ty'' + y' = 1$ for $t > 0$.
   *Hint:* You should get a separable equation for $v$.
   Since $y'' = v'$ the equation is $v'' + v = 1$. This is a separable equation: $v''/(v - 1) = -1/t$. The solution is $\ln(v - 1) = -\ln t + c$. Thus $v = 1 + C_1/t$ and $\left\{ y = \int v dt = C_2 + t + C_1 \ln t \right\}$

Printed: February 24, 2019

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Differential Equations MATH 2073 Test-1-2019a *Key*

**Instructions.** Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve $y' + y = 3$, $y(0) = 1$.
   This is a linear and separable equation: \[
   \frac{dy}{y - 3} = -dt
   \]
   $\ln |y - 3| = -t + c$ or $y = 3 + Ce^{-t}$. The equation for constant is $3 + C = 1$, so $y = 3 - 2e^{-t}$

2. Solve the exact equation $2 + 3y + (3 + 3x)y' = 0$. Give explicit answer.
   \[
   \frac{\partial v}{\partial x} = 2 + 3y \Rightarrow 2x + 3xy + h(y) = 3 + 3x \Rightarrow h(y) = 3y
   \]
   Answer: Implicit: $2x + 3xy + 3y = C$ Explicit: $y = \frac{C - 2x}{3 + 3x}$

3. Find the solution of the initial value problem $y'' - 2y' + 2y = 2t$, $y(0) = 0$, $y'(0) = 1$.
   This is linear equation with constant coefficients. Characteristic equation $r^2 - 2r + 2$ has two roots: $r = 1 \pm i$. The general solution is $y = C_1e^t \cos t + C_2e^t \sin t$. Particular solution $y_p = A + Bt$ gives $y_p = 1 + t$.
   The general solution is $y = C_1e^t \cos t + C_2e^t \sin t + 1 + t$. Now we can compute $C_1, C_2$ from the initial condition.
   Answer: $\left\{ y = 1 + t - e^t \cos t + e^t \sin t \right\}$

4. Find the general solution of the equation $y'' + 4y = 15e^t$.
   Answer: $\left\{ y(t) = C_1 \cos 2t + C_2 \sin 2t + 3e^t \right\}$

5. Find the general solution of nonhomogeneous equation $y'' - 4y' + 4y = 25 \sin t$.
   The general solution of homogeneous equation is $y = C_1e^{2t} + C_2te^{2t}$. By undetermined coefficients method we seek particular solution of the form $y_p = A \cos Bt$. We get $A = 4, B = 3$ so the general solution is $y = C_1e^{2t} + C_2te^{2t} + 4 \cos t + 3 \sin t$.

6. Use the reduction-of-order substitution $v(t) = y'(t)$ to find the general solution of the (non-homogeneous) equation $ty'' + y' = 1$ for $t > 0$.
   *Hint:* You should get a separable equation for $v$.
   Since $y'' = v'$ the equation is $v'' + v = 1$. This is a separable equation: $v''/(v - 1) = -1/t$. The solution is $\ln(v - 1) = -\ln t + c$. Thus $v = 1 + C_1/t$ and $\left\{ y = \int v dt = C_2 + t + C_1 \ln t \right\}$

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Differential Equations MATH 2073 Test-1-2019

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve $y' + y = 4$, $y(0) = 1$
   This is a linear and separable equation:
   \[
   \frac{dy}{y-4} = -dt
   \]
   \[\ln |y-4| = -t + C \text{ or } y = 4 + Ce^{-t}. \text{ The equation for constant is } y = 4 - Ce^{-t} \]

2. Solve the exact equation $2 + 4y + (3 + 4x)y' = 0$. Give explicit answer.
   \[
   \frac{dy}{dx} = 2 + 4y \quad \text{so} \quad \psi = 2x + 4xy + h(y). \text{ This gives } \frac{dy}{h(y)} = 3 + 4x \quad \text{so} \quad h(y) = 3y
   \]
   Answer: Implicit: $\int 2x + 4xy + 3y = C$.Explicit: $y = \frac{C - 2x}{3 + 4x}$

3. Find the solution of the initial value problem $y'' - 2y' + 2y = 2t$, $y(0) = 1$, $y'(0) = 0$.
   This is linear equation with constant coefficients. Characteristic equation $r^2 - 2r + 2$ has two roots: $r = 1 \pm i$. The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t$. Particular solution $y_p = A + Bt$ gives $y_p = 1 + t$
   The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t + 1 + t$. Now we can compute $C_1, C_2$ from the initial condition.
   Answer: $\{y = 1 + t - e^t \sin t\}$

4. Find the general solution of the equation $y'' + 4y = 15e^t$
   Answer: $\{y(t) = C_1 \cos 2t + C_2 \sin 2t + 3e^t\}$

5. Find the general solution of nonhomogeneous equation $y'' + 4y' + 4y = 25 \cos t$.
   The general solution of homogeneous equation is $y = C_1 e^{-2t} + C_2 e^{-2t}$. By undetermined coefficients method we seek particular solution of the form $y_p = A \cos B + B \sin t$. We get $A = 3, B = 4$ so the general solution is $\{y = C_1 e^{-2t} + C_2 e^{-2t} + 3 \cos t + 4 \sin t\}$

6. Use the reduction-of-order substitution $v(t) = y'(t)$ to find the general solution of the (non-homogeneous) equation $ty'' + y' = 1$ for $t > 0$.
   Hint: You should get a separable equation for $v$.
   Since $y'' = v'$ the equation is $v'' + v = 1$. This is a separable equation: $v'(v - 1) = -1/t$. The solution is $\ln(v - 1) = -\ln t + C$. Thus \(v = 1 + C_1 / t\) and $\{y = \int v dt = C_2 + t + C_1 \ln t\}$

Differential Equations MATH 2073 Test-1-2019

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve $y' + y = 6$, $y(0) = 1$
   This is a linear and separable equation:
   \[
   \frac{dy}{y-6} = -dt
   \]
   \[\ln |y-6| = -t + C \text{ or } y = 6 + Ce^{-t}. \text{ The equation for constant is } y = 6 - 5e^{-t} \]

2. Solve the exact equation $2 + 6y + (3 + 6x)y' = 0$. Give explicit answer.
   \[
   \frac{dy}{dx} = 2 + 6y \quad \text{so} \quad \psi = 2x + 6xy + h(y). \text{ This gives } \frac{dy}{h(y)} = 3 + 6x \quad \text{so} \quad h(y) = 3y
   \]
   Answer: Implicit: $\int 2x + 6xy + 3y = C$.Explicit: $y = \frac{C - 2x}{3 + 6x}$

3. Find the solution of the initial value problem $y'' - 2y' + 2y = 2t$, $y(0) = 0$, $y'(0) = 1$.
   This is linear equation with constant coefficients. Characteristic equation $r^2 - 2r + 2$ has two roots: $r = 1 \pm i$. The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t$. Particular solution $y_p = A + Bt$ gives $y_p = 1 + t$
   The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t + 1 + t$. Now we can compute $C_1, C_2$ from the initial condition.
   Answer: $\{y = 1 + t - e^t \sin t\}$

4. Find the general solution of the equation $y'' + 4y = 10e^t$
   Answer: $\{y(t) = C_1 \cos 2t + C_2 \sin 2t + 2e^t\}$

5. Find the general solution of nonhomogeneous equation $y'' - 4y' + 4y = 25 \sin t$.
   The general solution of homogeneous equation is $y = C_1 e^{2t} + C_2 e^{-2t}$. By undetermined coefficients method we seek particular solution of the form $y_p = A \cos B + B \sin t$. We get $A = 4, B = 3$ so the general solution is $\{y = C_1 e^{2t} + C_2 e^{-2t} + 4 \cos t + 3 \sin t\}$

6. Use the reduction-of-order substitution $v(t) = y'(t)$ to find the general solution of the (non-homogeneous) equation $ty'' + y' = 1$ for $t > 0$.
   Hint: You should get a separable equation for $v$.
   Since $y'' = v'$ the equation is $v'' + v = 1$. This is a separable equation: $v'(v - 1) = -1/t$. The solution is $\ln(v - 1) = -\ln t + C$. Thus $v = 1 + C_1 / t$ and $\{y = \int v dt = C_2 + t + C_1 \ln t\}$