Differential Equations MATH 2073 Test-1-2019 Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve $y'' + y = 5$, $y(0) = 1$
   This is a linear and separable equation:
   \[
   \frac{dy}{y-5} = -dt
   \]
   \[
   \ln |y - 5| = -t + c \quad \text{or} \quad y = 5 + Ce^{-t}. \quad \text{The equation for constant is } 3 + C = 1, \quad \text{so} \quad y = 5 - 4e^{-t}
   \]

2. Solve the exact equation $2 + 5y + (3 + 5x)y' = 0$. Give explicit answer.
   \[
   \frac{dy}{dx} = 2 + 5y \quad \text{so} \quad \psi = 2x + 5xy + h(y). \quad \text{This gives } \frac{d\psi}{dy} = 5x + h'(y) = 3 + 5x \quad \text{so} \quad h(y) = 3y
   \]
   Answer: Implicit: \[2x + 5xy + 3y = C\]
   Explicit: \[y = \frac{C-3x}{3+5x}\]

3. Find the solution of the initial value problem $y'' - 2y' + 2y = 2t$, $y(0) = 1$, $y'(0) = 0$.
   This is linear equation with constant coefficients. Characteristic equation $r^2 - 2r + 2$ has two roots: $r = 1 \pm i$. The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t$. Particular solution $y_p = A + Bt$ gives $y_p = 1 + t$
   The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t + 1 + t$. Now we can compute $C_1, C_2$ from the initial condition.
   Answer: \[y = 1 + t - e^t \sin t\]

4. Find the solution of the equation $y''' + 4y = 10e^t$
   The homogeneous equation has characteristic equation $r^3 + 4r = 1$ with roots $r = \pm 2i$ that gives $y_1 = \cos 2t, y_2 = \sin 2t$. Using method of undetermined parameters, we seek the particular solution $y_p = Ate^t$.
   Answer: \[y(t) = C_1 \cos 2t + C_2 \sin 2t + 2e^t\]

5. Find the general solution of nonhomogeneous equation $y''' + 4y' + 4y = 25 \cos t$
   The general solution of homogeneous equation is $y = C_1 e^{-2t} + C_2 te^{-2t}$. By undetermined coefficients method we seek particular solution of the form $y_p = C e^t \sin t$. We get $A = 3, B = 4$ so the general solution is $y = C_1 e^{-2t} + C_2 te^{-2t} + 3 \cos t + 4 \sin t$

6. Use the reduction-of-order substitution $v(t) = y'(t)$ to find the general solution of the (non-homogeneous) equation $ty'' + y' = 1$ for $t > 0$.
   Hint: You should get a separable equation for $v$. Since $y'' = v'$ the equation is $v'' + v = 1$. This is a separable equation: $v'/(v-1) = -1/t$. The solution is $\ln (v-1) = -\ln t + c$. Thus $v = 1 + C_1/t$ and \[y = \int v dt = C_2 + t / C_1 \ln t\]

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Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve $y'' + y = 3$, $y(0) = 1$
   This is a linear and separable equation:
   \[
   \frac{dy}{y-3} = -dt
   \]
   \[
   \ln |y - 3| = -t + c \quad \text{or} \quad y = 3 + Ce^{-t}. \quad \text{The equation for constant is } 3 + C = 1, \quad \text{so} \quad y = 3 - 2e^{-t}
   \]

2. Solve the exact equation $2 + 3y + (3 + 3x)y' = 0$. Give explicit answer.
   \[
   \frac{dy}{dx} = 2 + 3y \quad \text{so} \quad \psi = 2x + 3xy + h(y). \quad \text{This gives } \frac{d\psi}{dy} = 3x + b'(y) = 3 + 3x \quad \text{so} \quad h(y) = 3y
   \]
   Answer: Implicit: \[2x + 3xy + 3y = C\]
   Explicit: \[y = \frac{C-3x}{3+3x}\]

3. Find the solution of the initial value problem $y'' - 2y' + 2y = 2t$, $y(0) = 0$, $y'(0) = 1$.
   This is linear equation with constant coefficients. Characteristic equation $r^2 - 2r + 2$ has two roots: $r = 1 \pm i$. The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t$. Particular solution $y_p = A + Bt$ gives $y_p = 1 + t$
   The general solution is $y = C_1 e^t \cos t + C_2 e^t \sin t + 1 + t$. Now we can compute $C_1, C_2$ from the initial condition.
   Answer: \[y = 1 + t - e^t \cos t\]

4. Find the general solution of the equation $y'' + 4y = 15e^t$
   Answer: \[y(t) = C_1 \cos 2t + C_2 \sin 2t + 3e^t\]

5. Find the general solution of nonhomogeneous equation $y'' - 4y' + 4y = 25 \sin t$
   The general solution of homogeneous equation is $y = C_1 e^{2t} + C_2 te^{2t}$. By undetermined coefficients method we seek particular solution of the form $y_p = A \cos Bt$. We get $A = 4, B = 3$ so the general solution is \[y = C_1 e^{2t} + C_2 te^{2t} + 4 \cos t + 3 \sin t\]

6. Use the reduction-of-order substitution $v(t) = y'(t)$ to find the general solution of the (non-homogeneous) equation $ty'' + y' = 1$ for $t > 0$.
   Hint: You should get a separable equation for $v$. Since $y'' = v'$ the equation is $v'' + v = 1$. This is a separable equation: $v'/(v-1) = -1/t$. The solution is $\ln (v-1) = -\ln t + c$. Thus $v = 1 + C_1/t$ and \[y = \int v dt = C_2 + t / C_1 \ln t\]
Differential Equations MATH 2073 Test-1-2019

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve \( y' + y = 4 \), \( y(0) = 1 \)

   This is a linear and separable equation:
   \[
   \frac{dy}{y - 4} = -dt
   \]
   \[
   \ln |y - 4| = -t + C \quad \text{or} \quad y = 4 + Ce^{-t}.
   \]
   The equation for constant is \( 4 + C = 1 \), so \( y = 4 - 3e^{-t} \)

2. Solve the exact equation \( 2 + 6y' + (3 + 4x)y'' = 0 \). Give explicit answer.

   \[
   \frac{\partial v}{\partial x} = 2 + 4y \quad \text{so} \quad v = 2x + 4xy + h(y).
   \]
   This gives \( \frac{\partial v}{\partial y} = 4x + h'(y) = 3 + 4x \quad \text{so} \quad h(y) = 3y \)

   Answer: Implicit: \( 2x + 4xy = C \quad \text{Explicit:} \quad y = \frac{C - 2x}{4x + 3} \)

3. Find the solution of the initial value problem \( y'' - 2y' + 2y = 2t \), \( y(0) = 1 \), \( y'(0) = 0 \).

   This is linear equation with constant coefficients. Characteristic equation \( r^2 - 2r + 2 \) has two roots: \( r = 1 \pm i \). The general solution is
   \[ y = C_1e^t \cos t + C_2e^t \sin t. \]
   Particular solution \( y_p = A + Bt \) gives \( y_p = 1 + t \)

   The general solution is \( y = C_1e^t \cos t + C_2e^t \sin t + 1 + t \). Now we can compute \( C_1, C_2 \) from the initial condition.

   Answer: \( y = 1 + t - e^t \sin t \)

4. Find the general solution of the equation \( y'' + 4y = 15e^t \)

   Answer: \( y(t) = C_1 \cos 2t + C_2 \sin 2t + 3e^t \)

5. Find the general solution of nonhomogeneous equation \( y'' + 4y' + 4y = 25 \cos t \).

   The general solution of homogeneous equation is \( y = C_1e^{-2t} + C_2te^{-2t} \). By undetermined coefficients method we seek particular solution of the form \( y_p = A \cos + B \sin t \). We get \( A = 3, B = 4 \) so the general solution is \( y = C_1e^{-2t} + C_2te^{-2t} + 3 \cos t + 4\sin t \)

6. Use the reduction-of-order substitution \( v(t) = y'(t) \) to find the general solution of the (non-homogeneous) equation \( ty'' + y' = 1 \) for \( t > 0 \).

   Hint: You should get a separable equation for \( v \).

   Since \( y'' = v' \) the equation is \( v'/v + 1 = 1 \). This is a separable equation: \( v'/v - 1 = -1/t \). The solution is \( \ln(v - 1) = -\ln t + c \). Thus \( v = 1 + C_1/t \) and \( y = \int v \, dt = C_2 + t + C_1 \ln t \)

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Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve \( y' + y = 6 \), \( y(0) = 1 \)

   This is a linear and separable equation:
   \[
   \frac{dy}{y - 6} = -dt
   \]
   \[
   \ln |y - 6| = -t + C \quad \text{or} \quad y = 6 + Ce^{-t}.
   \]
   The equation for constant is \( 6 + C = 1 \), so \( y = 6 - 5e^{-t} \)

2. Solve the exact equation \( 2 + 6y + (3 + 6x)y' = 0 \). Give explicit answer.

   \[
   \frac{\partial v}{\partial x} = 2 + 6y \quad \text{so} \quad v = 2x + 6xy + h(y).
   \]
   This gives \( \frac{\partial v}{\partial y} = 6x + h'(y) = 3 + 6x \quad \text{so} \quad h(y) = 3y \)

   Answer: Implicit: \( 2x + 6xy = C \quad \text{Explicit:} \quad y = \frac{C - 2x}{4x + 3} \)

3. Find the solution of the initial value problem \( y'' - 2y' + 2y = 2t \), \( y(0) = 0 \), \( y'(0) = 1 \).

   This is linear equation with constant coefficients. Characteristic equation \( r^2 - 2r + 2 \) has two roots: \( r = 1 \pm i \). The general solution is
   \[ y = C_1e^t \cos t + C_2e^t \sin t. \]
   Particular solution \( y_p = A + Bt \) gives \( y_p = 1 + t \)

   The general solution is \( y = C_1e^t \cos t + C_2e^t \sin t + 1 + t \). Now we can compute \( C_1, C_2 \) from the initial condition.

   Answer: \( y = 1 + t - e^t \cos t \)

4. Find the general solution of the equation \( y'' + 4y = 10e^t \)

   Answer: \( y(t) = C_1 \cos 2t + C_2 \sin 2t + 3e^t \)

5. Find the general solution of nonhomogeneous equation \( y'' - 4y' + 4y = 25 \sin t \).

   The general solution of homogeneous equation is \( y = C_1e^{2t} + C_2te^{2t} \). By undetermined coefficients method we seek particular solution of the form \( y_p = A \cos + B \sin t \). We get \( A = 4, B = 3 \) so the general solution is \( y = C_1e^{2t} + C_2te^{2t} + 4 \cos t + 3\sin t \)

6. Use the reduction-of-order substitution \( v(t) = y'(t) \) to find the general solution of the (non-homogeneous) equation \( ty'' + y' = 1 \) for \( t > 0 \).

   Hint: You should get a separable equation for \( v \).

   Since \( y'' = v' \) the equation is \( v'/v + 1 = 1 \). This is a separable equation: \( v'/v - 1 = -1/t \). The solution is \( \ln(v - 1) = -\ln t + c \). Thus \( v = 1 + C_1/t \) and \( y = \int v \, dt = C_2 + t + C_1 \ln t \).