Differential Equations MATH 2073 Quiz-8. Key

Question. Use the power series \( y = \sum_{n=0}^{\infty} a_n x^n \) to determine the solution of the initial value problem

\[
y'' - 2xy' - 2y = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0.
\]

Recommended procedure:

- Use the initial condition to determine the first two coefficients \( a_0 \) and \( a_1 \) (1pt)
- Use the differential equation to determine the recursion for \( a_n \) (5pts)
- Use the recursion to compute the first couple of the coefficients (like \( a_2 \) through say \( a_6 \)) (1pt)
- Solve the recursion for \( a_n \), e.g. by noticing a pattern (1pt)
- Express your answer using summation notation (1pt)
- If possible, identify the function in your answer (1pt)

Solution: Recall the formulas for the series expansion: (Are we allowed to use \( c_n \) instead of \( a_n \)?)

\[
y = \sum_{n=0}^{\infty} c_n x^n
\]

\[
y' = \sum_{n=1}^{\infty} n c_n x^{n-1}
\]

\[
y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}
\]

Note that \( x_0 = 0 \) is an ordinary point so the initial condition at \( x_0 = 0 \) makes sense. With \( y = c_0 + c_1 x + \ldots \) and \( y' = c_1 + 2c_2 x + \ldots \) we get \( y(0) = c_0 \) and \( y'(0) = c_1 \) so \( c_0 = 1 \) and \( c_1 = 0 \)

Inserting the series expansions into the left hand side of the differential equation we get

\[
y'' - 2xy' - 2y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x \sum_{n=1}^{\infty} nc_n x^{n-1} - 2 \sum_{n=0}^{\infty} c_n x^n
\]

\[
= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2nc_n x^{n-1} - \sum_{n=0}^{\infty} 2c_n x^n
\]

\[
= \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} 2nc_n x^n - \sum_{n=0}^{\infty} 2c_n x^n
\]

\[
= 2c_2 - 2c_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - 2nc_n - 2c_n] x^n
\]

\[
= 2(c_2 - c_0) + \sum_{n=1}^{\infty} [(n+2)c_{n+2} - 2c_n] (n+1)x^n
\]

Since the right hand of the differential equation is zero, all the coefficients at the consecutive powers of \( x \) must be zero. Therefore \( c_2 = c_0 \) and for \( n \geq 1 \) we have \( (n+2)c_{n+2} - 2c_n = 0 \). This gives \( c_{n+2} = \frac{2}{n+2} c_n \). In particular,

- \( c_0 = 1 = \frac{1}{0!} \) (from the initial condition)
- \( c_1 = 0 \) (from the initial condition)
- \( c_2 = 1 = \frac{1}{1!} \) (from the "irregular" term in the recursion)
- \( c_3 = 0 \) (from \( c_{n+2} = \frac{2}{n+2} c_n \) and hence we notice a pattern \( c_{2k+1} = 0 \) for all odd coefficients)
- \( c_4 = \frac{2}{3} c_2 = \frac{2}{3} \) (from \( c_{n+2} = \frac{2}{n+2} c_n \))
- \( c_6 = \frac{2}{5} c_4 = \frac{2}{5} \times \frac{2}{3} c_2 = \frac{4}{15} \) (from \( c_{n+2} = \frac{2}{n+2} c_n \))

So we notice the pattern! All even coefficients follow the same pattern \( c_{2k} = \frac{2}{2k} c_{2k-2} = \frac{1}{k} c_{2k-2} = \frac{1}{k(k-1)} c_{2k-4} = \ldots = \frac{1}{k!} c_2 = \frac{1}{k!} \)

Inserting our new formulas for \( c_n \) back into the series expansion \( y = \sum_{n=0}^{\infty} c_n x^n \) we get

\[
y = \left( 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots \right) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = e^{x^2}
\]