Worksheet on models  Key

Instructions. In each of the problems below, setup the differential equation. Then solve it and use your solution (and a calculator) to answer the question. (Be sure to convert the data to convenient units!)

1. A tank originally contains 100 gal of fresh water. Then water containing \( \frac{1}{2} \) lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. Setup the differential equation for the amount \( A(t) \) of salt in the tank at time \( t \). At what moment of time the amount of salt in the tank exceeds 25 lb?

   Equation: The “rate in - rate out” formula gives \( A'(t) = 1 - A/50 \)

Solving the equation, we get:

\[
d\frac{dA}{dt} = -\frac{A - 50}{50}
\]

So

\[
\frac{dA}{A - 50} = -\frac{dt}{50}
\]

Integrating, we get \( \ln |A - 50| = -t/50 + c \) or \( A(t) = 50 + Ce^{-t/50} \). Since \( A(0) = 0 \) (fresh water), we get

\[
A(t) = 50 - 50e^{-t/50}
\]

\( A(t) = 25 \) when \( e^{-t/50} = 1/2 \), or \( t = 50 \ln 2 \approx 34.7 \) min.

2. A student graduates from college with a $10,000 loan, at 3% yearly rate. Determine what monthly payments will pay off her loan by the time she retires in 40 years. In your solution use the model with continuous compounding and continuous payment. (For the usual monthly schedule, payments are $35.80.)

Equation: Let use years and K$ as units. Then the equation is \( y' = .03y - 12m, y(0) = 10 \)

\( y(t) = e^{0.03t}(10 - 400m) + 400m \), so \( y(40) = 0 \) when \( m = 35.7753 \). EXCEL function PMT returns monthly payments of $36.05 for yearly compounding, $35.80 (=35.7984) for monthly, $35.78(=35.7761) for daily.
3. It is known that cigarette smoke contains about 4\% of poisonous carbon monoxide $CO$. A smoker stops at my door, trying to get some help on a complicated differential equation of order 5. While listening to my answer, the smoker is exhaling smoke into my office at a rate of 0.1 ft$^3$ per minute, and well circulated mixture of air escapes the room at the same rate. The air conditioning is off. My office (FH 4316) has the dimensions 10 $\times$ 14 $\times$ 15 ft and is smoke-free.

(a) Set up the differential equation for the amount of carbon monoxide in my office.

Let $Q(t)$ denote the amount of CO. Then $Q(0) = 0$ and $Q'(t) = \frac{0.04}{10 \times 14 \times 15} - 0.1 Q(t)$.

(b) Find an expression for the concentration of carbon monoxide in the room at any time $t > 0$.

Let $y$ denote the concentration. Clearly, $y = \frac{Q}{2100}$ so the equation is $y' = \frac{0.04}{2100} - \frac{0.1}{2100} y$, $y(0) = 0$. The solution is $y(t) = \frac{0.04 (1 - e^{-0.000476 t})}{10 \times 14 \times 15}$.

(c) Find the time $T$ at which harmful concentration of 0.012\% is reached. Solving the equation $4(1 - e^{-0.000476 t}) = 0.012$ we get $t = 63.09$ minutes.

4. A turkey is ready to eat when its temperature is 180F. Baking instruction on the wrapping says the following

- thaw the turkey (40F)
- preheat oven to 350F
- put the turkey into the oven for 3 hours.

I need to bake a frozen turkey (0F) in the same amount of time. What temperature should I set to preheat the oven? Use Newtons law of cooling: the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings.

Let $T$ denote the temperature of the baking turkey. Let $T_o$ denote the unknown temperature of the oven. The Newton law of cooling says that $T' = k(350 - T)$, $T(0) = 40$ in the first case, and $T' = k(T_o - T)$, $T(0) = 0$ in the second case. Thus $T(t) = 350 - 310e^{-kt}$ for the first case and $T(t) = T_o(1 - e^{-kt})$ for the second one. Now compute $T_o$ from the condition that $T(3) = 180$ in both cases. From the first equation $e^{-3k} = 17/31$ thus $T_o = \frac{31}{14} \times 180 = 398.6 \approx 400F$. 