Differential Equations MATH 2073 Exam-4 Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Find the general solution of the equation $y^{iv} - y = 0$ (with the fourth derivative denoted by $y^{iv} := \frac{d^4y}{dt^4}$).
   The characteristic equation $r^4 - 1 = 0$ factors as $r^4 - 1 = (r^2 - 1)(r^2 + 1) = (r - 1)(r + 1)(r - i)(r + i) = 0$ with roots $\pm 1, \pm i$. The general solution is $y = C_1e^t + C_2e^{-t} + C_3\sin t + C_4\cos t$

Then find particular solution for each of the equations

(a) $y^{iv} - y = 10e^t$ Seek $y_p = Ae^t$. Answer: \( y_p = \frac{5}{2}te^t \)

(b) $y^{iv} - y = 12t$
   Seek $y_p = A + Bt$. Answer: \( y_p = -12t \)

(c) $y^{iv} - y = 12\cos 2t$
   Seek $y_p = A \cos(2t) + B \sin(2t)$. Answer: \( y_p = \frac{1}{5} \cos(2t) \)
2. The general solution of the homogeneous equation \( y^{\prime\prime\prime} + y^{\prime\prime} + y^{\prime} + y = 0 \) is \( y = C_1 \cos t + C_2 \sin t + C_3e^{-t} \) and a particular solution of the equation \( y^{\prime\prime\prime} + y^{\prime\prime} + y^{\prime} + y = t \) is \( y_p = t - 1 \).

Use this information to solve the initial value problem
\[
y^{\prime\prime\prime} + y^{\prime\prime} + y^{\prime} + y = t, \ y(0) = 1, y'(0) = 0, y''(0) = 0
\]

The general solution of the equation \( y^{\prime\prime\prime} + y^{\prime\prime} + y^{\prime} + y = t \) is \( y = C_1 \cos t + C_2 \sin t + C_3e^{-t} + t - 1 \). So we need to solve the system of equations:
\[
\begin{align*}
y(0) &= C_1 + C_3 - 1 = 1 \\
y'(0) &= C_2 - C_3 + 1 = 0 \\
y''(0) &= -C_1 + C_3 = 0
\end{align*}
\]

The solution is \( C_1 = 1, C_2 = 0, C_3 = 1 \) so the answer is \( y = \cos t + e^{-t} + t - 1 \)
3. Find the general solution of the homogeneous equation \( y''' + 4y' = 0 \).

**Solution:** We first find the general solution of the homogeneous equation \( y''' + 4y' = 0 \). Characteristic equation \( r^3 + 4r = 0 \) factors as \( r(r - 2i)(r + 2i) = 0 \). The roots are 0, \(-2i, 2i\). **Answer:** \( y = C_1 + C_2 \cos 2x + C_3 \sin 2x \)

Then find particular solution for each of the equations

(a) \( y''' + 4y' = 10e^t \)

Seek \( y_p = Ae^t \). Answer: \( y_p = 2e^t \)

(b) \( y''' + 4y' = 12t \)

Seek \( y_p = (A + Bt)t \). Answer: \( y_p = \frac{3}{2}t^2 \)

(c) \( y''' + 4y' = 3 \cos t \)

**Solution:**

We search for the particular solution of the form \( Y = A \cos t + B \sin t \). This gives \( Y' = -A \sin t + B \cos t, Y''' = A \sin t - B \cos t \). So \( L[Y] = -3A \sin t + 3B \cos t \) and

\[
A = 0, \quad B = 1
\]

**Answer:** \( y_p = \sin t \) (The general solution is thus \( y = C_1 + C_2 \cos 2x + C_3 \sin 2t + \sin t \)).

(d) \( y''' + 4y' = 12 \cos 2t \)

Seek \( y_p = At \cos(2t) + Bt \sin(2t) \). Answer: \( y_p = -\frac{3}{2}t \cos(2t) \)
4. Find the general solution of the equation $y''' - y'' - y' + y = 0$.
   Characteristic equation $r^3 - r^2 - r + 1 = 0$ factors as $(r - 1)(r^2 - 1) = (r - 1)^2(r + 1)$ so the general solution of the homogeneous equation is $y = C_1 e^t + C_2 te^t + C_3 e^{-t}$

Then find particular solution for each of the equations

(a) $y''' - y'' - y' + y = 12t$
   Seek $y_p = A + Bt$. Answer: $y_p = 12(t + 1)$

(b) $y''' - y'' - y' + y = 10e^t$
   Seek $y_p = At^2 e^t$. Answer: $y_p = \frac{5}{2} t^2 e^t$

(c) $y''' - y'' - y' + y = 3 \cos t$
   Seek $y_p = A \cos t + B \sin t$. Answer: $y_p = \frac{3}{4} (\cos t - \sin t)$

(d) $y''' - y'' - y' + y = 12 \cos 2t$
   Seek $y_p = A \cos(2t) + B \sin(2t)s$. Answer: $y_p = \frac{12}{25} \cos(2t) - \frac{24}{25} \sin(2t)$